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$\vec{P}_{4k-1}$ –Factorization of Symmetric Complete Bipartite Digraph

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ABSTRACT
In path factorization, Ushio K [1] gave the necessary and sufficient conditions for P_k –design when k is odd. P_{2p} –Factorization of a complete bipartite graph for p, an integer was studied by Wang [2]. Further, Beiling [3] extended the work of Wang [2], and studied P_{2k} –factorization of complete bipartite multigraphs. For even value of k in P_k –factorization the spectrum problem is completely solved [1, 2, 3]. However, for odd value of k i .e. P_3, P_5, P_7, P_9 and P_{4k-1}, the path factorization have been studied by a number of researchers [4, 5, 6, 7, 8]. The necessary and sufficient conditions for the existences of P_3 –factorization of symmetric complete bipartite digraph were given by Du B [9]. Earlier we have discussed the necessary and sufficient conditions for the existence of $\vec{P}_5$ and $\vec{P}_7$ –factorization of symmetric complete bipartite digraph [10, 11]. Now, in the present paper, we give the necessary and sufficient conditions for the existence of $\vec{P}_{4k-1}$ –factorization of symmetric complete bipartite digraph of $K^*_{m,n}$.

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Key words: Complete bipartite Graph, Factorization of Graph, Symmetric Graph.

1. INTRODUCTION
Let $K^*_{m,n}$be a complete bipartite symmetric digraph with two partite sets having m and n vertices. A spanning sub graph $\vec{F}$ of $K^*_{m,n}$ is called a path factor if each component of $\vec{F}$ is a path of order at least two. In particular, a spanning sub graph $\vec{F}$ of $K^*_{m,n}$ is called a $\vec{P}_{4k-1}$ –factor if each component of $\vec{F}$ is isomorphic to $\vec{P}_{4k-1}$. If $K^*_{m,n}$ is expressed as an arc disjoint sum of $\vec{P}_{4k-1}$ –factors, then this sum is called $\vec{P}_{4k-1}$ –factorization of $K^*_{m,n}$. Here, we take path of order $4k-1$. A $\vec{P}_{4k-1}$ is the directed path on $4k-1$ vertices.

2. MATHEMATICAL ANALYSIS
The necessary and sufficient conditions for the existence of $\vec{P}_{4k-1}$ –factorization of complete bipartite symmetric digraph are given below in theorem 2.1.

Theorem 2.1: Let m and n be the positive integers then $K^*_{m,n}$ has a $\vec{P}_{4k-1}$ –factorization iff:

(1) $2kn \geq (2k-1)m$,

(2) $2km \geq (2k-1)n$,

(3) $m+n \equiv 0 (mod\ 4k-1)$, and

(4) $(4k-1)mn/[(2k-1)(m+n)]$ is an integer.

Proof of necessity of theorem 2.1
Proof: Let r be the number of $\vec{P}_{4k-1}$ –factor in the factorization, and e be the number of copies of $\vec{P}_{4k-1}$ –factor in a factorization, which can be computed by using

$$r = \frac{(4k-1)mn}{[(2k-1)(m+n)]} \quad \ldots (1)$$

and

$$e = \frac{m+n}{4k-1} \quad \ldots (2)$$

respectively.

Obviously, r and e will be integers. Thus conditions (3) and (4) in theorem 2.1 are necessary. Let a and b be the number of copies of $\vec{P}_{4k-1}$ with its end points in Y and X, respectively in a particular $\vec{P}_{4k-1}$ –factor. Then by simple arithmetic we can obtain, $2kb + (2k-1)a = m$ and $2ka + (2k-1)b = n$.

From this, we can compute a and b which are as follows:

$$a = \frac{2kn-(2k-1)m}{4k-1} \quad \ldots (3)$$

$$b = \frac{2km-(2k-1)n}{4k-1} \quad \ldots (4).$$

Since, by definition a and b are integers, therefore equation (1) and (2) imply,

$$\frac{2kn-(2k-1)m}{4k-1} \geq 0$$

and

$$\frac{2km-(2k-1)n}{4k-1} \geq 0,$$

this implies that $2kn \geq (2k-1)m$ that $2km \geq (2k-1)n$. Therefore condition (1) and

(2) in theorem 2.1 are necessary. This proves the necessity of theorem 2.1.

Proof of sufficiency of theorem 2.1

Further, we need the following number theoretic result (lemma 2.2) to prove the sufficiency of theorem 2.1. Its proof can be found in any good text related to number theory [12].

Lemma 2.2: If $\gcd(xu, yv) = 1$ then $\gcd(uv, ux + vy) = 1$, where x, y, u and v are positive integers.

We prove the following result of lemma 2.3, which will be used further.

Lemma 2.3: If $K^*_{m,n}$ has $\vec{P}_{4k-1}$ –factorization, then $K^*_{sm,sn}$ has a $\vec{P}_{4k-1}$ –factorization for every positive integer s.

Proof: Let $K_{s,s}$ is 1- factorable [13] and $\{F_1, F_2, \dots, F_s\}$ be a 1- factorization of it. For each i with $1 \le i \le s$, replace every edge of F_i by a $K^*_{m,n}$to get a spanning sub-graph G_i of $K^*_{sm,sn}$ such that the graph G_i's $\{1 \le i \le s\}$ are pair wise edge disjoint , and there union is $K^*_{sm,sn}$. Since $K^*_{m,n}$ has a $\vec{P}_{4k-1}$ –factorization, it is clear that G_i is also $\vec{P}_{4k-1}$ –factorable, and hence $K^*_{sm,sn}$, $\vec{P}_{4k-1}$ –factorization.

Now to prove the sufficiency of theorem 2.1, there are three cases to consider:

Case (i) $2kn = (2k-1)m$; In this case from lemma 2, $K^*_{m,n}$ has $\vec{P}_{4k-1}$ –factorization. Consider the trivial case at $k = 1, m = 1$ and $n = 2$, then number of copy $e = 1$ and total number of factor $r = 2$. Path factor is given below:

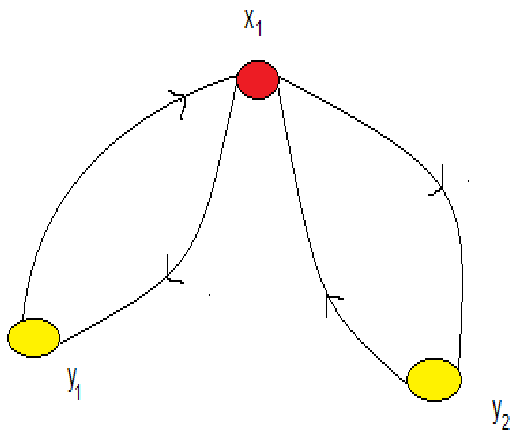

Fig :1 shows the path $y_1x_1y_2$; $y_2x_1y_1$.

Case (ii) $2km = (2k-1)n$; Obliviously, $K^*_{m,n}$ has $\vec{P}_{4k-1}$ –factorization since in this case position of m and n changes only from previous case.

Case (iii): $2km > (2k-1)n$ and $2kn > (2k-1)m$. In this case, let

$$a = \frac{2kn - (2k-1)m}{4k-1}, \qquad b = \frac{2km - (2k-1)n}{4k-1},$$

$$r = \frac{(4k-1)mn}{[(2k-1)(m+n)]} \quad \text{and} \quad e = \frac{m+n}{4k-1}.$$

Then from condition (1)-(4) of theorem 2.1, a, b, e and r are integers and $0 < a < m$ and $0 < b < n$. As obtained previously $2kb + (2k-1)a = m$ and $2ka + (2k-1)b = n$. Hence

$$r = 2k(a+b) + \frac{ab}{[(2k-1)(a+b)]}.$$

Let

$$z = \frac{ab}{[(2k-1)(a+b)]},$$

be a positive integer. Again let $\gcd\big((2k-1)a, 2kb\big) = d$ then $(2k-1)a = dp$ and $2kb = dq$, where p and q are integer and $\gcd(p, q) = 1$.

Then

$$z = \frac{dpq}{(2k-1)[2kp + (2k-1)q]}.$$

These equality implies the following equality:

$$d = \frac{(2k-1)[2kp + (2k-1)q]z}{pq},$$

$$r = \frac{(p+q)[4k^2p + (2k-1)^2q]z}{pq},$$

$$m = \frac{(2k-1)(p+q)[2kp + (2k-1)q]z}{pq},$$

$$n = \frac{[4k^2p + (2k-1)^2q][2kp + (2k-1)q]z}{2kpq},$$

$$a = \frac{p[2kp + (2k-1)q]z}{pq},$$

$$b = \frac{(2k-1)q[2kp + (2k-1)q]z}{2kpq}.$$

Let $2k - 1 = P_1^{k_1}, P_2^{k_2}, \dots, P_r^{k_r}$ where $P_1, P_2, \dots, P_r$ are distinct prime number.

Let $k_1, k_2, \dots, k_r$ are positive number, and $2k = q_1^{h_1}, q_2^{h_2}, \dots, q_w^{h_w}$ are positive integers, where $h_1, h_2, \dots, h_w$are positive integer.

If
$\gcd(p, (2k-1)^2) =$
${p_1}^{i_1}{p_2}^{i_2} \dots {p_\alpha}^{i_\alpha}{p_{\alpha+1}}^{2k_{\alpha+1}-i_{\alpha+1}}{p_{\alpha+2}}^{2k_{\alpha+2}-i_{\alpha+2}} \dots {p_\beta}^{2k_\beta - i_\beta}{p_{\beta+1}}^{2k_{\beta+1}}\,{p_{\beta+2}}^{2k_{\beta+2}} \dots {p_\gamma}^{2k_\gamma}.$

Where $1 \le \alpha \le \beta \le \gamma$, $0 \le i_j \le k_i$(when $0 \le j \le \alpha$) or $0 < i_j < k_j$(when $\alpha + 1 \le j \le \beta$),
$\gcd(q, 4k^2) =$
${q_1}^{j_1}{q_2}^{j_2} \dots {q_\mu}^{j_\mu}{q_{\mu+1}}^{2h_{\mu+1}-j_{\mu+1}}{q_{\mu+2}}^{2h_{\mu+2}-j_{\mu+2}} \dots {q_\vartheta}^{2h_\vartheta - j_\vartheta}{q_{\vartheta+1}}^{2h_{\vartheta+1}}\,{q_{\vartheta+2}}^{2h_{\vartheta+2}} \dots {q_w}^{2h_w}.$

Where $1 \le \mu \le \vartheta \le w$, $0 \le j_i \le h_i$(when $0 \le i \le \mu$) or $0 < j_i < h_i$(when $\mu + 1 \le i \le \vartheta$), and let

$$s = {p_1}^{i_1}{p_2}^{i_2} \dots {p_\alpha}^{i_\alpha}, \qquad t = {p_1}^{k_1-i_1}{p_2}^{k_2-i_2} \dots {p_\alpha}^{k_\alpha - i_\alpha}$$

$$u = {p_{\alpha+1}}^{i_{\alpha+1}}{p_{\alpha+2}}^{i_{\alpha+2}} \dots {p_\beta}^{i_\beta},$$

$$v = {p_{\alpha+1}}^{k_{\alpha+1}-i_{\alpha+1}}{p_{\alpha+2}}^{k_{\alpha+2}-i_{\alpha+2}} \dots {p_\beta}^{k_\beta - i_\beta},$$

$$w = p_{\beta+1}{}^{k_{\beta+1}} p_{\beta+2}{}^{k_{\beta+2}} \dots p_{\gamma}{}^{2k_{\gamma}},$$

and

$$s' = q_1{}^{j_1} q_2{}^{j_2} \dots q_{\mu}{}^{j_{\mu}}, \qquad t' = q_1{}^{h_1-j_1} q_2{}^{h_2-j_2} \dots q_{\mu}^{h_{\mu}-j_{\mu}},$$

$$u' = q_{\mu+1}{}^{j_{\mu+1}} q_{\mu+2}{}^{j_{\mu+2}} \dots q_{\vartheta}{}^{j_{\vartheta}}, v' = q_{\mu+1}{}^{h_{\mu+1}-j_{\mu+1}} q_{\mu+2}{}^{h_{\mu+2}-j_{\mu+2}} \dots q_{\vartheta}{}^{h_{\vartheta}-j_{\vartheta}},$$

and $w' = q_{\vartheta+1}{}^{h_{\vartheta+1}} q_{\vartheta+2}{}^{h_{\vartheta+2}} \dots q_w{}^{h_w}$.

Then $2k-1 = stuvw$ and $2k = s't'u'v'w'$.

Also let $p = suv^2w^2p'$ and $q = s'u'v'^2w'^2q'$. Now by using $p, q, (2k-1)$ and $2k$ the parameter m, n, a, b and r satisfying the condition $(1)-(4)$ are expressed in the following lemma(2.4). The purpose of lemma (2.4) is to discuss the detail of $\vec{P}_{4k-1}$ −factorization, and reduce it to number of base cases, which are then solve in later lemma.

Lemma 2.4:

Case 1: If $t' \equiv 1 (\text{mod } 2)$ and $v'w' \equiv 1 (\text{mod } 2)$, then

$$m = stut'(suv^2w^2p' + s'u'v'^2w'^2q')(t'vwp' + tv'w'q')z',$$

$$n = suvwv'w'(s't'^2u'p' + st^2uq')(t'vwp' + tv'w'q')z',$$

$$a = suvwt'p'(vwt'p' + tv'w'q')z',$$

$$b = stuv'w'q'(vwt'p' + v'w'tq')z',$$

$$d = stut'(vwt'p' + v'w'tq')z',$$

$$r = t'v'w'(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z',$$

for some positive integer z'.

Case 2: If $t' \equiv 0 (\text{mod } 2)$, and $v'w' \equiv 1 (\text{mod } 2)$, then

$$m = stut'(suv^2w^2p' + s'u'v'^2w'^2q')(t'vwp' + tv'w'q')z'/2,$$

$$n = suvwv'w'(s't'^2u'p' + st^2uq')(t'vwp' + tv'w'q')z'/2,$$

$$a = suvwt'p'(vwt'p' + tv'w'q')z'/2,$$

$$b = stuv'w'q'(vwt'p' + v'w'tq')z'/2,$$

$$r = t'v'w'(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z'/2,$$

$$d = stut'(vwt'p' + v'w'tq')z'/2,$$

for some positive integer z'.

Case 3: If $t' \equiv 1 (\text{mod } 2)$, and $v'.w' \equiv 0 (\text{mod } 2)$, then

$$m = stut'(suv^2w^2p' + s'u'v'^2w'^2q')(t'vwp' + tv'w'q')z'/2,$$

$$n = suvwv'w'(s't'^2u'p' + st^2uq')(t'vwp' + tv'w'q')z'/2,$$

$$a = suvwt'p'(vwt'p' + tv'w'q')z'/2,$$

$$b = stuv'w'q'(vwt'p' + v'w'tq')z'/2,$$

$$r = t'v'w'(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z'/2,$$

$$d = stut'(vwt'p' + v'w'tq')z'/2,$$

for some positive integer z'.

Proof: Let us assume that $\gcd(p, (2k-1)^2) = suv^2w^2$ and $\gcd(q, 4k^2) = s'u'v'^2w'^2$.

If $\gcd(p,q) = 1$ and $p = suv^2w^2p'$ and $q = s'u'v'^2w'^2q'$ hold,

then

$\gcd(suv^2w^2p', s'u'v'^2w'^2q') = 1.$

Which implies that, if $\gcd(4k^2p, (2k-1)^2 = 1$ and $(2k-1) = stuvw$ and

$2k = s't'u'v'w'$ hold, then $\gcd(s't'^2u'p', st^2uq') = 1$.

Since

$$r = \frac{(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z}{p'q'}$$

is an integer, therefore by using lemma 2.2, we see that

$\gcd(suv'w^2p', s'u'v'^2w'^2q') = 1$

implies that

$\gcd(suv^2w^2p' + s'u'v'^2w'^2q', p'q') = 1,$

and

$\gcd(s't'^2u'p', st'^2uq') = 1$

implies that

$\gcd(s't'^2u'p' + st^2uq', p'q') = 1.$

Since r is an integer therefore $\frac{z}{p'q'}$ must be an integer.

Let $z_1 = \frac{z}{p'q'}$ then we have

$$d = \frac{stu(vwt'p' + v'w'tq')z_1}{v'w'}$$

is an integer.

Now for the values of t'.and $v'w'$ there are three cases will possible.

Case I: When $t' \equiv 1 (\text{mod } 2)$ and $v'w' \equiv 1 (\text{mod } 2)$.

Since,

$\gcd(2, v'w') = \gcd(stu, v'w') = \gcd(vwt'p' + tv'w'q', v'w') = 1,$

therefore $z_1 = \frac{z}{v'w'}$ is an integer. Letting $z_2 = \frac{z_1}{v'w'}$ we have

$$b = \frac{stuv'w'q'(vwt'p' + tv'w'q')z_2}{t'}.$$

Since

$\gcd(2, t') = \gcd(stuv'w'q', t') = \gcd(vwt'p' + tv'w'q', t') = 1.$

Therefore $\frac{z_2}{t'}$ is an integer. Let $z' = \frac{z_2}{t'}$. Then all the values $m, n\ a, b, r$ and d in case (1) hold.

Case 2: When $t' \equiv 0 (\text{mod } 2)$, and $v'w' \equiv 1 (\text{mod } 2)$. Since $\gcd(2, v'w') = 2$,

$\gcd(stu, v'w') = \gcd(vwt'p' + tv'w'q', v'w') = 1,$

therefore $\frac{z'}{v'w'}$ is an integer. Let $z_2 = \frac{z_1}{v'w'}$, then

$$b = \frac{stuv'w'q'(vwt'p' + tv'w'q')}{2t'} z_2.$$

Since $\gcd(2,t') = 2, \gcd(st'uv'w'q',t') = \gcd(vwt'p' + tv'w'q',t') = 1$, therefore $\frac{z_2}{t'}$ is an integer. Let $z' = \frac{z_2}{t'}$. Then all the values of m,n,a,b,r and d in case (2), hold.

Case 3: When $t' \equiv 1 \pmod 2$, and $v'w' \equiv 0 \pmod 2$.
Since

$$\gcd(2,v'w') = 2, \gcd(stu,v'w') = \gcd(vwt'p' + tv'w'q',v'w') = 1,$$

Therefore $\frac{z_2}{2t'}$ is an integer. Let $z' = \frac{z_2}{t'}$, then

$$b = \frac{stuv'w'q'(vwt'p' + tv'w'q')z_2}{2t'}.$$

Since $\gcd(stuv'w'q',t') = \gcd(vwt'p' + t'v'w'q',t') = 1$, therefore $\frac{z_2}{2t'}$ is an integer. Let $z' = \frac{z_2}{t'}$.
Then all the values of m,n,a,b,r and d in case (3), hold.

This proves the lemma 2.4.

For the parameters m and n in lemma 2.4 case (1) - case (3) when $s = 1$, we can construct a $\vec{P}_{4k-1}$ −factorization of $K^*_{m,n}$.
It is easy to see that the existence of a P_{4k-1} −factorization of $K_{m,n}$ implies the existence of a $\vec{P}_{4k-1}$ −factorization of $K^*_{m,n}$.
For our main result we need to prove the following lemma:

Lemma 2.5: For any positive integers $s,t,u,v,w,s',t',u',v',w',p$, and q, let

$$m = stut'(suv^2w^2p + s'u'v'^2w'^2q)(t'vwp + tv'w'q),$$
$$n = suvwv'w'(s't'^2u'p + st^2uq)(t'vwp + tv'w'q).$$

Then $K^*_{m,n}$ has a $\vec{P}_{4k-1}$ −factorization if $s.t.u.v.w + 1 = s't'u'v'w'$, where $4k - 1 = s\,t\,u\,v\,w + s't'u'v'w'$.

Proof. The proof is by construction (case 1 of lemma 2.4).
Let $a = suvwt'p(vwt'p + tv'w'q)$,
$b = stuv'w'q(vwt'p + v'w'tq)$, hence
$r = t'v'w'(suv^2w^2p + s'u'v'^2w'^2q)(s't'^2u'p + st^2uq)$,
and $r = r_1 . r_2$, where

$$r_1 = t'(suv^2w^2p + s'u'v'^2w'^2q),$$

and

$$r_2 = v'w'(s't'^2u'p + st^2uq).$$

Let X and Y be the two partite sets of vertices of $K^*_{m,n}$ such that:

$$X = \{x_{i,j}: 1 \le i \le r_1, 1 \le j \le m_0,$$
$$Y = \{y_{i,j}: 1 \le i \le r_2, 1 \le j \le n_0.$$

Where first subscript of $x_{i,j}$'s and $y_{i,j}$'s taken additional modulo r_1 and r_2 respectively and the second subscript of $x_{i,j}$'s and $y_{i,j}$'s taken additional modulo m_0 and n_0 respectively, where $m_0 = \frac{m}{r_1}$ and $n_0 = \frac{n}{r_2}$ i.e.,

$$m_0 = stu(t'vwp + tv'w'q),$$

and

$$n_0 = suvw(t'vwp + tv'w'q).$$

Now we construct a model of $\vec{P}_{4k-1}$ −factor of $K^*_{m,n}$, here type M copies of $\vec{P}_{4k-1}$ denote the $\vec{P}_{4k-1}$ with its end point in Y and type W with its end point in X.
Type M copies of $\vec{P}_{4k-1}$.

For each i,x,y,z and x', $1 \le i \le t'p$, $1 \le x \le vw$, $1 \le y \le suvw$, $1 \le z \le t$ and $0 \le x' \le 1$,
let

$$f\{i,x,y\} = suv^2w^2(i-1) + suvw(x-1) + y,$$

$$g(i,y,z,x') = s't'u'v'w'(i-1) + suvw(z-1) + y + x',$$

and

$$h(i,x,y,x') = suvw(i-1) + su(vwt'p + tv'w'q)(x-1) + y + x' - 1.$$

Hence set

$$E_i = \{ x_{f(i,x,y),j+su(vwt'p+tv'w'q)(z-1)} y_{g(i,y,z,x'),j+h(i,x,y,x')}: 1 \le j \le su(vwt'p + tv'w'q), 1 \le x \le vw, 1 \le y \le suvw, 1 \le z \le t, 0 \le x' \le 1\}.$$

Each of $E_i (1 \le i \le t'p)$, consists of n_0 vertex disjoint type M copies. And $\cup_{1 \le i \le t'p} E_i$ contains $a = suvwt'p(vwt'p + tv'w'q)$, vertex disjoint type M copies of $\vec{P}_{4k-1}$.

Type W copies $\vec{P}_{4k-1}$.

For each i,x,y,z and x', $1 \le i \le v'w'q$, $1 \le x \le stu$, $1 \le y \le vw$, $1 \le z \le t$ and $0 \le x' \le 1$,

let

$$\psi\{i,x,z\} = s't'^2u'v'w'p + st^2u(i-1) + stu(z-1) + x,$$

$$\varphi(i,x,y,x') = suv^2w^2t'p + s'u'v'w't'(i-1) + vw(x-1) + y + x',$$

and

$$\phi(i,x,y,x') = suvwt'p + x + stu(i-1) + su(vwt'p + tv'w'q)(y-1) + x' - 1.$$

Hence set

$$E_{t'p+i} = \{ x_{\varphi(i,x,y,x'),j+su(vwt'p+tv'w'q)(x-1)} y_{\psi\{i,x,z\},j+\phi(i,x,y,x')}: 1 \le j \le su(vwt'p + tv'w'q), 1 \le x \le stu, 1 \le y \le vw, 1 \le z \le t, 0 \le x' \le 1\}.$$

Each of $E_{t'p+i} (1 \le i \le v'w'q)$, consists of m_0 vertex disjoint type W copies of $\vec{P}_{4k-1}$. And $\cup_{(1 \le i \le v'w'q)} E_{t'p+i}$, contains $b = stuv'w'q(vwt'p + v'w'tq)$ vertex disjoint type W copies of $\vec{P}_{4k-1}$. It is important that $stuvw + 1 = s't'u'v'w'$.

Let $F = \cup_{(1 \le i \le t'p+v'w'q)} E_i$ then F contains $t = a + b$ number of vertex disjoint and
edge disjoints $\vec{P}_{4k-1}$ components, and spans $K^*_{m,n}$. Then the graph F is $\vec{P}_{4k-1}$factor of $K^*_{m,n}$. Further, in the graph $F = \cup_{(1 \le i \le t'p+v'w'q)} E_i$ each of the second subscript of $x_{i,j}$'s meets each of the second subscripts of $y_{i,j}$'s once and only once.

Define a bijection σ such that $\sigma: X \cup Y \xrightarrow{\text{onto}} X \cup Y$ in such a way that:

$$\sigma(x_{i,j}) = x_{i+1,j},\ \sigma(y_{i,j}) = y_{i+1,j}$$

For each $i \in (1,2,\ldots,r_1)$ and each $j \in (1,2,\ldots,r_2)$, let

$$F_{i,j} = \{\sigma^i(x)\sigma^j(y): x \in X, y \in Y, xy \in F\}.$$

It is shown that the graphs $F_{i,j}(1 \le i \le r_1,\ 1 \le j \le r_2)$, are edge disjoints $\vec{P}_{4k-1}$ − factor of $K^*_{m,n}$ and there union is $K^*_{m,n}$.
Thus $(F_{i,j}: 1 \le i \le r_1,\ 1 \le j \le r_2)$ is a $\vec{P}_{4k-1}$ −factorization of $K^*_{m,n}$.
This proves the lemma 2.5.
By Similar manner we can also prove the other two cases of lemma 2.4.

Applying lemma 2.3 – 2.4 and 2.5, we see that for parameter m and n satisfying conditions in theorem 2.1, $K^*_{m,n}$ has a $\vec{P}_{4k-1}$ −factorization.

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