On $\pi g\mathcal{I}$-Closed Sets in Ideal Topological Spaces

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ABSTRACT

In this paper, we define and investigate the notions of $\pi g\mathcal{I}$-closed sets and $\pi g\mathcal{I}$-open sets in ideal topological spaces. Then, we define $\vee_\pi$-sets and $\wedge_\pi$-sets and discuss the relation between them. Also, we give characterizations of $\pi g\mathcal{I}$-closed sets and πgs-closed sets. A separation axiom stronger than $\pi T_{\mathcal{I}}$-space is defined and various characterizations are given.

Keywords:

$\pi g\mathcal{I}$-closed, $\pi g\mathcal{I}$-open, $\vee_\pi$-set, $\wedge_\pi$-set, $\pi T_{\mathcal{I}}$-space

1. INTRODUCTION AND PRELIMINARIES

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $cl(A)$ and $int(A)$ denote the closure and interior of A in (X, τ) respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ (2)$A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.
If $(X, \tau)$is a topological space and $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space or an ideal space.
Let $P(X)$ be the power set of X. Then the operator $()^* : P(X) \to P(X)$ called a local function [6] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open set U containing $x\}$. We simply write A^* instead of $A^*(\mathcal{I}, \tau)$ in case there is no confusion. For every ideal topological space $(X, \tau, \mathcal{I})$ there exists topology τ^* finer than τ, generated by $\beta(\mathcal{I}, \tau) = \{U \setminus J : U \in \tau$ and $J \in \mathcal{I}\}$ but in general $\beta(\mathcal{I}, \tau)$ is not always a topology. Additionally $cl^*(A) = A \cup A^*$ defines Kuratowski closure operator for a topology τ^* finer than τ. Throughout this paper X denotes the ideal topological space $(X, \tau, \mathcal{I})$ and also $int^*(A)$ denotes the interior of A with respect to τ^*.

DEFINITION 1. *Let (X, τ) be a topological space. A subset A of X is said to be semi-open [7] if there exists an open set U in X such that $U \subseteq A \subseteq cl(U)$. The complement of a semi-open set is said to be semi-closed. The collection of all semi-open (resp. semi-closed) sets in X is denoted by SO(X) (resp. SC(X)). The semi-closure of A in (X, τ) is denoted by the intersection of all semi-closed sets containing A and is denoted by $scl(A)$.*

DEFINITION 2. *For $A \subseteq X$, $A_*(\mathcal{I}, \tau)$ = $\{ x \in X / U \cap A \notin \mathcal{I}$ for every $U \in SO(X)\}$ is called the semi-local function [4] of A with respect to $\mathcal{I}$ and τ, where $SO(X, x) = \{U \in SO(X) : x \in U\}$. We simply write A_* instead of $A_*(\mathcal{I}, \tau)$ in this case there is no ambiguity.*

It is given in [1] that $\tau^{*s}(\mathcal{I})$ is a topology on X, generated by the sub basis $\{U - E : U \in SO(X)$ and $E \in I\}$ or equivalently $\tau^{*s}(\mathcal{I}) = \{U \subseteq X : cl^{*s}(X - U) = X - U\}$. The closure operator cl^{*s} for a topology $\tau^{*s}(\mathcal{I})$ is defined as follows: for A $\subseteq X, cl^{*s}(A) = A \cup A_*$ and int^{*s} denotes the interior of the set A in $(X, \tau^{*s}, \mathcal{I})$. It is known that $\tau \subseteq \tau^*(\mathcal{I}) \subseteq \tau^{*s}(\mathcal{I})$. A subset A of $(X, \tau, \mathcal{I})$ is called semi-$*$-perfect [5] if $A = A_*$. $A \subseteq (X, \tau, \mathcal{I})$ is called $*$-semi dense in-itself [5](resp. semi-$*$-closed [5]) if $A \subset A_*$ (resp. $A_* \subseteq A$).A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be gI-closed [5] if $A_* \subseteq U$ whenever U is open and $A \subseteq U$. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be gI-open [5] if $X - A$ is gI-closed. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_{\pi g}$-closed [10] if $A^* \subseteq U$ whenever U is π-open and $A \subseteq U$.
A subset A of a space (X, τ) is said to be regular open [11] if $A = int(cl(A))$ and A is said to be regular closed [11] if $A = cl(int(A))$. Finite union of regular open sets in (X, τ) is π-open [13] in(X, τ). The complement of a π-open [13] set in (X, τ) is π-closed in (X, τ) A subset A of a space (X, τ) is said to be g-closed [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. A subset A of a space (X, τ) is said to be πgs-closed [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open.

LEMMA 3. *[4] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X. Then for the semi-local function the following properties hold:*

(a) If $A \subseteq B$ then $A_* \subseteq B_*$.
(b) If $U \in \tau$ then $U \cap A_* \subseteq (U \cap A)_*$
(c) $A_* = scl(A_*) \subseteq scl(A)$ and A_* is semi-closed in X.
(d) $(A_*)_* \subseteq A_*$.
(e) $(A \cup B)_* = A_* \cup B_*$.
(f) If $\mathcal{I} = \{\phi\}$, then $A_* = scl(A)$.

2. $\pi g\mathcal{I}$- CLOSED SETS

DEFINITION 4. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\pi g\mathcal{I}$-closed if $A_* \subseteq U$ whenever $A \subseteq U$ and U is π-open.*

The complement of $\pi g\mathcal{I}$-closed is said to be $\pi g\mathcal{I}$-open. The family of all $\pi g\mathcal{I}$-closed(resp. $\pi g\mathcal{I}$-open) subsets of a space $(X, \tau, \mathcal{I})$ is denoted by $\pi GIC(X)$ (resp. $\pi GIO(X)$).

THEOREM 5. *In an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

(a) Every $\mathcal{I}_{\pi g}$-closed set is $\pi g\mathcal{I}$-closed.

(b) Every $g\mathcal{I}$-closed set is $\pi g\mathcal{I}$-closed.

(c) Every πg-closed set is $\pi g\mathcal{I}$-closed.

REMARK 6. *Converse of the Theorem 5 need not be true as seen from the following example.*

EXAMPLE 7. *(a) Let $X = \{a, b, c, d\}, \tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Then the set A = {a, c, d} is$\pi g\mathcal{I}$-closed but it is not $\mathcal{I}_{\pi g}$-closed.*
(b)Let X = {a, b, c, d}, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\phi, \{a\}\}$. Then A = $\{b\}$ is $\pi g\mathcal{I}$-closed but it is not $g\mathcal{I}$-closed.
(c)Let X = {a, b, c, d, e}, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, d, e\}$, X} and $\mathcal{I} = \{\phi, \{b\}, \{d\}, \{b, d\}\}$. Then A = {a, b} is $\pi g\mathcal{I}$-closed but it is not πg-closed.

THEOREM 8. *In an ideal space $(X, \tau, \mathcal{I})$, the union of two $\pi g\mathcal{I}$-closed set is an $\pi g\mathcal{I}$-closed set.*

Proof. Suppose that $A \cup B \subseteq U$ and U is π-open in $(X, \tau, \mathcal{I})$, then $A \subseteq U$ and $B \subseteq U$. Since A and B are $\pi g\mathcal{I}$-closed. $A_* \subseteq U$ and $B_* \subseteq U$. By Lemma 3, $(A \cup B)_* \subseteq A_* \cup B_* \subseteq U$. Thus $A \cup B$ is $\pi g\mathcal{I}$-closed.

THEOREM 9. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If A is $\pi g\mathcal{I}$-closed and B is π-closed in X, then $A \cap B$ is $\pi g\mathcal{I}$-closed.*

Proof. Let U be an π-open set in X containing $A \cap B$. Then $A \subseteq U \cup (X - B)$. So A is $\pi g\mathcal{I}$-closed, $A_* \subseteq U \cup (X - B)$ and $B \cap A_* \subseteq U$. By Lemma 3, $(A \cap B)_* = A_* \cap B_* \subseteq A_* \cap B \subseteq U$, because every π-closed set is closed. This proves $A \cap B$ is $\pi g\mathcal{I}$-closed.

THEOREM 10. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be a $\pi g\mathcal{I}$-closed set if and only if $cl^{*s}(A) - A$ contains no-nonempty π-closed set.*

Proof. Let A be an $\pi g\mathcal{I}$-closed set of $(X, \tau, \mathcal{I})$. Suppose π-closed set F contained in $cl^{*s}(A) - A = cl^{*s}(A) \cap (X - A)$. Since $F \subseteq X - A$, we have $A \subseteq F$ and $X - F$ is π-open. Therefore $cl^{*s}(A) \subseteq X - F$ and so $F \subseteq X - cl^{*s}(A)$. Already we have $F \subseteq cl^{*s}(A)$. Thus $F \subseteq cl^{*s}(A) \cap (X - cl^{*s}(A)) = \phi$. Hence $cl^{*s}(A) - A$ contains no-nonempty π-closed set.
Conversely, Let $A \subseteq U$ and U be a π-open subset of X such that $A_* \nsubseteq U$. This gives $A_* \cap (X - U) \neq \phi$ or $A_* - U \neq \phi$. Moreover, $A_* - U = A_* \cap (X - U)$ is π-closed in X. Since $A_* - U \subseteq A_* - A$ and $A_* - A = cl^{*s}(A) - A$ contains nonempty π-closed set. This is a contradiction. This proves A is a $\pi g\mathcal{I}$-closed set.

THEOREM 11. *If A is an $\pi g\mathcal{I}$-closed subset of an ideal space $(X, \tau, \mathcal{I})$ and $A \subseteq B \subseteq A_*$, then B is also $\pi g\mathcal{I}$-closed.*

Proof. Suppose $B \subseteq U$and U is π-open. Since A is $\pi g\mathcal{I}$-closed and $A \subseteq U$, $A_* \subseteq U$. By Lemma 3, $B_* \subseteq (A_*)_* \subseteq A_* \subseteq U$ and so B is $\pi g\mathcal{I}$-closed.

THEOREM 12. *In an ideal space $(X, \tau, \mathcal{I})$, a $\pi g\mathcal{I}$-closed and $*$-semi dense in-itself is πgs-closed.*

Proof. Suppose A is $*$-semi dense in-itself and $\pi g\mathcal{I}$-closed in X. Let U be any π-open set containing A. Since A is $\pi g\mathcal{I}$-closed, $A_* \subseteq U$ and hence by Lemma 3, $scl(A_*) \subseteq U$. Since A is $*$-semi dense in-itself, $A \subset A_*$ and hence $scl(A) \subseteq U$ whenever $A \subseteq U$. This proves that A is πgs-closed.

COROLLARY 13. *Let A and B be subsets of an ideal space $(X, \tau, \mathcal{I})$ such that $A \subset B \subset A_*$.If A is $\pi g\mathcal{I}$-closed, then A and B are πgs-closed.*

Proof. Since $A \subset B \subset A_*$ and A is $\pi g\mathcal{I}$-closed. By Theorem 11, B is $\pi g\mathcal{I}$-closed. Since $A \subset B \subset A_*, B_* = A_*$. Therefore A and B are $*$-semi dense in-itself. By Theorem 12, A and B are πgs-closed.

THEOREM 14. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be an $\pi g\mathcal{I}$-closed set. Then the following are equivalent.*

(a) A is semi-$*$-closed set.

(b) $cl^{*s}(A) - A$ is a π-closed set.

(c) $A_* - A$ is a π-closed set.

Proof. $(a) \Longrightarrow (b)$. If A is a semi-$*$-closed, then $cl^{*s}(A) - A = \phi$ and so $cl^{*s}(A) - A$ is π-closed.
$(b) \Longrightarrow (a)$. Suppose $cl^{*s}(A) - A$ is π-closed. Since A is $\pi g\mathcal{I}$-closed, by Theorem 10, $cl^{*s}(A) - A = \phi$ and so A is semi-$*$-closed.
$(b) \Longleftrightarrow (c)$. The proof follows from the fact that $cl^{*s}(A) - A = A_* - A$.

THEOREM 15. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) Every subset of $(X, \tau, \mathcal{I})$ is $\pi g\mathcal{I}$-closed

(b) Every π-open set is semi-$*$-closed.

Proof. $(a) \Longrightarrow (b)$.Suppose every subset of X is $\pi g\mathcal{I}$-closed. If U is π-open then by hypothesis, U is $\pi g\mathcal{I}$-closed and so $U_* \subseteq U$. Hence U is semi-$*$-closed.
$(b) \Longrightarrow (a)$.Suppose every π-open set is semi-$*$-closed. Let A be a subset. If U is π-open set such that $A \subseteq U$, then $A_* \subseteq U_* \subseteq U$ and so A is $\pi g\mathcal{I}$-closed.

THEOREM 16. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be an $\pi g\mathcal{I}$-closed set if and only if $A \cup (X - A_*)$ is $\pi g\mathcal{I}$-closed set.*

Proof. Suppose that A is an $\pi g\mathcal{I}$-closed set. If U is any π-open set such that $A \cup (X - A_*) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A_*)) = (X - A) \cap A_* = A_* - A$. Since $X - U$ is π-closed and A is $\pi g\mathcal{I}$-closed. By Theorem 10, $X - U = \phi$ and so X=U. Hence X is the only π-open set containing $A \cup (X - A_*)$ and so $A \cup (X - A_*)$ is $\pi g\mathcal{I}$-closed.
Conversely, suppose $A \cup (X - A_*)$ is $\pi g\mathcal{I}$-closed. Let F be any π-closed set such that $F \subseteq A_* - A$. Since $A_* - A = X - A \cup (X - A_*)$, we have $A \cup (X - A_*) \subseteq X - F$ and $X - F$is π-open. Therefore $(A \cup (X - A_*))_* = A_* \cup (X - A_*)_* \subseteq X - F$ and hence $F \subseteq X - A_*$. But $F \subseteq A_* - A$ implies $F = \phi$. By Theorem 10 $A_* - A = cl^{*s}(A) - A$ contains no non-empty π-closed set, hence A is $\pi g\mathcal{I}$-closed.

THEOREM 17. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $A \cup (X - A_*)$ is $\pi g\mathcal{I}$-closed set if and only if $A_* - A$ is $\pi g\mathcal{I}$-open.*

Proof. Since $X - (A_* - A) = A \cup (X - A_*)$, the proof follows immediately.

THEOREM 18. *Let $(X, \tau, \mathcal{I})$ be an ideal space. A subset $A \subseteq X$ is $\pi g\mathcal{I}$-open if and only if $F \subseteq int^{*s}(A)$ whenever F is π-closed and $F \subseteq A$.*

Proof. Let A be an $\pi g\mathcal{I}$-open set of $(X, \tau, \mathcal{I})$ and F be a π-closed set contained in A. Then $X - A \subseteq X - F$ and hence $(X - A)_* \subseteq X - F$. Hence, we have $(X - int^{*s}(A)) = cl^{*s}(X - A) = (X - A) \cup (X - A)_* \subseteq X - F$. This proves that $F \subseteq int^{*s}(A)$.

Conversely, let $F \subseteq int^{*s}(A)$ whenever $F \subseteq A$ and F is a π-closed subset of X. Let $X - A \subseteq V$ and V is a π-open set. Then $X - V \subseteq A$and $X - V$ is π-closed. By the assumption, $X - V \subseteq int^{*s}(A)$ and $V \supseteq X - int^{*s}(A) = cl^{*s}(X - A) \supseteq (X - A)_*$. This proves that $(X - A)$ is $\pi g\mathcal{I}$-closed and A is an $\pi g\mathcal{I}$-open subset of X.

THEOREM 19. *In an ideal space $(X, \tau, \mathcal{I})$, if A is an $\pi g\mathcal{I}$-open set, then G = X whenever G is π-open and $int^{*s}(A) \cup (X - A) \subseteq G$.*

Proof. Let A be an $\pi g\mathcal{I}$-open set. Suppose G is π-open set and $int^{*s}(A) \cup X - A \subseteq G$. $X - G \subseteq (X - int^{*s}(A)) \cap A = (X - int^{*s}(A)) - (X - A) = cl^{*s}(X - A) - (X - A)$. Since $X - A$ is $\pi g\mathcal{I}$-closed, by Theorem 10, $X - G = \phi$and so $G = X$.

THEOREM 20. *If A is an $\pi g\mathcal{I}$-closed set in an ideal space $(X, \tau, \mathcal{I})$, then $cl^{*s}(A) - A$ is $\pi g\mathcal{I}$-open.*

Proof. A is $\pi g\mathcal{I}$-closed, by Theorem 10, ϕ is the only π-closed set contained in $cl^{*s}(A) - A$ and so by Theorem 18, $cl^{*s}(A) - A$ is $\pi g\mathcal{I}$-open.

EXAMPLE 21. *Let $X = \{a, b, c, d\}, \tau = \{\phi, \{b\}, \{c, d\}, \{b, c, d\}, X\}$ and $\mathcal{I} = \{\phi\}$. Let A = {b, c, d}. Then $cl^{*s}(A) - A = X - \{b, c, d\} = \{a\}$. Since ϕ is the only π-open set contained in $cl^{*s}(A) - A$, by Theorem 18 $cl^{*s}(A) - A$ is $\pi g\mathcal{I}$-open but A is not $\pi g\mathcal{I}$-closed.*

THEOREM 22. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. If A is $\pi g\mathcal{I}$-open and $int^{*s}(A) \subseteq B \subseteq A$, then B is $\pi g\mathcal{I}$-open and $cl^{*s}(A) - A$ is $\pi g\mathcal{I}$-open.*

Proof. Since $int^{*s}(A) \subseteq B \subseteq A$, we have $int^{*s}(A) = int^{*s}(B)$. Suppose F is π-closed and $F \subseteq B$, then $F \subseteq A$. Since A is $\pi g\mathcal{I}$-open, by Theorem 18. $F \subseteq int^{*s}(A) = int^{*s}(B)$.So again by Theorem 18, B is $\pi g\mathcal{I}$-open.

DEFINITION 23. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

(a) a $\mathcal{N}_{\mathcal{I}_s}$-set if $A = U \cap V$, where U is a π-open and V is a semi-$*$-perfect set.

(b) a $\mathcal{P}_{\mathcal{I}_s}$-set if $A = U \cap V$, where U is a π-open and V is a semi-$*$-closed set.

THEOREM 24. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is a $\mathcal{N}_{\mathcal{I}_s}$-set and a $\pi g\mathcal{I}$-closed set, then A is a semi-$*$-closed set.*

Proof. Let A be a $\mathcal{N}_{\mathcal{I}_s}$-set and a $\pi g\mathcal{I}$-closed set. Since A is a $\mathcal{N}_{\mathcal{I}_s}$-set, $A = U \cap V$, where U is a π-open and V is a semi-$*$-perfect set. Now $A = U \cap V \subseteq U$ and A is a $\pi g\mathcal{I}$-closed set implies that $A_* \subseteq U$. Also $A = U \cap V$ and V is semi-$*$-perfect set implies that $A_* \subseteq V$. Thus $A_* \subseteq U \cap V = A$ Hence A is a semi-$*$-closed set.

EXAMPLE 25. *Let $X = \{a, b, c, d\}, \tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Set A = {b, c} is semi-$*$-closed set which is not $\mathcal{N}_{\mathcal{I}_s}$-set.*

THEOREM 26. *A subset A of an ideal space $(X, \tau, \mathcal{I})$, the following are equivalent.*

(a) A is semi-$*$-closed set.

(b) A is a $\mathcal{P}_{\mathcal{I}_s}$-set and a $\pi g\mathcal{I}$-closed set.

Proof. $(a) \Longrightarrow (b)$. Let A be a semi-$*$-closed set and $A = X \cap V$, where X is π-open and V is a semi-$*$-closed set. Hence A is a $\mathcal{P}_{\mathcal{I}_s}$-set. Let U be a open set such that $A \subseteq U$. Then $A_* \subseteq A \subseteq U$ and hence A is a $\pi g\mathcal{I}$-closed set.

$(b) \Longrightarrow (a)$. Let A be a $\mathcal{P}_{\mathcal{I}_s}$-set and a $\pi g\mathcal{I}$-closed set. $A = U \cap V$, where U is a π-open and V is a semi-$*$-closed set. Now $A \subseteq U$ implies that $A_* \subseteq U$, since A is $\pi g\mathcal{I}$-closed set. Also $A \subseteq V$, V is semi-$*$-closed set implies that $A_* \subseteq V_* \subseteq V$. Thus $A_* \subseteq U \cap V = A$. Hence A is a semi-$*$-closed set.

REMARK 27. *The notions of $\mathcal{P}_{\mathcal{I}_s}$-set and a $\pi g\mathcal{I}$-closed set are independent as shown from the following examples.*

EXAMPLE 28. *In Example 25, $A = \{b, c, d\}$ is a $\mathcal{P}_{\mathcal{I}_s}$-set but it is not $\pi g\mathcal{I}$-closed and $A = \{a, c\}$ is a a $\pi g\mathcal{I}$-closed but it is not $\mathcal{P}_{\mathcal{I}_s}$-set.*

3. $\vee_\pi$ND $\wedge_\pi$SETS

Let (X, τ) be a space. If $B \subseteq X$. we define $B^\vee_\pi = \bigcup\{F : F \subseteq B$and F is π-closed$\}$ and $B^\wedge_\pi = \bigcap\{U : B \subseteq U$and U is π-open$\}$.

THEOREM 29. *Let (X, τ) be a space. If A and B are subsets of X, then the following hold.*

(a) $\phi^\vee_\pi = \phi$ and $\phi^\wedge_\pi = \phi$.

(b) $X^\vee_\pi = X$ and $X^\wedge_\pi = X$.

(c) $A^\vee_\pi \subseteq A$ and $A \subseteq A^\wedge_\pi$.

(d) $(A^\vee_\pi)^\vee_\pi = A^\vee_\pi$.

(e) $(A^\wedge_\pi)^\wedge_\pi = A^\wedge_\pi$.

(f) $A \subseteq B \Longrightarrow A^\vee_\pi \subseteq B^\vee_\pi$

(g) $A \subseteq B \Longrightarrow A^\wedge_\pi \subseteq B^\wedge_\pi$

(h) $A^\vee_\pi \cup B^\vee_\pi \subseteq (A \cup B)^\vee_\pi$

(i) $A^\wedge_\pi \cup B^\wedge_\pi \subseteq (A \cup B)^\wedge_\pi$

(j) $(A \cap B)^\vee_\pi \subseteq A^\vee_\pi \cap B^\vee_\pi$

(k) $(A \cap B)^\wedge_\pi \subseteq A^\wedge_\pi \cap B^\wedge_\pi$

(l) $A^\vee_\pi \subseteq A^\vee$ and $A^\wedge_\pi \supseteq A^\wedge$

A subset B of a space $(X, \tau, \mathcal{I})$ is said to be $\vee$-set[9] (resp. $\wedge$-set) if $B = B^\vee$ (resp. $B = B^\wedge$) where $B^\vee = \bigcup\{F/F \subseteq B, X - F \in \tau\}$ and $B^\wedge = \bigcap\{U/B \subseteq U, U \in \tau\}$. A subset B of a space (X, τ) is said to be $\vee_\pi$-set if $B = B^\vee_\pi$. A subset B of X is said to be $\wedge_\pi$-set if $B = B^\wedge_\pi$. Every π-closed set is a $\vee_\pi$-set and every π-open set is $\wedge_\pi$-set.

THEOREM 30. *Let (X, τ) be a space and A be a subset of X. Then the following hold.*

(a) If A is a $\vee_\pi$-set, then it is a $\vee$-set.

(b) If A is a $\wedge_\pi$-set, then it is a $\wedge$-set.

Proof. (a) Always $A^\vee \subseteq A$. Since A is a $\vee_\pi$-set, $A = A^\vee_\pi \subseteq A^\vee$, by Theorem 4. Therefore, $A = A^\vee$ and so A is a $\vee$-set.
(b) clearly, $A \subseteq A^\wedge$. Since A is a $\wedge_\pi$-set, $A = A^\wedge_\pi \subseteq A^\wedge$ and so A is a $\wedge$-set.
The following Example 31 shows that a $\vee$-set need not be a $\vee_\pi$-set.

EXAMPLE 31. *Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$. Set A = {c, d}. Since A is closed, it is a $\vee$-set. But $A^\vee_\pi = \phi$, since there is no π-closed set contained in A and so A is not a $\vee_\pi$-set.*

THEOREM 32. *Let (X, τ) be a space. Then $(X - B)^\wedge_\pi = (X - B^\vee_\pi)$ for every subset B of X.*

Proof. The proof follows from the definition.

COROLLARY 33. *Let (X, τ) be a space. Then $(X - B)^\vee_\pi = (X - B^\wedge_\pi)$ for every subset B of X.*

COROLLARY 34. *Let (X, τ) be a space. Then a subset B of X is $\vee_\pi$-set if and only if $X - B$ is a $\wedge_\pi$-set.*

REMARK 35. *Let $(X, \tau, \mathcal{I})$ be an ideal space. It is clear that a subset A of X is $\pi g\mathcal{I}$-closed if and only if $cl^{*s}(A) \subseteq A^\wedge_\pi$.*

COROLLARY 36. *Let A be a $\wedge_\pi$-set in $(X, \tau, \mathcal{I})$. Then A is $\pi g\mathcal{I}$-closed if and only if A is semi-$*$-closed.*

If $\mathcal{I} = \phi$, in Remark 35 and Corollary 36, we get the following corollary 37 which gives characterization of πgs-closed sets.

COROLLARY 37. *Let (X, τ) be a space and $A \subseteq X$. Then the following hold.*

(a) *A is πgs-closed if and only if $scl(A) \subseteq A^\vee_\pi$.*
(b) *If A is a $\wedge_\pi$-set, then A is πgs-closed if and only if A is semi-closed.*

THEOREM 38. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$. If $A^\wedge_\pi$ is $\pi g\mathcal{I}$-closed, then A is also $\pi g\mathcal{I}$-closed.*

Proof. Suppose that $A^\wedge_\pi$ is $\pi g\mathcal{I}$-closed. If $A \subseteq U$ such that U is π-open, then $A^\wedge_\pi \subseteq U$.Since $A^\wedge_\pi$ is $\pi g\mathcal{I}$-closed, $cl^{*s}(A^\wedge_\pi) \subseteq U$. Since $A \subseteq A^\wedge_\pi$, it follows that $cl^{*s}(A) \subseteq U$ and so A is $\pi g\mathcal{I}$-closed.

EXAMPLE 39. *Consider the same topology in Example 7(c) and $\mathcal{I} = \{\phi, \{d\}\}$. $A = \{a, b, d\}$ is semi-$*$-closed and hence $\pi g\mathcal{I}$-closed. $A^\wedge_\pi = \{a, b, c, d\}$ is π-open but it is not semi-$*$-closed. Therefore $A^\wedge_\pi$ is not an $\pi g\mathcal{I}$-closed set.*

In an ideal space $(X, \tau, \mathcal{I})$, a subset B of X is said to be an $\mathcal{I}.\wedge_\pi$-set if $B^\wedge_\pi \subseteq F$ whenever $B \subseteq F$ and F is semi-$*$-closed. A subset B of X is called $\mathcal{I}.\vee_\pi$-set if $X - B$ is an $\mathcal{I}.\wedge_\pi$-set. Every $\vee_\pi$-set is an $\mathcal{I}.\vee_\pi$ and every $\wedge_\pi$-set is an $\mathcal{I}.\wedge_\pi$-set. The following Example 40 shows that on $\mathcal{I}.\wedge_\pi$-set is not a $\wedge_\pi$-set.

EXAMPLE 40. *Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Let A = $\{a, b, c\}$. Then the only semi-$*$-closed set containing A is X and so A is an $\mathcal{I}.\wedge_\pi$-set. Since $A^\wedge_\pi = X$, A is not a $\wedge_\pi$-set.*

THEOREM 41. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is an $\mathcal{I}.\vee_\pi$-set if and only if $U \subseteq A^\vee_\pi$ whenever $U \subseteq A$ and U is a semi-$*$-open.*

Proof. The proof is obvious.

THEOREM 42. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then for each $x \in X$,$\{x\}$ is either semi-$*$-open or an $\mathcal{I}.\vee_\pi$-set.*

Proof. Suppose $\{x\}$ is not semi-$*$-open for some $x \in X$. Then $X - \{x\}$ is not semi-$*$-closed set and so the only semi-$*$-closed set containing $X - \{x\}$ is X. Therefore, $X - \{x\}$ is an $\mathcal{I}.\wedge_\pi$-set and hence $\{x\}$ is an $\mathcal{I}.\vee_\pi$-set

THEOREM 43. *Let B be an $\mathcal{I}.\vee_\pi$-set in $(X, \tau, \mathcal{I})$. Then for every semi-$*$-closed set F such that $B^\vee_\pi \cup (X - B) \subseteq F$, F=X holds.*

Proof. Let B be an $\mathcal{I}.\vee_\pi$-set. Suppose F is a semi-$*$-closed set such that $B^\vee_\pi(X - B) \subseteq F$. Then $X - F \subseteq X - (B^\vee_\pi \cup (X - B)) = (X - B^\vee_\pi) \cap B$. Since B is an $\mathcal{I}.\vee_\pi$-set and the semi-$*$-open set $X - F \subseteq B$, by Theorem 41, $X - F \subseteq B^\vee_\pi$. Also, $X - F \subseteq X - B^\vee_\pi$. Therefore, $X - F \subseteq B^\vee_\pi \cap (X - B^\vee_\pi) = \phi$ and hence F = X.

COROLLARY 44. *Let B be an $\mathcal{I}.\vee_\pi$-set in an ideal space $(X, \tau, \mathcal{I})$. Then $B^\vee_\pi \cup (X - B)$ is semi-$*$-closed if and only if B is a $\vee_\pi$-set.*

Proof. Let B be an $\mathcal{I}.\vee_\pi$-set in $(X, \tau, \mathcal{I})$. If $B^\vee_\pi \cup (X - B)$ is semi-$*$-closed, then by Theorem 43, $B^\vee_\pi \cup (X - B) = X$ and so $B \subseteq B^\vee_\pi$. Therefore, $B = B^\vee_\pi$ which implies that B is a $\vee_\pi$-set. Conversely, suppose that B is an $\vee_\pi$-set. Then $B = B^\vee_\pi$ and so $B^\vee_\pi(X - B) = B \cup (X - B) = X$ is semi-$*$-closed.

THEOREM 45. *Let B be a subset of an ideal space $(X, \tau, \mathcal{I})$ such that $B^\vee_\pi$ is semi-$*$-closed. If X is the only semi-$*$-closed set containing $B^\vee_\pi \cup (X - B)$, then B is an $\mathcal{I}.\vee_\pi$-set.*

Proof. Let U be a semi-$*$-open set contained in B. Since $B^\vee_\pi$ is semi-$*$-closed, $B^\vee_\pi \cup (X - U)$ is semi-$*$-closed. Also, $B^\vee_\pi \cup (X - B) \subseteq B^\vee_\pi \cup (X - U)$. By hypothesis, $B^\vee_\pi \cup (X - U) = X$. Therefore, $U \subseteq B^\vee_\pi$ which implies by Theorem 41, that B is an $\mathcal{I}.\vee_\pi$-set.

DEFINITION 46. *An ideal space $(X, \tau, \mathcal{I})$ is said to be an $\pi T_\mathcal{I}$-space if every $\pi g\mathcal{I}$-closed set is a semi-$*$-closed set.*

THEOREM 47. *In an ideal space $(X, \tau, \mathcal{I})$, the following statements are equivalent.*

(a) *$(X, \tau, \mathcal{I})$ is an $\pi T_\mathcal{I}$-space.*
(b) *Every $\mathcal{I}.\vee_\pi$-set is a $\vee_\pi$-set.*
(c) *Every $\mathcal{I}.\wedge_\pi$-set is a $\wedge_\pi$-set.*

Proof. $(a) \Longrightarrow (b)$ If B is an $\mathcal{I}.\vee_\pi$-set which is not a $\vee_\pi$-set, then $B^\vee_\pi \not\subseteq B$. So, there exists an element $x \in B$ such that $x \notin B^\vee_\pi$. Then $\{x\}$ is not π-closed. Therefore, $X - \{x\}$ is not π-open and so it follows that $X - \{x\}$ is $\pi g\mathcal{I}$-closed. By hypothesis, $X - \{x\}$ is semi-$*$-closed. Since $x \in B$ and $x \notin B^\vee_\pi$, $B^\vee_\pi \cup (X - B) \subseteq X - \{x\}$. Since $X - \{x\}$ is semi-$*$-closed, by Theorem 43, $X - \{x\} = X$, a contradiction.
$(b) \Longrightarrow (a)$ Suppose that there exists an $\pi g\mathcal{I}$-closed set B which is not semi-$*$-closed. Then, there exists $x \in cl^{*s}(B)$ such that $x \notin B$. By Theorem 42, $\{x\}$ is either semi-$*$-open or an $\mathcal{I}.\vee_\pi$-set. If $\{x\}$ is semi-$*$-open, then $\{x\} \cap B = \phi$ is a contradiction to the fact that $x \in cl^{*s}(B)$. If $\{x\}$ is an $\mathcal{I}.\vee_\pi$-set, then $\{x\}$ is a $\vee_\pi$-set and hence it follows that $\{x\}$ is π-closed. Since $B \subseteq X - \{x\}$, $X - \{x\}$ is π-open and B is $\pi g\mathcal{I}$-closed, $cl^{*s}(B) \subseteq X - \{x\}$, a contradiction to the fact that $x \in cl^{*s}(B)$. Therefore, $(X, \tau, \mathcal{I})$ is an $\pi T_\mathcal{I}$-space.
$(b) \Longleftrightarrow (c)$ The proof follows from the definition of an $\mathcal{I}.\vee_\pi$-set and from Corollary 34.

THEOREM 48. *An ideal space $(X, \tau, \mathcal{I})$ is an $\pi T_\mathcal{I}$-space if and only if every singleton set in X is either semi-$*$-open or π-closed.*

Proof. If $x \in X$ such that $\{x\}$is not π-closed, then $X - \{x\}$ is not π-open and so it follows that $X - \{x\}$ is $\pi g\mathcal{I}$-closed. By hypothesis, $X - \{x\}$ is semi-$*$-closed and so $\{x\}$ is semi-$*$-open. Conversely, let A be an $\pi g\mathcal{I}$-closed set and $x \in cl^{*s}(A)$. Consider the following two cases:
Case (i): Suppose $\{x\}$ is π-closed. Since A is $\pi g\mathcal{I}$-closed, by Theorem 10, $cl^{*s}(A) - A$ does not contain a non-empty π-closed set which implies that $x \notin cl^{*s}(A) - A$and so $x \in A$.
Case (ii): Suppose $\{x\}$ is semi-$*$-open. Then $\{x\} \cap A \neq \phi$ and so $x \in A$.
Thus in both cases $x \in A$. Therefore, $A = cl^{*s}(A)$ which implies that A is semi-$*$-closed. So $(X, \tau, \mathcal{I})$ is an $\pi T_\mathcal{I}$ -space.

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