Complex Dynamics of Sine Function using Jungck Ishikawa Iterates

Suman Pant
Research Scholar
G. B. Pant Eng. College,
Pauri Garhwal

Yashwant S.Chauhan, Ph.D
Assistant Professor
G. B. Pant Eng. College,
Pauri Garhwal

Priti Dimri, Ph.D
Asso. Professor and Head
G. B. Pant Eng. College,
Pauri Garhwal

ABSTRACT
The dynamics of transcendental function is one of emerging and interesting field of research nowadays. We introduce in this paper the complex dynamics of sine function of the type $\{\sin(z^n) - z + c = 0\}$ and applied Jungck Ishikawa iteration to generate Relative Superior Mandelbrot set and Relative Superior Julia set. In order to solve this function by Jungck – type iterative schemes, we write it in the form of Sz = Tz, where the function T, S are defined as $Tz = \sin(z^n) + c$ and Sz= z. Only mathematical explanations are derived by applying Jungck Ishikawa Iteration for transcendental function in the literature but in this paper we have generated relative Mandelbrot sets and Relative Julia sets.

Keywords
Complex dynamics, Relative Superior Mandelbrot set, Relative Julia set, Jungck Ishikawa Iteration

1. INTRODUCTION
The study of dynamical behavior of the transcendental functions was initiated by Fatou [12]. For transcendental function, points with unbounded orbits are not in Fatou sets but they must lie in Julia sets. Attractive points of a function have a basin of attraction, which may be disconnected.

A point z in Julia for sine function has an orbit that satisfies

$|\text{Im} z| >= 50$.

The iteration of complex analytic function (f) decompose the complex plane into two disjoint sets

1. Stable Fatou sets on which the iterations are well behaved.

2. Julia sets on which the map is chaotic.

In this past literature the sine function was considered of the following forms:

(i) $\sin(z^n) + c = 0$

(ii) $(\sin z + c)^n = 0$

We are using in our paper sine function of the type $\sin(z^n) - z + c = 0$ where $n \geq 2$ and applied Jungck Ishikawa iterates to develop fractal images of this transcendental

function. Escape criteria of polynomials are used to generate Relative Superior Mandelbrot Sets and Relative Superior Julia Sets. Our results are different from existing results in literature.

2. PRELIMINARIES
The process of generating fractal images from $z \rightarrow \sin(z^n) - z + c$ is similar to the one employed for the self-squared function [17]. Briefly, this process consists of iterating this function up to N times.

Starting from a value z_0 we obtain $z1, z_2, z_3, z_4$ ... by applying the transformation $z \rightarrow \sin(z^n) - z + c$

2.1 Ishikawa Iteration [2]
Let X is a subset of real or complex numbers and $T: X \rightarrow X$ for $x_0 \in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$x_{n+1} = \alpha_n T y_n + (1-\alpha_n) x_n$$

$$y_n = \beta_n T x_n + (1-\beta_n) x_n$$

where $0 \leq \beta_n \geq 1$ and $0 \leq \alpha_n \geq 1$ and α_n & β_n both convergent to non zero number.

2.2 Definition [1]
The sequences $\{x_n\}$ and $\{y_n\}$ constructed above is called Ishikawa sequences of iteration or relative superior sequences of iterates. We denote it by $(x_0, \alpha_n, \beta_n, t)$ .Notice that RSO $(x_0, \alpha_n, \beta_n, t)$ with $\beta_n = 1$ is RSO(x_0, α_n, t) i.e. Mann's orbit and if we place $\alpha_n = \beta_n = 1$ then RSO $(x_0, \alpha_n, \beta_n, t)$ reduces to O (x_0, t) .We remark that Ishikawa orbit RSO$(x_0, \alpha_n, \beta_n, t)$ with $\beta_n = 1/2$ is Relative superior orbit. Now we define Julia set for function with respect to Ishikawa iterates. We call them as Relative Superior Julia sets.

2.3 Definition [1]
The set of points SK whose orbits are bounded under Relative superior iteration of function Q (z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is a boundary of Julia set RSK.

2.4 Jungck Ishikawa Iteration [2]
Let $(X, \|.\|)$ be a Banach space and Y an arbitrary set. Let S, T: $Y \rightarrow X$ be two non self-mappings such that $T(Y) \subseteq S(Y)$, S(Y) is a complete subspace of X and S is injective. Then for $x_o \in Y$, define the sequence $\{S x_n\}$ iteratively by

$$S x_{n+1} = \alpha_n T y_n + (1-\alpha_n) S x_n$$

$$S y_n = \beta_n T x_n + (1-\beta_n) S x_n$$

where $0 \leq \beta_n \geq 1$ and $0 \leq \alpha_n \geq 1$ and α_n & β_n both convergent to non zero number.

3. FIXED POINTS

3.1 Fixed points of quadratic function

Table 1: Orbit of F (z) for (z_o=0.1625+0.8125i) at α =0.5, β =0.5, c=0.1

No. of iterations	|Sz|	|Tz|
1	0.31249	0.43321
2	0.04783	0.01441
3	0.01464	0.07266
4	0.05462	0.09546
5	0.07897	0.10403
6	0.09330	0.10803
7	0.10160	0.11011
8	0.10636	0.11125
9	0.10909	0.11188
10	0.11064	0.11224
11	0.11153	0.11244
12	0.11203	0.11255
13	0.11232	0.11262
14	0.11249	0.11265
15	0.11258	0.11267
16	0.11263	0.11269
17	0.11266	0.11269
18	0.11268	0.11270
19	0.11269	0.11270
20	0.11269	0.11270
21	0.11270	0.11270
22	0.11270	0.11270
23	0.11270	0.11270
24	0.11270	0.11270

Here we observe that the value converges to a fixed point after 21 iterations.

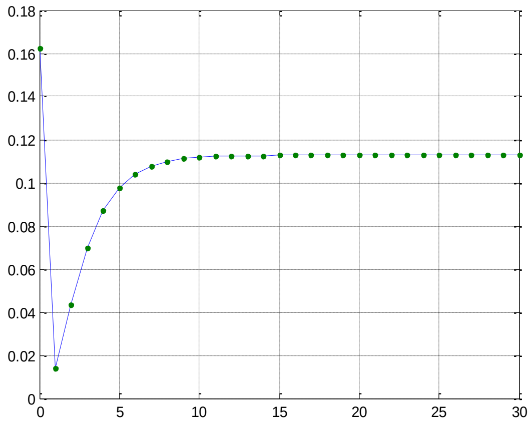

Figure 1: Orbit of F(x) for (z_o = 0.1625+0.8125i) at α =0.5, β =0.5, c=0.1

Table 2: Orbit of F (z) for (z_0=-2.55+0.375i) at α =0.3, β =0.7, c=0.1

No. of iterations	|Sz|	|Tz|
1	0.27560	2.46585
2	0.74621	1.37485
3	0.72702	0.55117
4	0.48395	0.17341
5	0.15496	0.02913
6	0.07996	0.09661
7	0.09854	0.10888
8	0.10783	0.11155
9	0.1111	0.11233
10	0.11218	0.11258
11	0.11253	0.11266
12	0.11265	0.11269
13	0.11268	0.11270
14	0.11270	0.11270
15	0.11270	0.11270
16	0.11270	0.11270
17	0.11270	0.11270
18	0.11270	0.11270
19	0.11270	0.11270
20	0.11270	0.11270

Here we observe that the value converges to a fixed point after 14 iterations

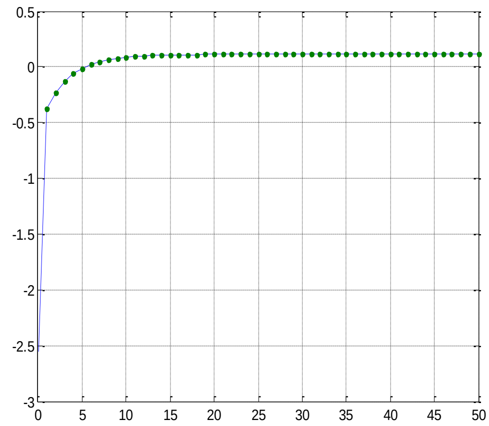

Figure 2. Orbit of F(x) for (z_o=-2.55+0.375i) at α =0.3, β =0.7, c=0.1

Table 3: Orbit of F (z) for (z_o=-0.1375-0.0625i) at α =0.5, β =0.8, c=0.1

No. of iterations	|Sz|	|Tz|
140	0.11269	0.1127
141	0.11269	0.1127
142	0.11269	0.1127
143	0.11269	0.1127
144	0.11269	0.1127
145	0.11269	0.1127
146	0.11270	0.1127
147	0.11270	0.1127
148	0.11270	0.1127
149	0.11270	0.1127
150	0.11270	0.1127

Here we skipped 139 iterations and observed that the value converges to a fixed point after 146 iterations.

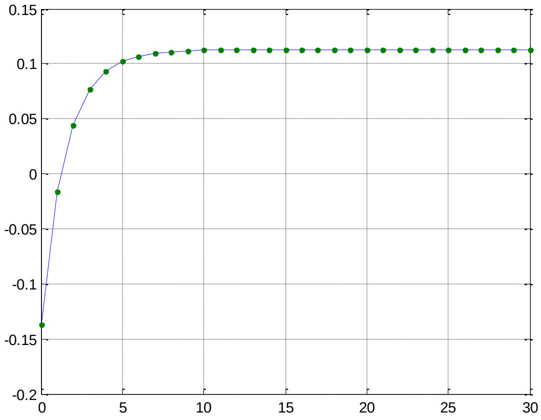

Figure 3: Orbit of F (z) for (z_o=-0.1375-0.0625i) at α =0.5, β =0.8, c=0.1

3.2 Fixed points of cubic function

Table 1: Orbit of F (z) for (z_o=-0.6125+0i) at α =0.5, β =0.5, c=0.1

No. of iterations	|Sz|	
1	0.09375	0.08144
2	0.09884	0.10013
3	0.10053	0.10098
4	0.10093	0.10103
5	0.10101	0.10103
6	0.10103	0.10103
7	0.10103	0.10103
8	0.10103	0.10103
9	0.10103	0.10103
10	0.10103	0.10103

Here we observe that the value converges to a fixed point after 6 iterations.

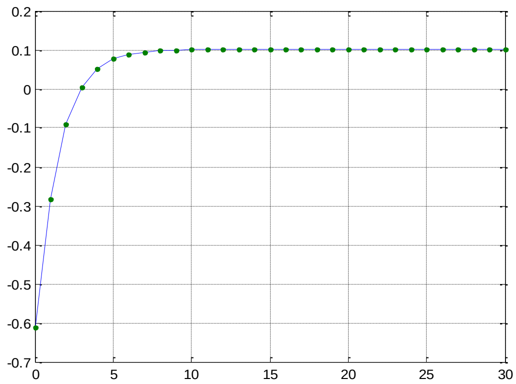

Figure 1 Orbit of F (z) for (z_o=-0.6125+0i) at α =0.5, β =0.5, c=0.1

Table 2: Orbit of F (z) for (z_o=-0.2625+1.10625i) at α =0.3, β =0.7, c=0.1

No. of iterations	|Sz|	|Tz|
1	0.02500	0.04437
2	0.06846	0.03949
3	0.08382	0.05806
4	0.08882	0.08958
5	0.09342	0.09785
6	0.09676	0.10012
7	0.09875	0.10076
8	0.09984	0.10094
9	0.10042	0.10100
10	0.10072	0.10102
11	0.10087	0.10103
12	0.10095	0.10103
13	0.10099	0.10103
14	0.10101	0.10103
15	0.10102	0.10103
16	0.10103	0.10103
17	0.10103	0.10103
18	0.10103	0.10103
19	0.10103	0.10103
20	0.10103	0.10103

Here we observe that the value converges to a fixed point after 16 iterations.

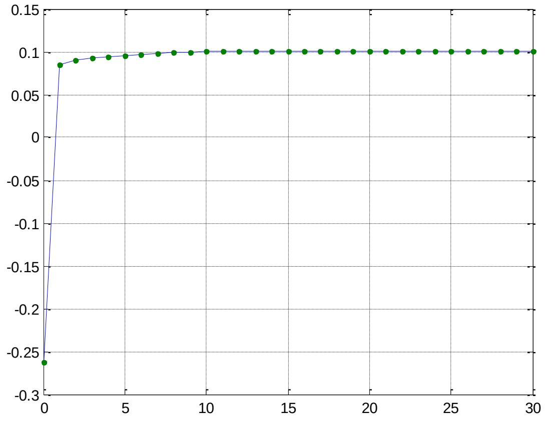

Figure 2: Orbit of F (z) for (z_o=-0.2625+1.10625i) at α =0.3, β =0.7, c=0.1

Table 3: Orbit of F (z) for (z_0 = 0.1875+0.175i) at α =0.5, β =0.8, c=0.1

No. of iterations	|Sz|	|Tz|
1	1.10625	-0.73928
2	0.13639	-0.1472
3	0.12193	0.02143
4	0.11184	0.07494
5	0.1062	0.09209
6	0.10355	0.09787
7	0.10232	0.0999
8	0.10172	0.10063
9	0.10142	0.10089
10	0.10125	0.10098
11	0.10116	0.10101
12	0.10111	0.10103
13	0.10108	0.10103
14	0.10106	0.10103
15	0.10105	0.10103
16	0.10104	0.10103
17	0.10104	0.10103
18	0.10103	0.10103
19	0.10103	0.10103
20	0.10103	0.10103

Here we observe that the value converges to a fixed point after 18 iterations.

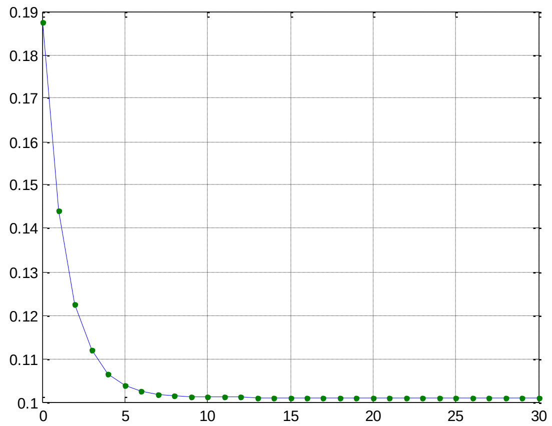

Figure 3: Orbit of F (z) for (z_0 = 0.1875+0.175i) at α =0.5, β =0.8, c=0.1

3.3 Fixed points of biquadratic function

Table 1: Orbit of F (z) for (z_0= 0.0375+0.625i) at α =0.5, β =0.5, c=0.1

No. of iterations	|Sz|	|Tz|
1	0.00625	0.29254
2	0.07642	0.09973
3	0.09532	0.10004
4	0.09914	0.10009
5	0.09991	0.1001
6	0.10006	0.1001
7	0.10009	0.1001
8	0.1001	0.1001
9	0.1001	0.1001
10	0.1001	0.1001

Here we observe that the value converges to a fixed point after 8 iterations.

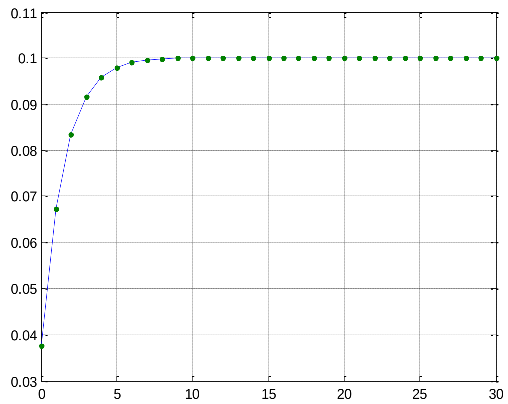

Figure 1: Orbit of F (z) for (z_0= 0.0375+0.625i) at α =0.5, β =0.5, c=0.1

Table 2: Orbit of F (z) for (z_0= 0.1-0.3i) at α =0.3, β =0.7, c=0.1

No. of iterations	|Sz|	|Tz|
1	0.06875	1.46516
2	0.48917	0.15118
3	0.20888	0.08937
4	0.05448	0.09998
5	0.02276	0.10000
6	0.06138	0.09999
7	0.08071	0.10003
8	0.09038	0.10006
9	0.09523	0.10008
10	0.09766	0.10009
11	0.09888	0.10010
12	0.09949	0.10010

13	0.09979	0.10010
14	0.09995	0.10010
15	0.10002	0.10010
16	0.10006	0.10010
17	0.10008	0.10010
18	0.10009	0.10010
19	0.10010	0.10010
20	0.10010	0.10010

Here we observe that the value converges to a fixed point after 19 iterations.

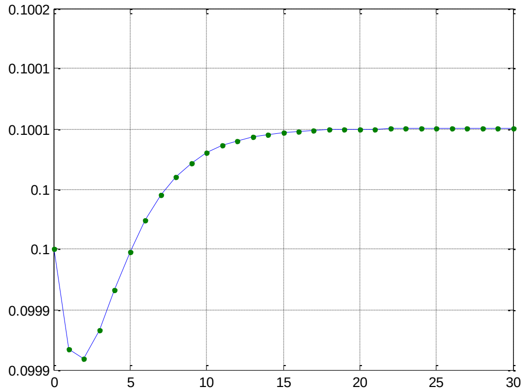

Figure 2: Orbit of F (z) for (z_o= 0.1-0.3i) at α =0.3, β =0.7, c=0.1

Table 3: Orbit of F (z) for (z_o = 0.2375+0i) at α =0.5, β =0.8, c=0.1

No. of iterations	|Sz|	|Tz|
1	0.26875	0.81993
2	0.04699	0.09997
3	0.06819	0.09994
4	0.08093	0.10000
5	0.08858	0.10004
6	0.09318	0.10007
7	0.09594	0.10008
8	0.09760	0.10009
9	0.09860	0.10009
10	0.09920	0.10010
11	0.09956	0.10010
12	0.09978	0.10010
13	0.09991	0.10010
14	0.09998	0.10010
15	0.10003	0.10010
16	0.10006	0.10010
17	0.10007	0.10010
18	0.10009	0.10010
19	0.10009	0.10010
20	0.10009	0.10010
21	0.10010	0.10010
22	0.10010	0.10010
23	0.10010	0.10010
24	0.10010	0.10010
25	0.10010	0.10010

Here we observe that the value converges to a fixed point after 21 iterations.

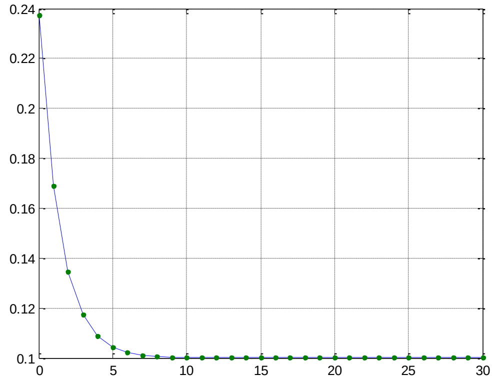

Figure 3: Orbit of F (z) for (z_o= 0.2375+0i) at α =0.5, β =0.8, c=0.1

4. GEOMETRY OF RELATIVE SUPERIOR MANDELBROT SETS AND RELATIVE SUPERIOR JULIA SETS

Relative Superior Mandelbrot Sets

- In case of quadratic function, the central body is divided into two parts. It is seen that the body is symmetric along the real axis only. Secondary lobes are very small initially for

  α = 1, β =1. As the value is changed to α =0.3 β =0.7, the central body is divided into four parts but the middle part is quite larger in comparison to the head and tail.

  Secondary lobes seem to appear larger than initial stage.

- In case of cubic function, the central body is divided into two equal parts, each part have one major secondary lobe and many minor secondary lobes. It is seen that the body is symmetric along the both axes. For α =0.3, β =0.7, the size of the major secondary lobes start enlarging and also a tiny bulb seems to occur along the real axis.

- In case of biquadratic function, the central body is divided into three parts, each part having one major secondary bulb along with large number of minor secondary bulbs. It is seen that the body is symmetric along the real axis only. For α =0.3, β =0.7, the two of the major secondary lobes are same in size but one of them grows larger in size and undergoes bifurcation along the real axis.

Relative Superior Julia Sets

- Relative Superior Julia Sets for the transcendental function sin (z) appears to follow law of having 2n wings. These sets are symmetric along both the axes i.e. along real and imaginary axis.

- The Relative Superior Julia Sets for quadratic function is divided into four wings having black central body. These sets are symmetric along both the axes.
- The Relative Superior Julia Sets for Cubic function is divided into six wings having reflectional and rotational symmetry, along with a larger black region.
- The Relative Superior Julia Sets for biquadratic function is divided into eight wings possessing the reflectional and rotational symmetry, along with a larger escape region as compared to quadratic and cubic function.
- It is also observed from the graphical study of fixed points of Relative Superior Julia Sets that the convergence for
- α =0.3, β =0.7 is quite fast for all polynomials in comparison to the convergence for α =0.5, β = 0.8.

5. GENERATION OF RELATIVE SUPERIOR MANDELBROT SETS

We generated the Relative Superior Mandelbrot sets. We present here some beautiful filled Relative Superior Mandelbrot sets for quadratic, cubic and biquadratic function.

6.1 Relative Superior Mandelbrot sets for Quadratic function

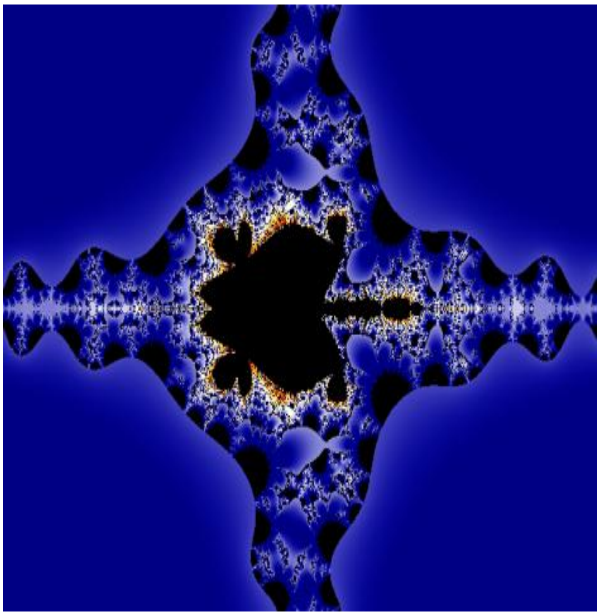

Figure 1: Relative Superior Mandelbrot Set for $\alpha = \beta$ =0.5 & c = -0.1625+0.8125i

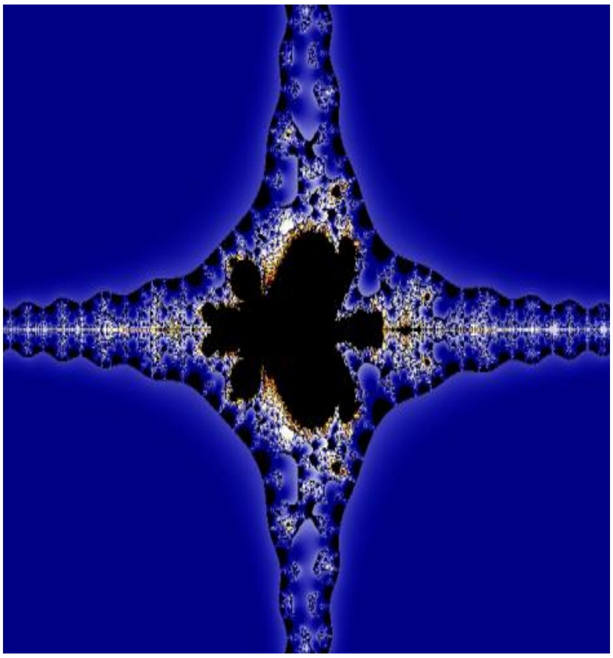

Figure 2: Relative Superior Mandelbrot Set for α =0.3, β =0.7, c=-2.55+0.375i

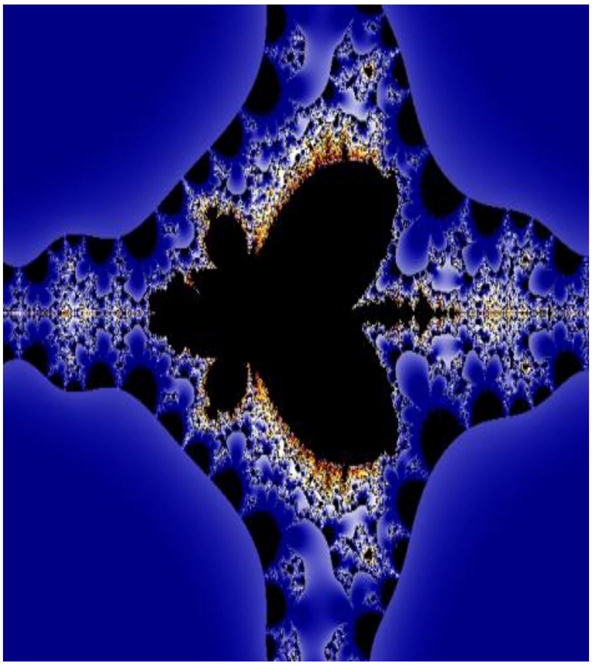

Figure 3: Relative Superior Mandelbrot Set for α =0.5, β =0.8, c=-0.1375-0.0625i

6.2 Relative Superior Mandelbrot Sets for Cubic function

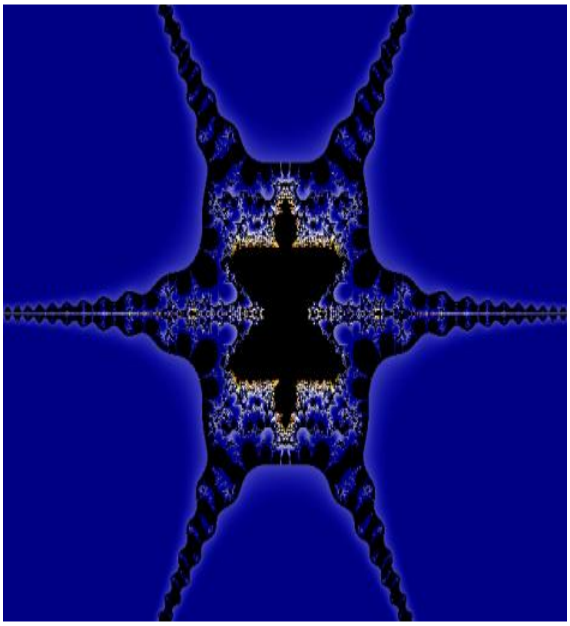

Figure 1: Relative Superior Mandelbrot Set for $\alpha = \beta$ =0.5, c=-0.6125+0i

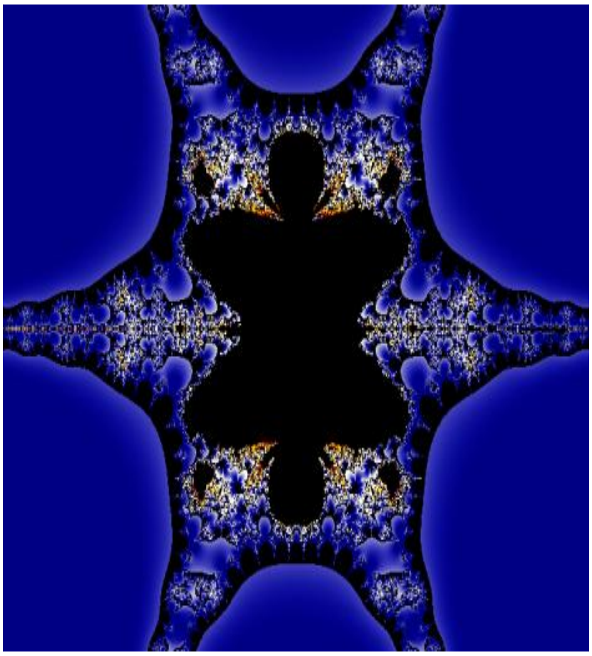

Figure 2: Relative Superior Mandelbrot Set for α =0.3, β =0.7, c = -0.2625+1.10625i

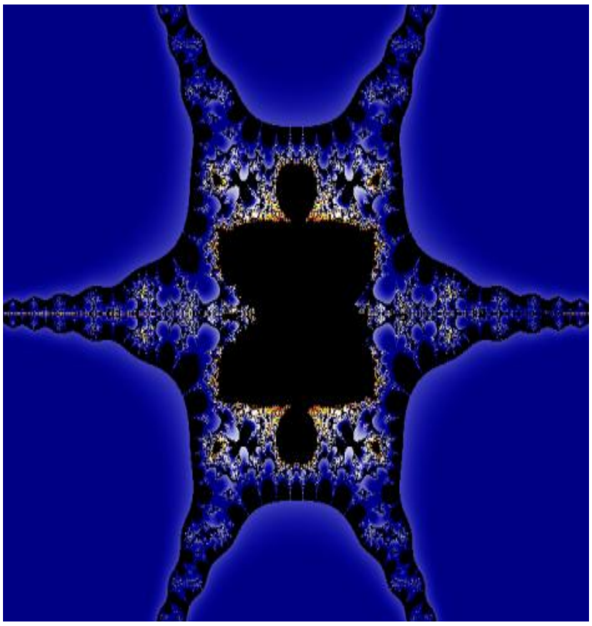

Figure 3: Relative Superior Mandelbrot Set for α =0.5, β =0.8, c = 0.1875+0.175i

6.3 Relative Superior Mandelbrot sets for biquadratic function

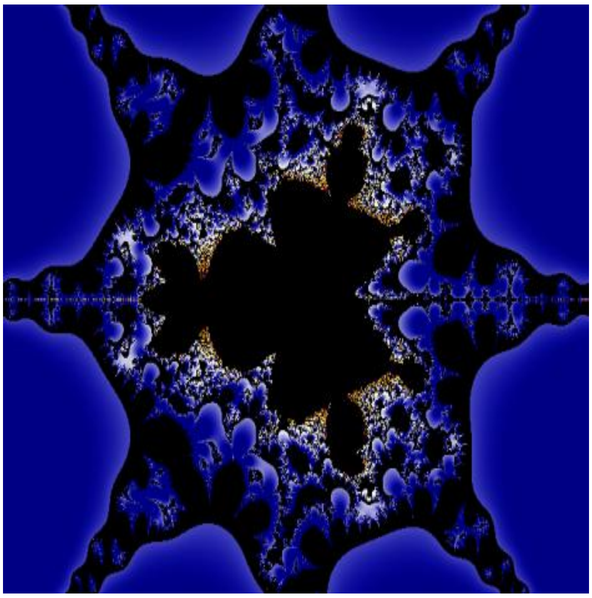

Figure 1: Relative Superior Mandelbrot Set for α =0.5, β =0.5, c = 0.0375+0.625i

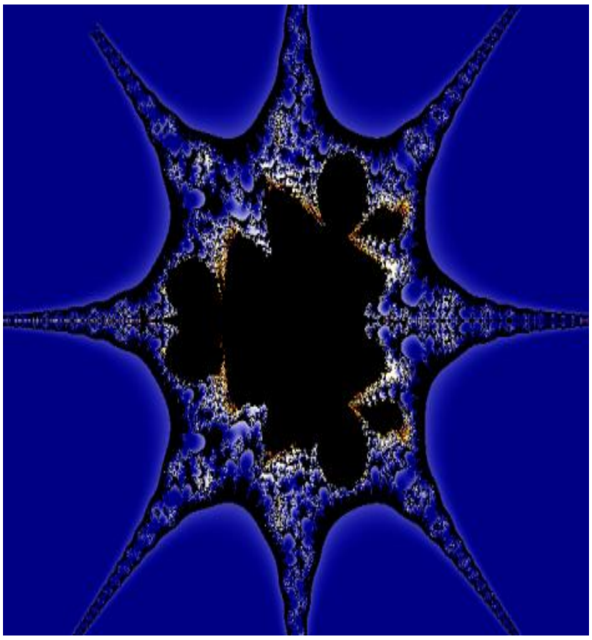

Figure 2: Relative Superior Mandelbrot Set for α =0.3, β =0.7, c = 0.1-0.3i

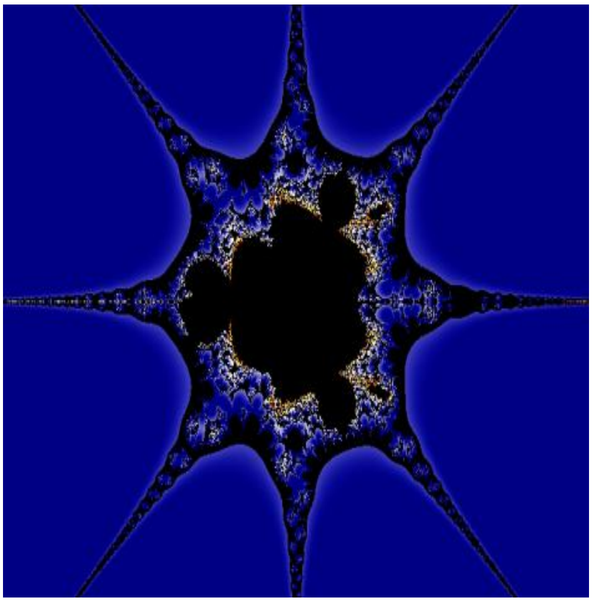

Figure 3: Relative Superior Mandelbrot Set for α =0.5, β =0.8, c = 0.2375+0i

6. GENERATION OF RELATIVE SUPERIOR JULIA SETS

We generated the Relative Superior Julia sets. We have presented here some beautiful filled Relative Superior Julia sets for quadratic, cubic and biquadratic function.

5.1 Relative Superior Julia sets for Quadratic function

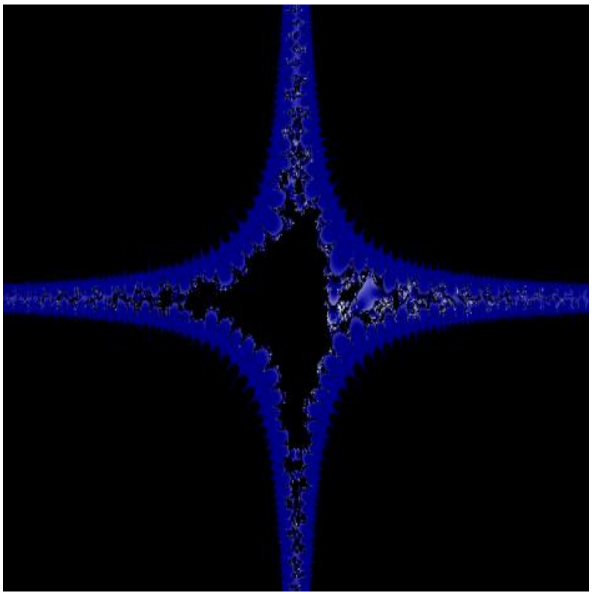

Figure 1: Relative Superior Julia Set for $\alpha = \beta$ =0.5, c=0.1625+0.8125i

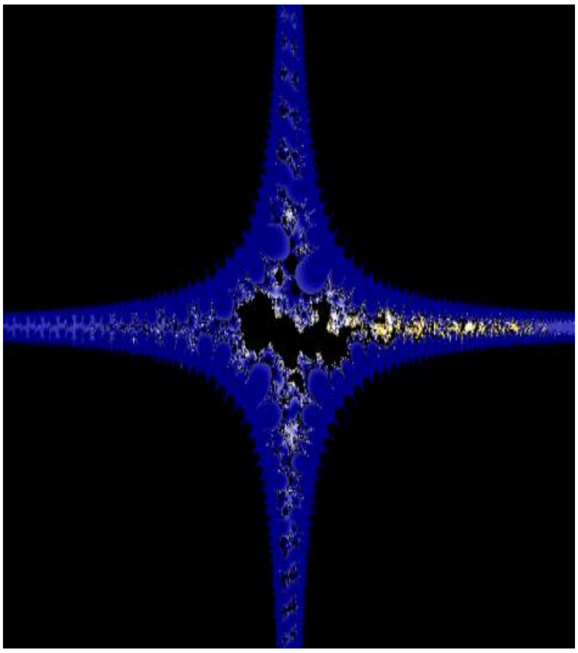

Figure 2: Relative Superior Julia Set for α =0.3, β =0.7, c=-2.55+0.375i

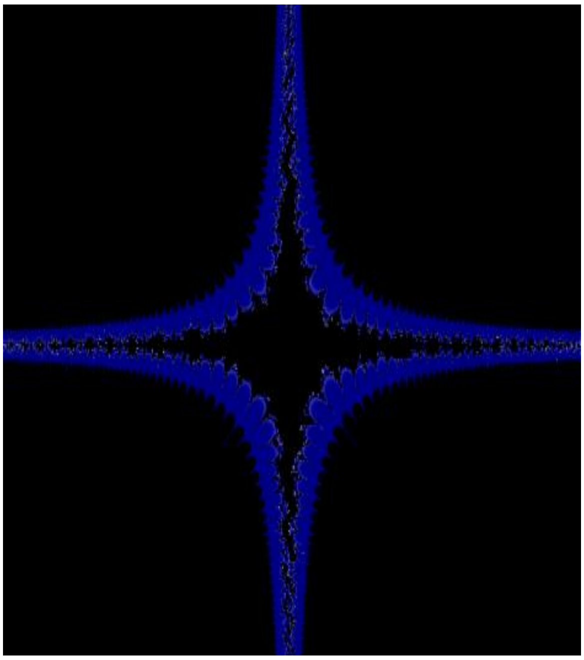

Figure 3: Relative Superior Julia Set for α =0.5, β =0.8, c=-0.1375-0.0625i

5.2 Relative Superior Julia Sets for Cubic function

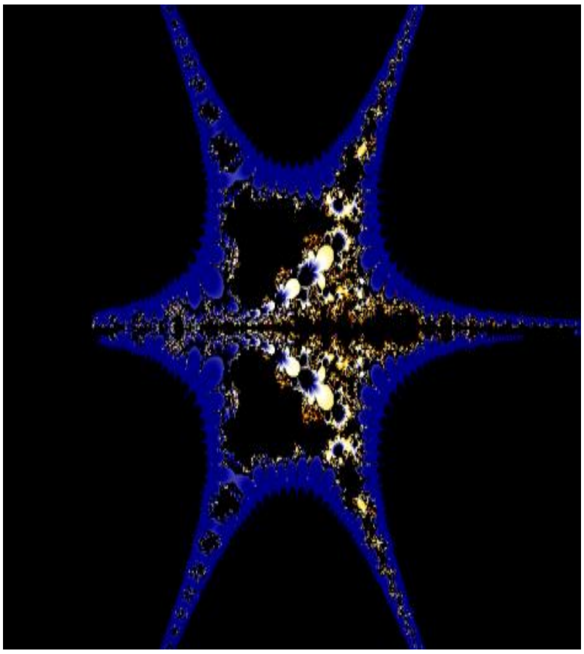

Figure 1: Relative Superior Julia Set for $\alpha = \beta$ =0.5, c=-0.612+0i

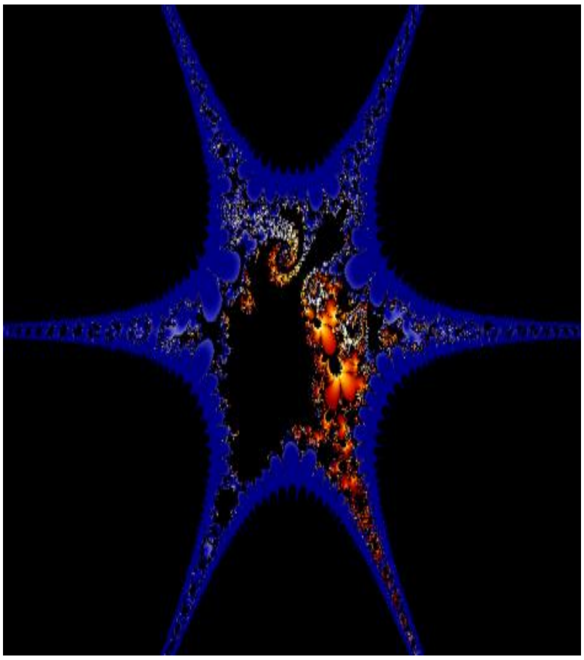

Figure 2: Relative Superior Julia Set for α =0.3, β =0.7, c = -0.2625+1.10625i

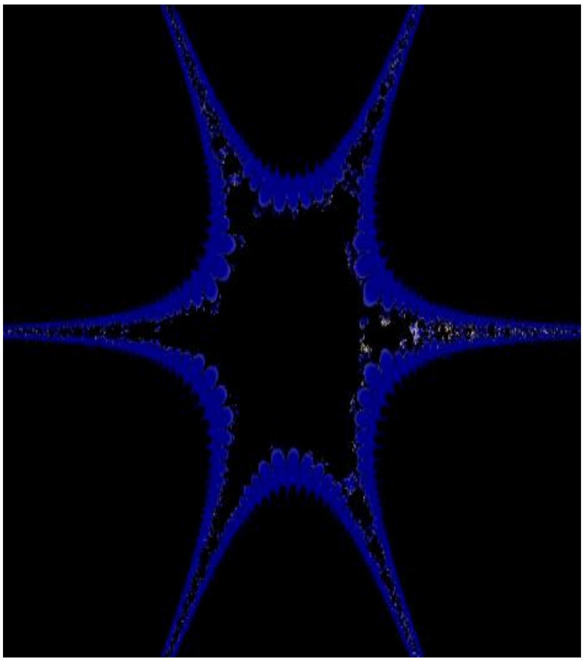

Figure 3: Relative Superior Julia Set for α =0.5, β =0.8, c = 0.1875+0.175i

5.3 Relative Superior Julia sets for biquadratic function

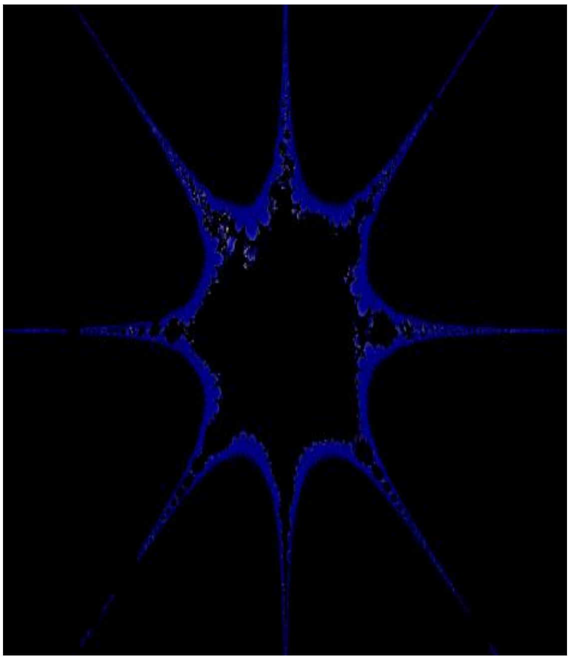

Figure 1: Relative Superior Julia Set for α = 0.5, β = 0.5, c=0.0375+0.625i

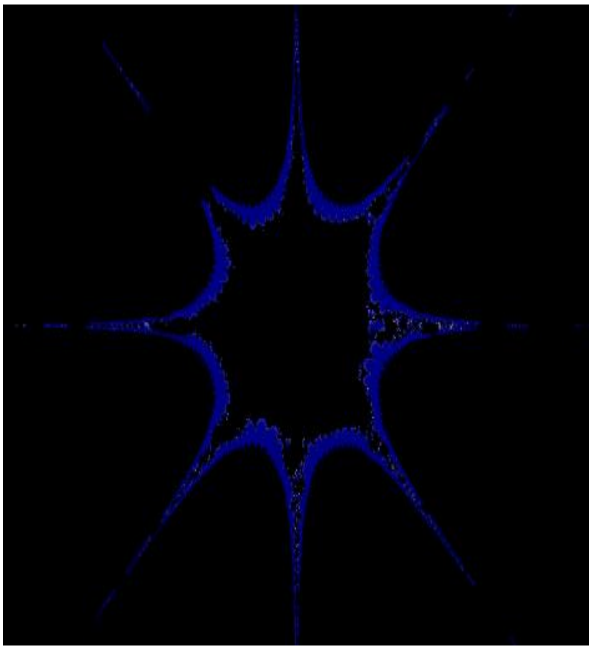

Figure 2: Relative Superior Julia Set for α = 0.3, β = 0.7, c=0.1-0.3i

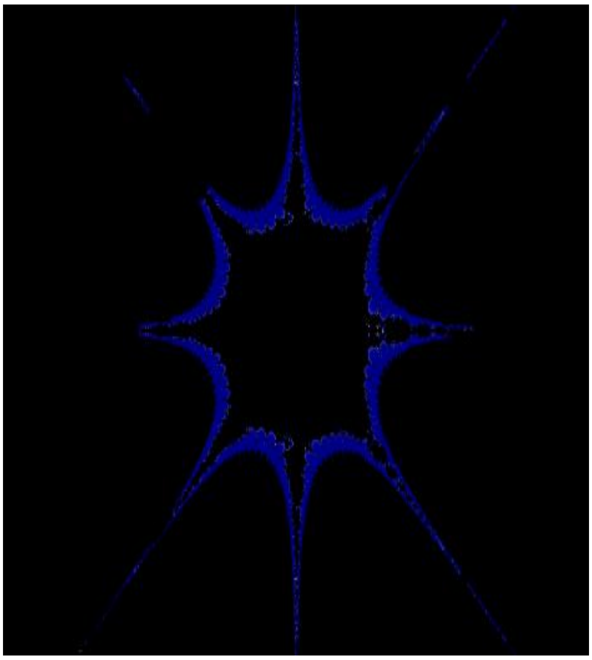

Figure 3: Relative Superior Julia Set for α = 0.5, β = 0.8, c=0.2375+0i

7. CONCLUSION

In this paper we studied the sine function which is one of the members of transcendental family. The fixed point 0 for S (z) = sin (z^n) – z + c = 0 also satisfies $S'(0) = 1$. Orbits on the real axis tend to 0 while orbits on the imaginary axis tend to infinity. Relative Superior Julia Sets for the transcendental function sin (z) appears to follow law of having 2n wings.

The surrounding region of the Mandelbrot set appears to be an invariant Cantor set in the form of curve or "hair" that extends to ∞. The orbit of any point on hair tends to infinity under iteration. Here the geometry of hairs is quite similar to that of exponential family and hence showed the property of transcendental function. The region filled up with large number of escaping points represents Julia set plane.

8. REFERENCES

[1] Suman Joshi, Dr.Yashwant Singh Chauhan and Dr. Ashish Negi, "New Julia and Mandelbrot Sets for Jungck Ishikawa Iterates" International Journal of Computer Trends and Technology (IJCTT), vol.9, no.5, pp.209-216, 2014.

[2] Suman Joshi, Dr.Yashwant Singh Chauhan and Dr.Priti Dimri ,"Complex Dynamics of Multibrot Sets for Jungck Ishikawa Iteration" International Journal of Research in Computer Applications and Robotics (IJRCAR), vol. 2, no. 4, pp. 12-22, 2014.

[3] M.O.Olatinwo, "Some stability and strong convergence results for the Jungck-Ishikawa iteration process,"Creative Mathematics and Informatics, vol. 17, pp. 33-42, 2008.

[4] R. Chugh and V. Kumar, "Strong Convergence and Stability results for Jungck-SP iterative scheme, International Journal of Computer Applications, vol. 36,no. 12, 2011.

[5] S. Ishikawa, "Fixed points by a new iteration method", Proc. Amer. Math. Soc.44 (1974), 147-150.

[6] G. Julia, "Sur 1' iteration des functions rationnelles", JMath Pure Appli. 8 (1918), 737-747

[7] B. B. Mandelbrot, The Fractal Geometry of Nature, W. H.Freeman, New York, 1983.

[8] Eike Lau and Dierk Schleicher, "Symmetries of fractals revisited." Math. Intelligencer (18) (1) (1996), 45-51.MR1381579 Zbl 0847.30018.

[9] J. Milnor, "Dynamics in one complex variable; Introductory lectures", Vieweg (1999).

[10] Shizuo Nakane, and Dierk Schleicher, "Non-local connectivity of the tricorn and multicorns", Dynamical systems and chaos (1) (Hachioji, 1994), 200-203, World Sci. Publ., River Edge, NJ, 1995. MR1479931.

[11] Rajeshri Rana, Yashwant S Chauhan and Ashish Negi.Article: Non Linear Dynamics of Ishikawa Iteration. International Journal of Computer Applications 7(13):43–49, October 2010. Published By Foundation of Computer Science.ISBN: 978-93-80746-97-5.

[12] Ashish Negi, "Generation of Fractals and Applications", Thesis, Gurukul Kangri Vishwvidyalaya, (2005).

[13] M.O.Osilike, "Stability results for Ishikawa fixed point iteration procedure", Indian Journal of Pure and Appl. Math., 26(1995), 937-945.

[14] A. G. D. Philip: "Wrapped midgets in the Mandelbrot set", Computer and Graphics 18 (1994), no. 2, 239-248.

[15] Shizuo Nakane, and Dierk Schleicher, "On multicorns and unicorns: I. Antiholomorphic dynamics. Hyperbolic components and real cubic polynomials", Internat. J. Bifur. Chaos Appl. Sci. Engrg, (13) (10) (2003), 2825-2844. MR2020986.