Statistical Properties of Kumaraswamy-Generalized Exponentiated Exponential Distribution

B. E. Mohammed
Department of Mathematics, Faculty of Science, Al-Azhar University
Nasr City 11884, Cairo, Egypt

ABSTRACT
In this paper, we present a new class of distributions called kumaraswamy Generalized Exponentiated Exponential Distribution, that is based upon the cumulative distribution function of Kumaraswamy (1980) distribution, which is more flexible and is a natural generalization of the exponential, Exponentiated Exponential and kumaraswamy Generalized exponential distributions as special cases found in literature. Also, the analytical shapes of the corresponding probability density function and hazard rate function are derived with graphical illustrations. Expressions for the r^{th} moments are calculated and the variation of the skewness and kurtosis measures is investigated. Likelihood estimators of the parameters are derived. Moreover, analysis of real data set, representing the breaking stress of carbon fibers, is conducted to demonstrate the usefulness of the proposed distribution.

Keywords
Kumaraswamy Distribution, Maximum likelihood estimation, Akaike information criterion, Baysian information criterion, Consistent Akaike Information Criteria, Kaplan-Meier estimator, likelihood ratio test, p-p plot.

1. INTRODUCTION
The Exponentiated Exponential (EE) distribution, a most attractive generalization of the exponential distribution, is defined as a particular case of Gompertz-Verhulst distribution function (see Ahuja and Nash (1967)). The EE distribution has been introdused and studied by Gupta and Kundu (1999, 2001 , 2003 , 2004, 2007). They observed that this distribution can be used in place of gamma and weibull distributions, since the two parameters of the gamma, wiebull, and EE distributions have increasing as well as decreasing hazard function depending on the value of the shape parameters, also they have a constant hazard function when the shape parameter is equal to one (Gupta and Kundu (1999)).

A random variable (rv) $X$is said to have the EE distribution if its cumulative distribution function (cdf) is defined by

$$F(x) = (1 - \exp(-\lambda x))^{\alpha}, x > 0, \alpha, \lambda > 0, \quad (1.1)$$

and the probability density function (pdf) is given by

$$f(x) = \alpha\lambda \exp(-\lambda x)\,(1 - \exp(-\lambda x))^{\alpha-1}, x > 0, \alpha, \lambda > 0, \quad (1.2)$$

whereα and$\lambda$are respectively shape and scale parameters. For different value of the shape parameters, the pdf can take different shapes.

Adding parameters to a well-established family of distributions are a time honored device for obtaining more flexible new families of distributions. Cordeiro and Castro (2011) defined the cdf$F(x)$ and the pdf $f(x)$ of the Kumaraswamy generalized (KwG) distribution by

$$G(x) = 1 - \{1 - F^{a}(x)\}^{b}, -\infty < x < \infty \quad (1.3)$$

and

$$g(x) = abf(x)F^{a-1}(x)\{1 - F^{a}(x)\}^{b-1} \quad (1.4)$$

respectively, where $f(x) = dF(x)/dx$ and$a, b > 0$ are additional shape parameters to the distribution F. Except for some special choices of the function$F(x)$,The density $g(x)$will be difficult to deal with some generality. One major benefit of the KwG distribution is its ability of fitting skewed data that cannot be properly fitted by existing distributions. This fact was demonstrated recently by Cordeiro et al. (2010) who apply the Kumaraswamyweibull distribution to failure data.

A physical interpretation of the KwG distribution given by Equations (1.3) and (1.4) (for a and b positive integers) is as follows. Consider that a system is formed by b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let$X_{j1}, \dots, X_{ja}$denote the lifetimes of the subcomponents within the j^{th} component, $j = 1, \dots, b$ having a common cdf$G(x)$. Let X_1denote the lifetime of thej^{th}component, for $j = 1, \dots, b$ and let X denote the lifetime of the entire system. Then, the cdf of X is

$$Pr(X \le x) = 1 - Pr(X_1 > x, X_2 > x, \dots, X_b > x)$$
$$= 1 - \{\Pr(X_1 > x)\}^{b} = 1 - \{1 - \Pr(X_1 \le x)\}^{b}$$
$$= 1 - \{1 - \Pr(X_{11} \le x, X_{12} \le x \dots, X_{1a} \le x)\}^{b}$$
$$= 1 - \{1 - [\Pr(X_{11} \le x)]^{a}\}^{b} = 1 - \{1 - F^{a}(x)\}^{b}$$

So, it follows that the KwG distribution given by (1.3) and (1.4) is precisely the time to failure distribution of the entire system.

In this paper, we introduce a new variant of KwG extended family of distribution by selecting in (1.3), the EE cdf (1.1) whichyields

$$G(x) = 1 - \{1 - [(1 - \exp(-\lambda x))^{\alpha}]^{a}\}^{b}, x > 0, \alpha, a, b > 0. \quad (1.5)$$

We shall write $X \sim KwGEE$ to denote an absolutely continuous rv$X$possessing theKwGextended EE distribution with parameters $\lambda, \alpha, a, b$and cdf given by (1.5).

The aim of this paper is to reveal some statistical properties of the $X \sim KwGEE$distribution.

2. DENSITY, MOMENTS AND QUANTILES

The pdf of the $KwGEE(\lambda, \alpha, a, b)$ distribution with cdf (1.4) is given by

$$g(x) = ab\alpha\lambda \exp(-\lambda x)\,(1-\exp(-\lambda x))^{a\alpha-1}$$
$$\{1-[(1-\exp(-\lambda x))^{\alpha}]^{a}\}^{b-1}, x>0. \quad (2.1)$$

Remarks

(i) For$a=1, b=1$(1.4) and (2.1) reduce to the case of the EE distribution.

(ii) For $a=1, b=1, \alpha=1,$ (1.4) and (2.1) reduce to the case of the exponential distribution.

(iii) For $\alpha=1$,(1.4) and (2.1) reduce to the case of the kumaraswamy Generalized exponential distributions distribution (see Nadarajah et al. (2012)).

The following theorem gives simple conditions under which the pdf (2.1) is decreasing or unimodal.

Theorem (1.2)

The pdf of the $KwGEE$, given by (2.1), is decreasing or unimodal if$a\alpha-1<0$or $a\alpha-1\geq 0$ respectively.

Proof

The first derivative of $g(x)$ is given by

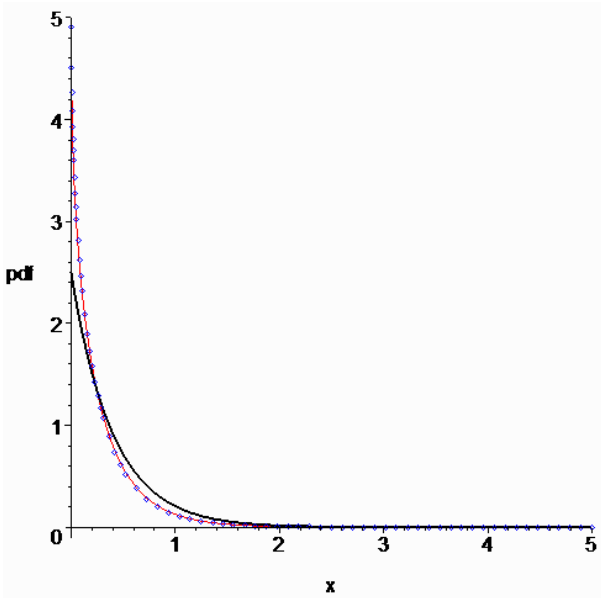
$$\dot{g}(x) = ab\alpha\lambda^2 \exp(-\lambda x)\,(1-\exp(-\lambda x))^{a\alpha-2}\{1 - [(1-\exp(-\lambda x))^{\alpha}]^{a}\}^{b-2}\eta(x), x>0,$$

where

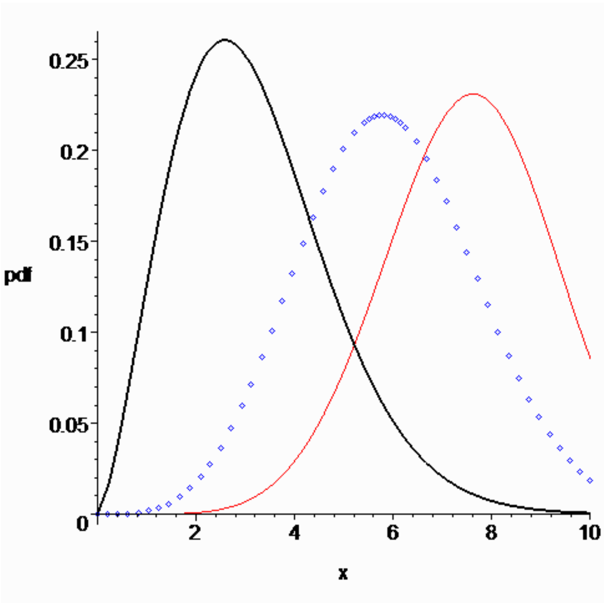
$$\eta(x) = \{1-(1-\exp(-\lambda x))^{a\alpha}\}\{(a\alpha-1)\exp(-\lambda x) - (1-\exp(-\lambda x))\} - a\alpha(b-1)\exp(-\lambda x)\,(1-\exp(-\lambda x))^{a\alpha},$$

the function $\eta(x)$has no (one) zero on $(0,\infty)$provided $\eta(0)=a\alpha-1\leq 0\ (>0)$that is,$g(x)$ has no (one) critical point provided $\eta(0)\leq 0\ (>0)$.Since $g(x)$is nonnegative and$g(x)=0$and, is decreasing (unimodal) provided $\eta(0)\leq 0\ (>0)$.

Fig. (2.1) below shows thepdfcurves for the $KwGEE$ distribution for selected values of the parameters $\lambda, \alpha, a$and b.



(a) a=0.5, α=2, (bold), 0.9, 1(plain), 0.7, 1.3(point), b=25, λ=0.1



(b) a=0.5, α=5, (bold), 1, 2(plain), 1.5, 1.5(point), b=40, λ=0.1

Fig. 2.1: Thepdf $g(x)$of the $KwGEE$distribution for selected values of the parameters. In Fig.2.1(a) $a\alpha<1$, showing that$g(x)$is decreasing. In Fig.2.1(b) $a\alpha>1$, showing that $g(x)$ is increasing-decreasing.

The r^{th}moment of the $KwGEE$ distribution is given by

$$E(X^r) = r\int_0^{\infty} x^{r-1}\,\bar{G}(x)dx = r\int_0^{\infty} x^{r-1}$$
$$\{1-[(1-\exp(-\lambda x))^{\alpha}]^{a}\}^{b}dx,$$

using the fact that

$$\{1-[(1-\exp(-\lambda x))^{\alpha}]^{a}\}^{b} = \sum_{i=1}^{\infty}(-1)^{i}\binom{b}{i}(1-\exp(-\lambda x))^{a\alpha i},$$

again,

$$(1-\exp(-\lambda x))^{a\alpha i} = \sum_{j=1}^{\infty}(-1)^{j}\binom{a\alpha i}{j}(\exp(-\lambda x))^{j},$$

then

$$E(X^{r}) = r\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{a\alpha i}{j}\int_{0}^{\infty}x^{r-1}(\exp(-\lambda x))^{j}\,dx,$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{a\alpha i}{j}\frac{\Gamma(r+1)}{(j\lambda)^{r}}. \qquad (2.2)$$

Since (2.2) is a convergent series for any $r \geq 0$, therefore putting $r = 1$, we obtain the mean as

$$E(X) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{a\alpha i}{j}\frac{\Gamma(2)}{(j\lambda)^{r}},$$

and putting $r = 2$, we obtain the second moment as

$$E(X^{2}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{i+j}\binom{b}{i}\binom{a\alpha i}{j}\frac{\Gamma(3)}{(j\lambda)^{r}}.$$

Theq^{th}quantile of the $KwGEE$ distribution is given by

$$Q(u) = G^{-1}(q) = \frac{1}{\lambda\log\left[1-\left(1-(1-q)^{\frac{1}{b}}\right)^{\frac{1}{a\alpha}}\right]}, \qquad 0 \leq q \leq 1,$$

where$G^{-1}(.)$is the inverse distribution function.

In particular, the median of the $KwGEE$ distribution is given by

$$median\ (X) = \frac{1}{\lambda\log\left[1-\left(1-\left(1-\frac{1}{2}\right)^{\frac{1}{b}}\right)^{\frac{1}{a\alpha}}\right]}.$$

For $a = 1, b = 1$ we get the corresponding results for the EE distribution.

3. QUANTILE MEASURES

To illustrate the effect of the shape parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ on skewness and kurtosis of the new distribution, we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile based measures stemmed from the non-existence of classical kurtosis for many generalized distributions.

The Bowley'sskewness (Kenney and Keeping, (1962)) is one of the earliest skewness measures defined by

$$Sk = \frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)},$$

The Moors kurtosis (Moors, (1988)) based on cotiles is defined by

$$Ku = \frac{Q(7/8)-Q(5/8)-Q(3/8)+Q(1/8)}{Q(6/8)-Q(2/8)},$$

where Q(.) represents the quantile function define in (2.3). The measures Sk and Ku are less sensitive to outliers and they exist even for distributions without moments. For symmetric unimodal distributions, positive kurtosis indicates heavy tails and peakedness relative to the normal distribution, whereas negative kurtosis indicates light tails and flatness. For the normal distribution, $Sk = Ku = 0$.

In figures 3.1 and 3.2, we plot the measures Sk and Ku for the $KwGEE(0.1, 0.5, a, b)$distribution, as functions of b (for fixed a) and as functions of a (for fixed b), respectively. These plots indicate that the Bowley skewness always decreases when an increases (for fixed b), and always increases when b increases (for fixed a). On the other hand, the Moors kurtosis always decreases when a increases (for fixed b) and always increases when b increases (for fixed a). So, these plots indicate that both measures can be very sensitive on these shape parameters, thus indicating the importance of the proposed distribution.

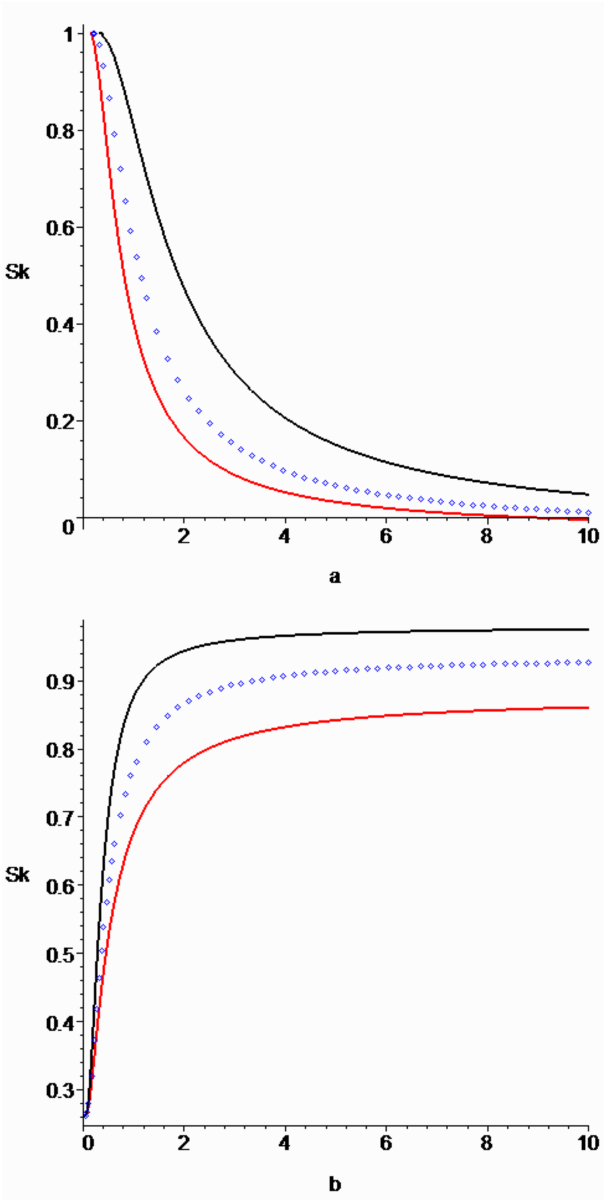


Fig. (3.1): The Bowleyskewness of the $KwGEE$distribution as function of b some values of a and as function of a for some values of b.

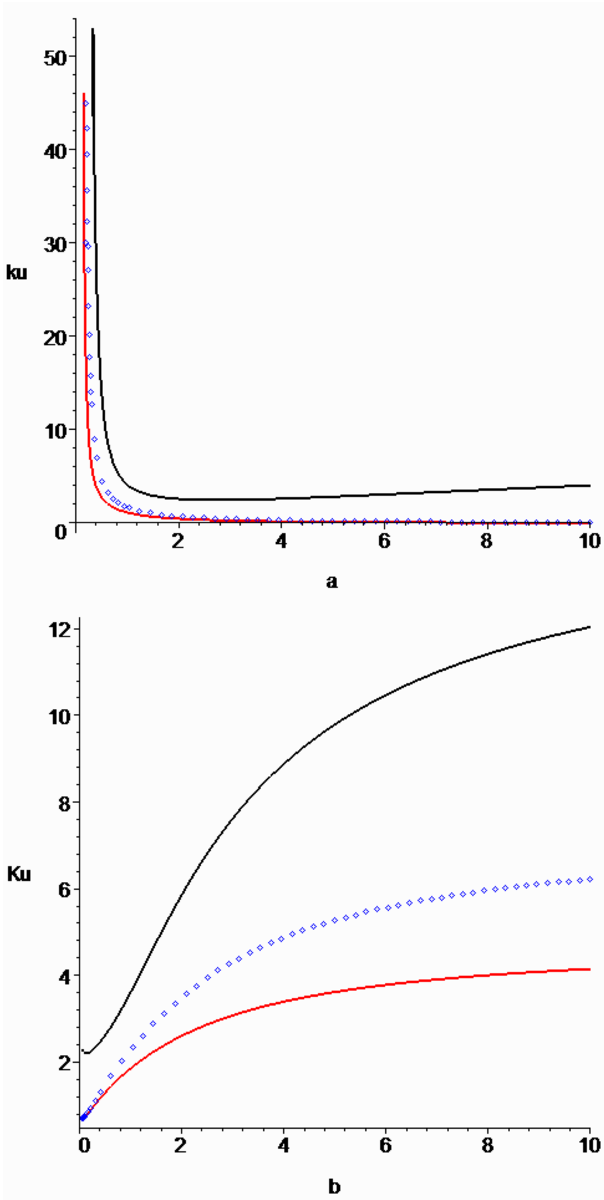


Fig. (3.2): The Moors kurtosis of the $KwGEE$ distribution as function of b for some values of a and as function of a for some values of b.

4. HAZARDRATE FUNCTION

The hazard rate function (hrf) of the $KwGEE$distribution is given by

$$h(x) = \frac{ab\alpha\lambda exp(-\lambda x)(1 - exp(-\lambda x))^{a\alpha --1}}{\{1 - [(1 - \exp(-\lambda x))^{\alpha}]^{a}\}}. \qquad (4.1)$$

Note that for all $b, \lambda,$ we have

$$h(0) = \begin{cases} 0 & \text{For } a\alpha > 1 \\ ab\alpha\lambda & \text{For } a\alpha = 1, \\ \infty & \text{For } a\alpha < 1 \end{cases}$$

$$h(\infty) = b\,\lambda \quad \text{For all } a\alpha.$$

The following theorem gives simple conditions under which hrf (4.1) is decreasing or increasing.

For any a, b, λ the hrf is an increasing if $a\alpha > 1$, and it is a decreasing function if $a\alpha < 1$. For $a\alpha = 1$, it is constant (Fig. (4.1)).

Remarks

(i) For $a = 1, b = 1$, h(x) is increasing if $\alpha > 1$, decreasing if $\alpha < 1$, and constant if $\alpha = 1$, which is a well-known results for the EE distribution (Gupta (2001)).

(ii) For $a = 1, b = 1, \lambda = 1$, h(x) is constant for all α, which is a well-known results for the exponential distribution (Venkatesan and Sundaram (2011)).

Fig. (4.1) below shows the hrf curves for the $KwGEE$ distribution for selected λ, α, a and b.

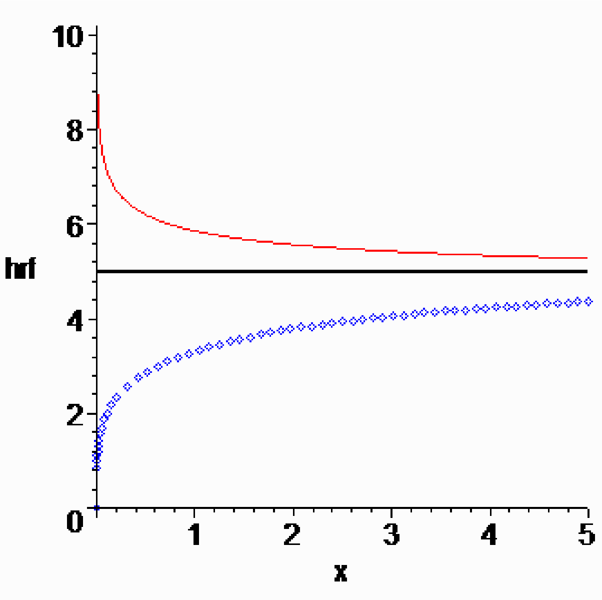


Fig. (4.1): aα<1 (plain), $a\alpha = 1$ (bold), $a\alpha > 1$ (point), $b = 40, \lambda = 0.1$

5. STOCHASTIC ORDERING

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. We will recall some basic definitions. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}_X(x) = \bar{F}_Y(x)$ for all x.

(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $h_X(x) = h_Y(x)$ for all x.

(iii) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ decreases in x.

The following implications [see Ross ((1996), Chap. 9)] are well known:

$$X \leq_{lr} Y (X \leq_{hr} Y, X \leq_{st} Y). \qquad (5.1)$$

The $KwGEE$ distributions are ordered with respect to the strongest likelihood ratio ordering, as shown in the following theorem.

Theorem (5.1)

Let $X \sim \mathrm{KwGEE}(\lambda, \alpha, a, b_1)$ and $Y \sim \mathrm{KwGEE}(\lambda, \alpha, a, b_2)$. If $b_2 < b_1$ then

$$X \leq_{lr} Y (X \leq_{hr} Y, X \leq_{st} Y).$$

Proof

First note that

$$\frac{g_X(x)}{g_Y(x)} = \frac{b_1\{1-(1-\exp(-\lambda x))^{a\alpha}\}^{b_1-1}}{b_2\{1-(1-\exp(-\lambda x))^{a\alpha}\}^{b_2-1}},$$

Since, for $b_2 < b_1$,

$$\frac{d}{dx}\frac{g_X(x)}{g_Y(x)} = a\alpha\lambda\frac{b_1}{b_2}(b_2-b_1)(1-\exp(-\lambda x))^{a\alpha}$$
$$\{1-(1-\exp(-\lambda x))^{a\alpha}\}^{b_1-1}$$

$g_X(x)/g_Y(x)$ is decreasing in x; that is $X \leq_{lr} Y$. The remaining statements follow from the implications (5.1).

6. ESTIMATION OF $KwGEE$ DISTRIBUTION

In this section, we determine the maximum likelihood estimates $(MLEs)$ of the parameters (λ, α, a, b) of the $KwGEE$ distribution. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from the $KwGEE$ distribution. Then the likelihood function is given by

$$\prod_{i=1}^{n} g_X(x_i) = \prod_{i=1}^{n} \mathrm{ab}\ \alpha\lambda \exp(-\lambda x_i)(1-\exp(-\lambda x_i))^{a\alpha-1}$$
$$\{1-(1-\exp(-\lambda x_i))^{a\alpha}\}^{b-1},$$

and the log-likelihood function is given by

$$L = \log\left[\prod_{i=1}^{n} g_X(x_i)\right] = n\, log(ab\alpha\lambda) - \lambda\sum_{i=1}^{n} x_i$$
$$+(a\alpha-1)\sum_{i=1}^{n}\log(1-\exp(-\lambda x_i))$$
$$+(b-1)\sum_{i=1}^{n}\log[1-(1-\exp(-\lambda x_i))^{a\alpha}].$$

The estimates of the parameters maximize the likelihood function. Taking the partial derivatives of the log-likelihood function with respect to λ, α, a, b respectively and equalizing the obtained expressions to zero yields to likelihood equations.

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (a\alpha-1)\sum_{i=1}^{n}\frac{x_i\exp(-\lambda x_i)}{(1-\exp(-\lambda x_i))}$$
$$-(b-1)\sum_{i=1}^{n}\frac{a\alpha x_i exp(-\lambda x_i)(1-exp(-\lambda x_i))^{a\alpha-1})}{1-(1-exp(-\lambda x_i))^{a\alpha}},$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + a\sum_{i=1}^{n}\log(1-\exp(-\lambda x_i))$$
$$-(b-1)\sum_{i=1}^{n}\frac{a x_i(1-\exp(-\lambda x_i))^{a\alpha}\log(1-\exp(-\lambda x_i))}{1-(1-\exp(-\lambda x_i))^{a\alpha}},$$

$$\frac{\partial L}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n} log[1-(1-\exp(-\lambda x_i))^{a\alpha}],$$

$$\frac{\partial L}{\partial a}=\frac{n}{a}+\alpha\sum_{i=1}^{n}\log(1-\exp(-\lambda x_i))$$

$$-(b-1)\sum_{i=1}^{n}\frac{\alpha x_i(1-\exp(-\lambda x_i))^{a\alpha}\log(1-\exp(-\lambda x_i))}{1-(1-\exp(-\lambda x_i))^{a\alpha}}.$$

The maximum likelihood estimates $(MLEs)\hat{\lambda},\hat{\alpha},\hat{a},\hat{b}$of the parameters $\lambda,\alpha,a,b$are obtained numerically by solving the non-linear equations$\frac{\partial L}{\partial \lambda}=0,\frac{\partial L}{\partial \alpha}=0,\frac{\partial L}{\partial a}=0,and\frac{\partial L}{\partial b}=0.$

The likelihood ratio test will be used to test the null hypothesis $H_0: a=1, b=1 (EE$ distribution). When H_0 is true, the deviance teststatistic$d_n=-2\{L(\tilde{\lambda},\tilde{\alpha},1,1)-L(\hat{\lambda},\hat{\alpha},\hat{a},\hat{b})\}$,where$\tilde{\lambda}$, $\tilde{\alpha}$are the MLEs ofλ,αunder$H_0: a=1, b=1$, has approximately a chi-square distribution with 2 degree of freedom. $H_0$is rejected at a significance level of αif $d_n>\chi^2_{2,\alpha}$.

In addition, for model selection, we use the Akiake Information Criterion (AIC), the Bayesian Information Criterion (BIC) and the Consistent Akaike Information Criteria $(CAIC)$ defined as:

$$AIC=-2\ log\ \ likelihood+2q$$

$$BIC=-2\ log\ \ likelihood+q log(n)$$

$$CAIC=-2\ log\ \ likelihood+\frac{2qn}{n-q-1}$$

where$q$ is the number of parameters in the model and n is the sample size. For more details about the AIC, BIC, and $CAIC$ see Akiake (1969), Schwarz (1978), and Bozdogan (1987) respectively. The model with smaller AIC, BIC and $CAIC$ is the one that better fits the data.

7. APPLICATION

In this section, we use a real data set to show that the $KwGEE$ distribution can be a better model than one based on the EE and exponential distribution. We make a results comparison of the models fit. We consider an uncensored data set corresponding an uncensored data set from consisting of 100 observations on breaking stress of carbon fibers as discussed by Shams (2013). The data are:

3.7	2.74	2.73	2.5	3.6	3.11	1.12
3.27	2.87	1.47	3.11	1.84	0.39	1.71
2.88	4.42	2.41	3.19	3.22	1.69	2.03
3.28	3.09	1.87	3.15	4.9	3.68	1.61
2.48	2.82	3.75	2.43	2.95	2.97	1.69
3.39	2.96	2.53	2.67	2.93	3.22	4.38
0.85	1.61	2.05	3.39	2.81	4.2	2.17
3.33	2.55	3.31	3.31	2.85	2.56	1.17
3.56	2.79	4.7	3.65	3.15	2.35	5.08
2.55	2.59	2.38	2.81	2.77	2.17	2.48
2.83	1.92	2.03	1.8	1.89	1.41	1.18
3.68	2.97	1.36	0.98	2.76	4.91	1.25
3.68	1.84	1.59	1.57	1.08	2.12	3.51
3.19	1.57	0.81	5.56	1.73	1.59	2.17
2	1.22					

The following tablegives a comparison between the $MLEs$, log-likelihood, AIC, BIC and $CAIC$ for the fitted $KwGEE, EE$ and exponential distributions to the given data. The table shows small values of both AIC, BIC, and $CAIC$ which favour selecting the $KwGEE$ distribution.

Model	parameter	MLE	L	AIC	BIC	CAIC
Exponential	**λ**	0.381476	-196.371	394.742	397.347	394.783
EE	**λ**	1.01317	-146.182	296.365	301.575	296.488
	α	7.78824				
KwGEE	λ	0.110961	-141.318	290.637	301.057	291.058
	α	4.95024				
	a	0.660032				
	b	66.246				

The results in the above table show that the fitted $KwGEE$ distribution should be selected based on either the $CAIC$ orBIC or AIC procedure.

For the given data, underH_0, $L(\tilde{\lambda}, \tilde{\alpha}, 1,1) = -146.182$thus $d_n = 9.728 > \chi^2_{2.0.05} = 5.991$,

therefore, we cannot accept the null hypothesis, i.e. The likelihood ratio test rejects the assumption that the EE model is sutable for the given data.

Also, letH_0, $L(\tilde{\lambda}, 1,1,1) = -196.371$,(exponential distribution) thus

$d_n = 110.106 > \chi^2_{3.0.05} = 7.815$, therefore, we cannot accept the null hypothesis, i.e. The likelihood ratio test rejects the assumption that the exponential model is sutable for the given data.

Let n be the total number of breaking stress of carbon fibers whose survival times, uncensored data, are available. Relabel the n survival times in order of increasing magnitude such that $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$. The Kaplan-Meier (1958) estimator (KME), also known as the product limit estimator, of a survival function is defined as

$$\overline{G}_n(t) = \prod_{t:\, t_{(i)} \leq t} \left\{1 - \frac{1}{n - r + 1}\right\}, \qquad t > 0.$$

Figs. (7.1), (7.2) and (7.3) show, respectively, the p-p plot of the KME versus the fitted exponential, EE and $KwGEE$ survival functions for the given data.

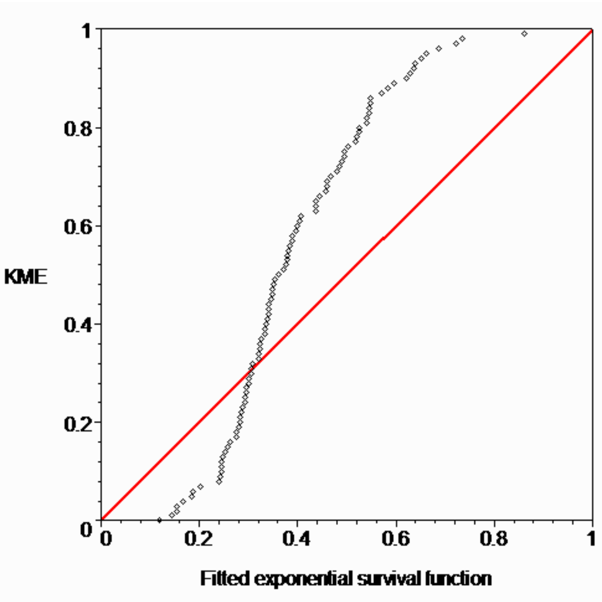


Fig. (7.1): p-p plot of KME versus fitted exponential survival function

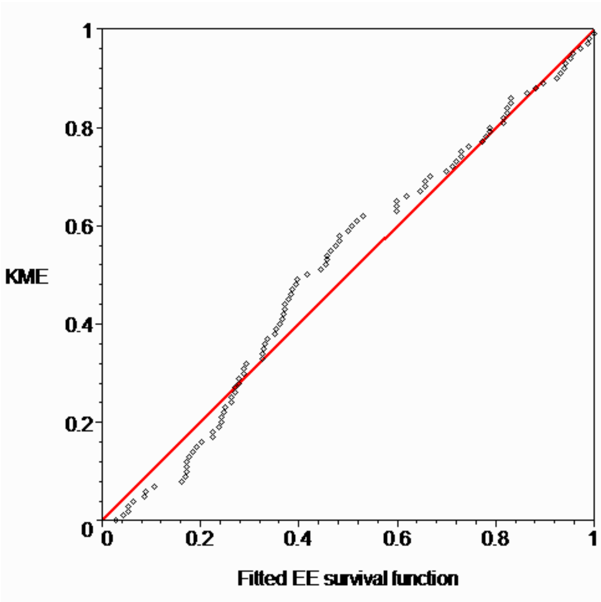


Fig. (7.2): p-p plot of KME versus fitted EE survival function

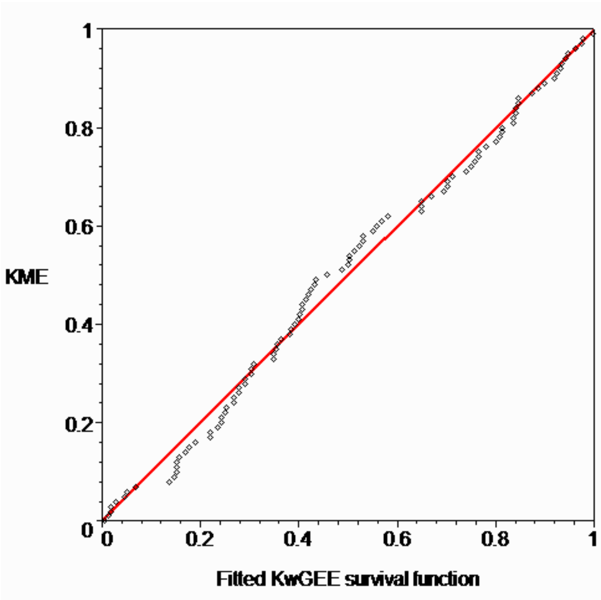


Fig. (7.3): p-p plot of KME versus fitted $KwGEE$ survival function

Visually, the depicted points for fitted $KwGEE$ survival function are very near the 45^o line, indicating very good fit as compared with the fitted EE survival function.

Since $\hat{\lambda} = 0.110961$, $\hat{\alpha} = 4.95024, \hat{a} = 0.660032$, and $\hat{b} = 66.246$, then the estimated hrt $h(x)$ is as shown in the following figure.

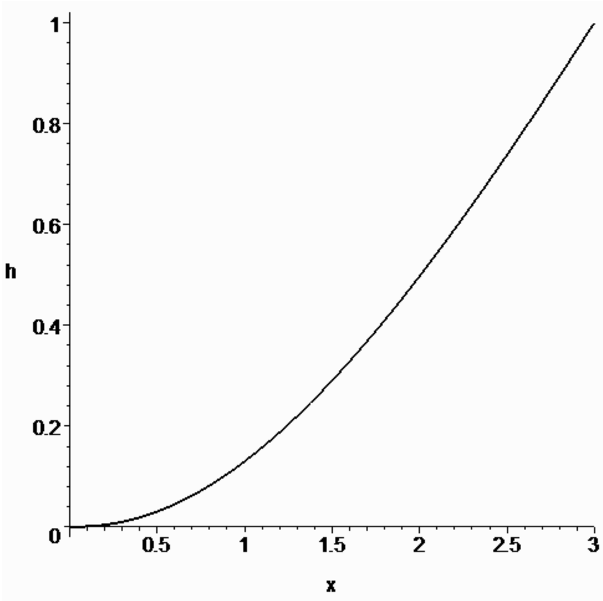


Fig. (7.4): The estimated hazard rate function of $KwGEE$ distribution based on observations the breaking stress of carbon fibers.

8. CONCLUSION

We note that for $KwGEE$ distribution, the AIC, BIC and $CAIC$ are smaller than the corresponding AIC, BIC and $CAIC$ of the EE and exponential distributions. Also the fitted $KwGEE$ survival function indicates strong linear relationship between the empirical and fitted survival functions comparing with the fitted EE and exponential survival functions. All these results lead us to the real data set was analyzed and the $KwGEE$ has provided a good fit for the given data and was more appropriate model.

9. ACKNOWLEDGMENTS

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10. REFERENCES

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