Haar Wavelet Matrices for the Numerical Solutions of Differential Equations

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ABSTRACT
Haar Wavelets has become important tool for solving number of problems of science and engineering. In this paper a computational scheme is implemented using Haar matrices to find the numerical solution of differential equations with known initial and boundary conditions. We also presented exact solution, numerical solution and absolute error. Numerical experiments presented here are comparable with the available data. The algorithm used in this is very simple and easy to implement.

Keywords
Haar wavelets, Haar functions, Operational matrix, Differential equation.

1. INTRODUCTION
It has been observed from the literature that many researchers are developing fast and accurate numerical schemes to handle the different problems arising in various fields of science and engineering. In the past finite element methods and finite difference methods were commonly used for solving such problems. Nowadays wavelet methods are extensively applied to the problems for numerical solutions as wavelets methods have several advantages over FEM and FDM. . FEM is one of the successful and dominant numerical methods in last century. Wavelet analysis is a new technique that can be performed in several ways, a continuous wavelet transform, a discretized continuous wavelet transform, and a true discrete wavelet transform. With the rapid development of computer technology in the past few decades a broad range of numerical methods have been developed for different types of problems and achieved a great success like Haar wavelets methods. Haar wavelet method is simple and possesses less computational cost. In comparison with existing numerical schemes used to solve the PDE's, the Haar wavelet methods is an improvement over other methods in terms of accuracyIt is extensively used in modeling and simulation of engineering and science due to its versatility and flexibility.
A wavelet is a mathematical function used to divide a given function or continuous-time signal into different scale components. The word wavelet is due to Morlet and Grossmann. In the early 1980s they used the French word ondelette, meaning "small wave". Soon it was transferred to English by translating "onde" into "wave", giving "wavelet". The study of wavelets has attained the present growth due to mathematical analysis of wavelets by Stromberg [1], Grossmann and Morlet and Meyer [2,3]. The concept of Multiresolution Analysis (MRA) was introduced by Mallat and Meyer [4]. Daubechies in 1988 presented a method to construct wavelets with compact support and scale functions [5].

A review of the basic properties of the wavelets and the decomposition and the reconstruction of functions in terms of the wavelet bases is given by Strang [6]. Many families of wavelets have been proposed in the literature. If one wants to use wavelets for the solution of differential equations, one therefore has to choose one specific family which is most advantageous for the intended application. Within one family there are also members of different degree. All these wavelet families can be classified as either being an orthogonal or biorthogonal family. Each orthogonal wavelet family is characterized by two functions- the mother scaling function and the mother wavelet. With a solid historical as well as practical background, Among the wavelet families, which are defined by an analytical expression, special attention deserves the Haar wavelets.
In 1910, Alfred Haar introduced the notion of wavelets. The Haar wavelet transform is one the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. Haar wavelets are made up of pairs of piecewise constant functions and are mathematically the simplest among all the wavelet families. A good feature of the Haar wavelets is the possibility to integrate them analytically arbitrary times. The Haar wavelets are very effective for treating singularities, since they can be interpreted as intermediate boundary conditions.
Haar wavelets are easy to handle from the mathematical aspect. Haar wavelets are very effective for solving ordinary differential and partial differential equations. Therefore the idea, to apply Haar wavelet technique was quite popular [7-10]. One property of the Haar wavelet is that it has compact support, which means that it vanishes outside of a finite interval.
In this paper, we apply Haar wavelet matrices to solve ordinary differential equations with initial or boundary condition known and we compare the results of numerical and exact solutions. This has been designed to promote the study of wavelets to beginners. Thus, simplified calculations are presented with necessary basic knowledge of haar functions and their generation.

2. HAAR WAVELET MATRICES
The Haar wavelet transform is the first known wavelet and was proposed in 1909 by Alfred Haar. A haar wavelet is a system of square wave having the first curve h_0 known as scaling function

$$h_0(x) = \begin{cases} 1 \; if \; 0 \le x \le 1 \\ 0 \quad otherwise \end{cases} \tag{1}$$

and second curve is h_1 is given by

$$h_1(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0 & otherwise \end{cases} \tag{2}$$

This is also known as mother wavelet. To perform the wavelet transform, haar wavelets uses translation and dilation using.

$$h_n(x) = h_1(2^j x - k), n = 2^j + k, j \geq 0, 0 \leq k < 2^j \tag{3}$$

Moreover, for any square integrals function y(x) approximation can be made using the haar functions as

$$y(x) = \sum_{i=1}^{2m} a_i h_i(x) \tag{4}$$

Where $h_n(x) = [h_0(x), h_1(x), , , , , h_{m-1}(x)]$. To express the haar matrix, we used the general notation as

$H_m = [h_m(1/2m), h_m(3/2m), , , , , h_m(2m - 1/2m)]$.

Thus we have

$$H_1 = (1), H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5}$$

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{6}$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{7}$$

Integration over the vector h_m is given by

$$h_m = \int_0^x h_m(t)dt \approx P_m h_m(x), x \in [0,1] \tag{8}$$

Operational matrix P_m obtains the values as

$$P_1 = \left(\tfrac{1}{2}\right), \tag{9}$$

$$P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

$$P_4 = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{10}$$

$$P_3 = \begin{pmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

3. NUMERICAL METHOD

In this section, we present some numerical examples that are considered to apply the solution procedure presented in the previous section. Basis functions considered here are family of Haar wavelets. In the first example we solve the equation

$$y'' + y = e^x$$

With $y(0) = 0$, $y'(0) = 0$ exact solution for this case is given by

$$y(x) = \frac{1}{2}(e^x - \cos x - \sin x) \tag{12}$$

To start the solution procedure, we have

$$y(x) = \sum_{n=1}^{2m} a_i h_i(x) \tag{13}$$

Integrate both the sides and put the values of $y'(0) = 0$ and we found the values of $y'(x)$, again integrate the side and put the value and we get the

$$y(x) = \sum_{i=1}^{2m} a_1 p_{2,i}(x)$$

after putting the values of $y'' + y = e^x$ in

$$\sum_{i=1}^{2m} a_i h_i(x) + \sum_{i=1}^{2m} a_i p_{2,i}(x) = e^x$$

We found after simplification of this equation we get the result

$$\sum_{i=1}^{2m} a_i\left[h_i(x) + p_{2,i}(x)\right] = e^x$$

then find and put the values of a_i then we have $y(x)$ after putting the values of $y''(x) = y(x) = e^x$ we obtain numerical solution then we compare with the exact solution and we have the absolute error.

Table 1. Comparison of Exact and Haar Solution

x	Exact Solution	Haar Solution	Absolute Error
0.125	0.0081	0.0088	0.0006
0.375	0.0791	0.0810	0.0019
0.625	0.2361	0.2399	0.0031
0.875	0.4952	0.4992	0.0041

If we take more collocation points and we will get more accurate results. Graphical representation of exact numerical solution is shown below.

Graphical Representation of first example

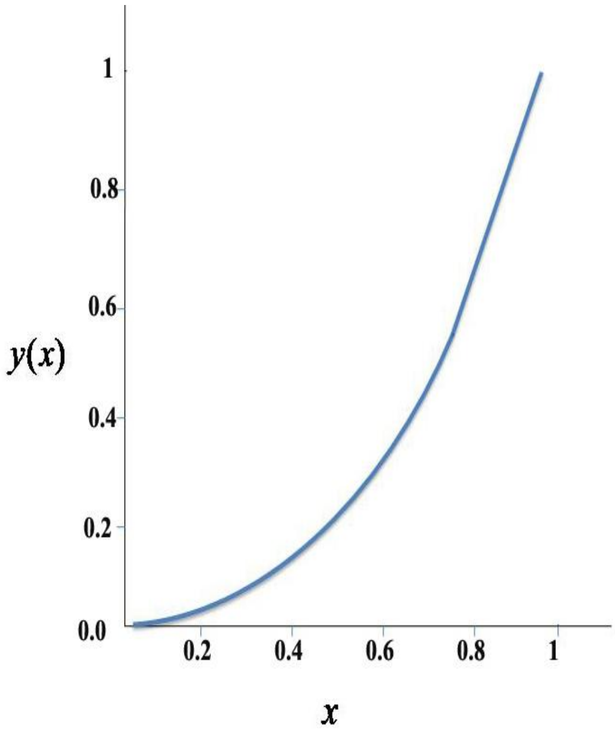


Figure 1(A): Exact Solution

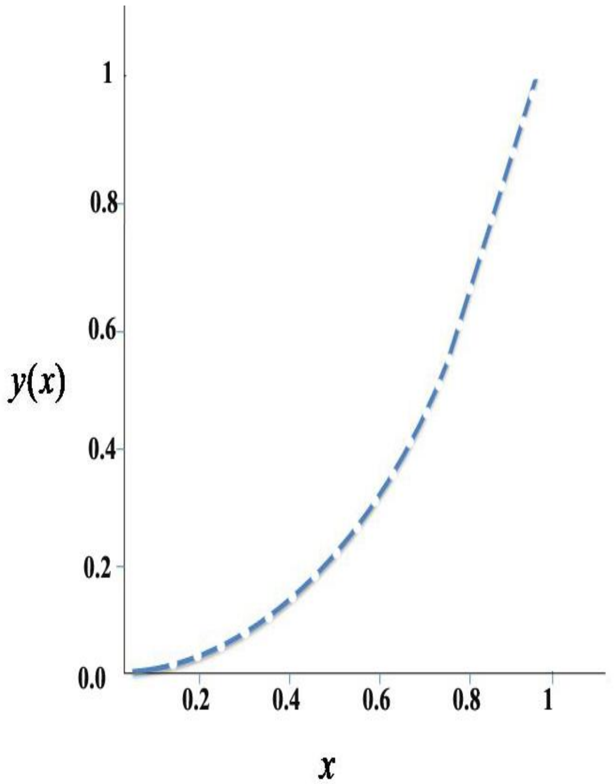


Figure 1(B) : Haar Solution

Another Example we will take

$$-y'' - y + x^2 = 0, 0 < x < 1$$

With $y(0) = 0,\ y(1) = 0$ and exact solution for this equation is

$$y(x) = \frac{\sin(x) + 2\sin(1-x)}{\sin(1)} + x^2 - 2$$

For numerical solution of above example we will take

$$y(x) = \sum_{n=1}^{2m} a_i h_i(x)$$

after solving this we get

$$y(x) = x + \sum_{i=1}^{2m} a_1 \left(p_{2,i}(x) - x p_{1,i}(1)\right)$$

substitute the values in the given equation, we obtained numerical solution and compared with exact Solution.

Table 2. Comparison of Exact and Haar Solution

x	Exact Solution	Haar Solution	Absolute Error
0.125	-0.0119	-0.0121	0.2159
0.375	-0.0334	-0.0340	0.5210
0.625	-0.0435	-0.0440	0.5022
0.875	-0.0259	-0.0261	0.2023

Graphical Representation of Second example

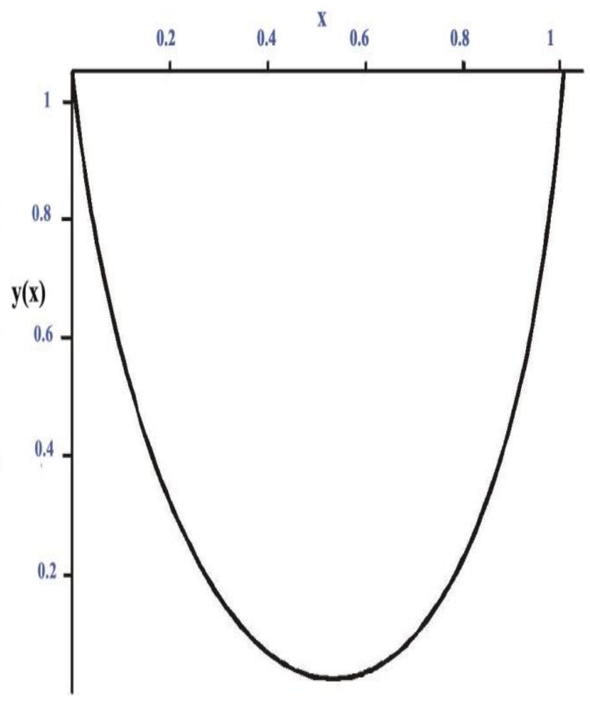
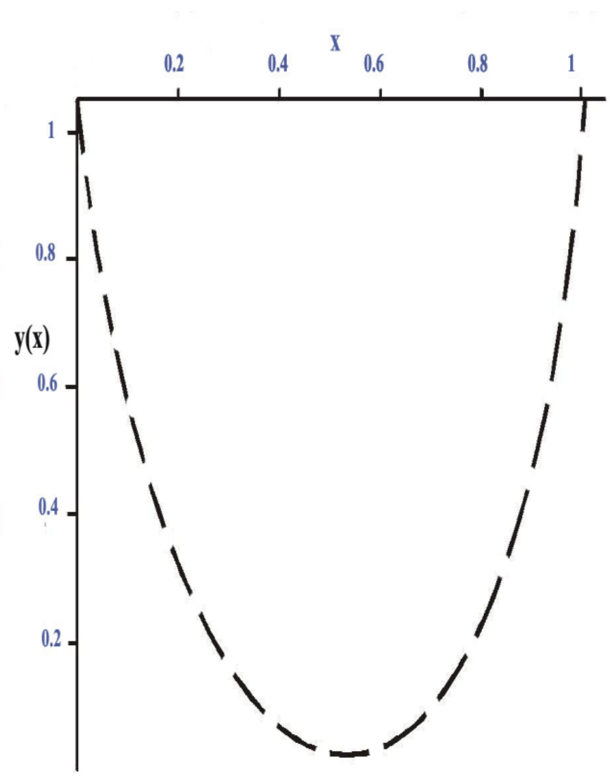


Figure 2(A): Exact Solution



4. CONCLUSION & FUTURE SCOPE

This paper represented simple and straight forward numerical technique based on Haar wavelet is proposed for solving the differential equation. This method is simple and has small computation cost and also very convenient for solving variety of boundary value problems. This paper presented numerical solution very close to the exact solution. So Haar wavelet method is very simple, fast and reliable. It is observed that Haar Wavelet method can be extended for more collocation points. Thus it will be interesting to study for what class of differential equations; the Haar wavelet will give better results.

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