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A Study on the Filtering Approach and Turbulence Modeling for LES

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ABSTRACT
During the past decades, Large-Eddy Simulation (LES) has been demonstrated to be a useful research tool for understanding the physics of turbulence as well as an accurate and sophisticated predictive method for flows of engineering interest. The LES is numerical technique and is based on the separation between large and small scales in which the large-scale motion is exactly calculated and the effects of small sales or so called sub grid-scale motions are modelled. It is also important to note that the explicit or implicit filter representations like spectral cut-offs or numerical discretizations are commonly used in LES of turbulent flows. Strictly we can say that in LES we need to filter the Navier-Stokes equations in turbulence. Therefore, the study on the filtering approach in turbulence is the main objects of the present research, and in this study we have elaborately studied on this filtering approach and analyzed some general algebraic properties of the filtered representations. It is shown that the averaged equations are the same in terms of the generalized central moments, and then we have defined the resolved turbulence using these average properties. The algebraic consistency rules related with the resolved quantities to the turbulent stresses are derived and their possible use in sub grid-scale modelling is examined. In this study, we have also discussed about the standard Smagorinsky model for LES and then we derived an expression to determine the Smagorinsky constant dynamically, which suppose to be assured the consistency between the filter and the sub grid-scale model. Finally, we have derived the governing equations for LES by applying the filtering approach to the Navier-Stokes equations.

General Terms
Large Eddy Simulation

Keywords
Large Eddy Simulation, Turbulence, Smagorinsky constant, Navier Stokes Equation, Central moments.

1. INTRODUCTION
Most flows encountered in engineering practice are turbulence. Still today, turbulence in fluids is considered as one of the most difficult problems in modern physics as well as in engineering applications. Understanding the structures in space of turbulent flow and their statistical properties remain challenges both for experimentalists and theoreticians.

Turbulence is an important and complicated kind of fluid. It is common experience that the flows observed in nature, such as those of rivers and winds usually differ from the streamline flow or the laminer flow of a viscous fluid. These kind of irregular flows occur at high Reynolds numbers and are often termed turbulent flows. In turbulent flow, the steady motion of the fluid is only steady in so far as the temporal mean values of the velocities and the pressure are concerned whereas actually both the velocities and the pressures are irregularly fluctuating. The velocity and the pressure distributions in turbulent flows as well as the energy losses are determined mainly by the turbulent fluctuations. the essential characteristics of turbulent flow is that the turbulent fluctuations are random in nature.

In 1937, Taylor and Von Karman [1] gave the following definition:

Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when the flow past solid surfaces or even neighbouring stream of the same fluid flow past over one another.

According to this definition the flow has to satisfy the condition of irregular. Indeed, this irregularity is a very important feature. Because of irregularity, it is impossible to describe the motion in all details as a function of time and space coordinates. But, fortunately, turbulent motion is irregular in the sense that it is possible to describe it by laws of probability. It appears possible to include distinct average values of various quantities, such as velocity, pressure, temperature etc, and this is very important. Therefore, it is not sufficient just to say that turbulence is an irregular motion. Yet we do not have a clear-cut definition of turbulence. This is rather difficult. Hinze (1975) [2] suggests in the book ‘Turbulence’

‘Turbulent fluid motion is an irregular condition of the flow in which various quantities show random variation with time and space coordinates, so that statistically distinct average can be discerned’

Although it is very difficult to give a precise definition of turbulence, however, yet most flows encountered in engineering practice are considered as turbulent flows, and they are characterized by the following properties:

Turbulent flows are highly unsteady. A plot of the velocity as a function of time would appear random to an observer unfamiliar with these flows.

They are three-dimensional. The time-averaged velocity may be a function of only two coordinates, but the instantaneous field appears essentially random.

The contain a great deal of velocity. Stretching of vortices is one of the principle mechanisms by which the intensity of the turbulence is increased.

Turbulence increases the rate at which conserved quantities are stirred. That is, particles of fluids width differing concentrations of the conserved properties are brought into contact. The actual mixing is accomplished by diffusion. Nonetheless, this behaviour is often called diffusive.

By increasing the mixing of momentum, turbulence brings fluids differing momentum content into contact. The reduction of the velocity gradients produced by the action of viscosity reduces the kinetic energy of the flow; in other words, it is dissipative. The lost energy is irreversibly converted into internal energy of the fluid.

These properties are important. The effects produced by turbulence may or may not be desirable. Turbulence is not a feature of fluids but of fluid flows. Most of the dynamics of turbulence is the same in all fluids. Whether they are liquids or gases, if the Reynolds number of the turbulence is large enough; the major characteristics of turbulent flows are not controlled by the molecular properties of the fluid in which the turbulent occurs. The characteristics of turbulence depend on its environment.

It is considered that the turbulent fluctuation is random in nature but it does not imply that the motion is 'completely random'. All turbulent flows exhibit characteristic structure with respectable features. However, those features show statistical variations in their size and strength in the time and space of their occurrence.

Turbulent flow is completely nonlinear and complex and recent studies suggest that this flow contains various types of verticals structures. The existence of these verticals structures in turbulent flow is the concept against random motion of fluid element. These verticals structures are also known as turbulent eddies. When these eddies move they affect the fluid surroundings them. These eddy and their random movements give rise to fluctuations in velocity components and pressure at any point in the flow field. The movement of those eddies is longitudinal as well as in the lateral direction impacts to the flow a greater ability diffusion and makes the analysis of such a flow extremely complex. The Navier-Stokes equations govern the evolution of the flows of interest.

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2. LARGE – EDDY SIMULATION

A simulation which treats the large eddies more exactly than small ones may make sense; Large – Eddy simulation (LES) is such an approach. LES are three dimensional, time depended and expensive but much less costly than a DNS. LES is the preferred method for flows in which the Reynolds number is too high or the geometry is too complex for the application of DNS.

The scale selection that the Large – Eddy simulation technique is based on a separation between large and small scales (Pierre sagaut, 1998 [3]). In order to define these two categories, a reference or cut-off length first has to be determined. Those scales that are of a characteristic size greater than the cut-off length are called large or resolves scales, and others are called small or sub grid scales and the way of a statistical model called a sub grid model.

The governing equation for Large Eddy Simulation is the filtered Navier-Stokes equations, so we have to filter the Navier-stokes equations with effective filter functions. There are several filter functions that are used to filter the Navier-Stokes equations, and using these filter-functions we can decompose the velocity fields into grid-scale and subgrid-scale velocity. Still today researchers are developing new SGS models. Therefore, it is clearly revealed that the filtering approach to the Navier-Stokes equations plays a vital role for formulating the governing equations as well as for developing SGS model for LES.

We have realized by the previous discussions that the Large-Eddy Simulation (LES) in turbulence is the more effective method than the other techniques such as Reynolds Averaged Navier – Stokes (RANS), Direct Numerical Simulation (DNS), etc. The LES is a numerical technique and is based on the separation between large and small scales in which the large-scale motion is exactly calculated and effects of small scale or so called sub grid-scale motions are modelled. Therefore, the study on the filtering approach in turbulence is the main objective and in this study we have elaborately explained this filtering approach which is previously done by Germano,M(1990). Then we discuss about the Smagorinsky model for LES and express a procedure for determination of the Smagorinsky constant dynamically. We also discuss on the consistent decomposition of the generalized central moments in turbulence. Finally, we derive the governing equations for LES by applying the filtering approach to the Navier – Stokes equations.

3. A FILTERED REPRESENTATION BY A CONVOLUTION INTEGRAL

A filtered representation of an original chaotic field, u_i generally can be expressed by a convolution integral (Leonard, 1974 [5]) given by:

$$\left(\left\langle u_i\left(x,t\right)\right\rangle_{l,\theta} = \int u_i\left(x',t'\right)\xi\left(x-x',\, t-t';\; l,\, \theta\right) d^3\; x' dt'\right) \tag{1.1}$$

with

$$\int \;\xi\left(x-x',\, t-t';\; l,\, \theta\right) d^3\;\, x' dt' = 1 \tag{1.2}$$

Where l and θ are characteristic filter length and characteristic filter time. The numerical discretizations are considered as the typical implicit filters and characteristic explicit filters are the following:

$$\langle u_i\,(x,t)\rangle_{\theta} = \frac{1}{\theta}\int_t^{t+\theta} u_i\,(x,t')dt' \tag{1.3}$$

$$\left\langle u_i\;\left(x,t\right)\right\rangle_t = \frac{1}{l^3}\int_x^{x+l}\int_y^{y+l}\int_z^{z+l} u_i\left(x',t\right) dx'\; dy'\; dz' \tag{1.4}$$

While typical filters commonly used in the large eddy simulation of turbulent flows are the spectral cut offs and the Gaussian filters given by the convolutional nucleus

$$\xi\left(x\,-\,x';l\right) = \left(\frac{6}{\pi\, l^2}\right)^{\frac{3}{2}}\;\exp\left(-\frac{6\left(x-x'\right)^2}{l^2}\right) \tag{1.5}$$

Formally, the filtering approach stands between the direct approach and the statistical approach and probably will produce in the future a unified theory linking the direct approach to the statistical one by a continuous interval of intermediate steps.

We notice that physical space-time averages are often substitutive of ensemble average, particularly when symmetries or homogeneities are present in the flow. As a particular example let us consider the class of time filters Eqn. (1.3) parameterized in terms of the characteristic filter time θ ; we see that they constitute a hierarchy of filters going from the identity $\theta = 0$ the direct approach, to the infinite time average, $\theta \rightarrow \infty$ and it is well known that for statistically steady flows this time average converges to the ensemble average.

From an operational point of view the filtering approach consists in applying explicitly or implicitly to the Navier-Stokes equations a linear operator commutative with the space-time derivatives. Its principal field of application is in computational fluid dynamics and the characteristic filter lengths and times are intimately related to the grid discretization or the spectral truncation. However the history of the filtering is very old, and dates from the first studies on turbulence. The first average proposed by Boussineq (1877) [6] is given by Eqn. 1.3 where, θ is the characteristic filter time, while Reynolds (1895) preferred the spatial average Eqn. (1.4) where l^3 is a certain volume of space. It is evident that only in the case $\theta \rightarrow \infty$ or $l \rightarrow \infty$ do these averaging operators satisfy the simple conditions that Reynolds himself stated as necessary for a well-behaved mean operator:

$$\langle f \langle g \rangle \rangle = \langle f \rangle \langle g \rangle \qquad (1.6)$$

$$\langle \langle f \rangle \rangle = \langle f \rangle \qquad (1.7)$$

and was subsequently directed to the statistical averages that obviously satisfy the Reynolds rules of the mean given in Eqns. 1.6 & 1.7.

This approach radically changed with the advent of the computer. The analogies between the filtering operators and the numerical discrimination were appreciated (Rogallo & Moin, 1984 [7]); it was shown that their characteristic lengths and times can be correlated with computational grid values and the filtering approach become the framework that permitted a formal theory of the Large-Eddy simulation in all its aspects.

The old idea of averaging the Navier-Stokes equations was almost always couple to the parallel idea of a decomposition of the turbulent signal into a mean part $\langle u_i \rangle$ that is to say the part generated by the average, and a fluctuation u_i', whose sum is the original quantity, $u_i = \langle u_i \rangle + u_i'$. The usual procedure of the statistical approach is to write equations for the fluctuating velocities u_i' and to produce evolutionary equations for the central moments defined in terms of the fluctuations as:

$$\langle u_i' \; u_j' \rangle, \; \langle u_i' \; u_j' \; u_k' \rangle \qquad (1.8)$$

Giving rise to the well-known problem of the closure This procedure, when extended to non-Reynolds averaging operators (Germano, 1987 [8]), produces a lot of problems mainly because the mean value of the fluctuations is now different from zero and the assumption that there is no correlation between the mean values and the fluctuations is no longer valid:

$$\langle u_i' \rangle \neq 0; \; \langle \langle u_i' \rangle u_j' \rangle \neq 0; \qquad (1.9)$$

As a consequence the classical relations between the moments $\langle u_i' \; u_j' \rangle, \langle u_i' \; u_j' \; u_k' \rangle$ and the central moments (Moin & Yaglom, 1971) [9]

$$\langle u_i' \; u_j' \rangle = \langle u_i \, u_j \rangle - \langle u_i \rangle \langle u_j \rangle \qquad (1.10a)$$

$$\langle u_i' \, u_j' \, u_k' \rangle = \langle u_i \, u_j \, u_k \rangle - \langle u_i \rangle \langle u_j' \, u_k' \rangle$$
$$\langle u_j \rangle \langle u_i' \, u_j' \rangle - \langle u_k \rangle \langle u_i' \, u_j' \rangle - \langle u_i \rangle \langle u_j \rangle \langle u_k \rangle \qquad (1.10b)$$

$$\langle u_i' \, u_j' \, u_k' \, u_l' \rangle \qquad (1.10\ c)$$

are no longer valid and new terms arising that considerably complicate the system of averaged equations and their closure. However, if we introduce a new set of generalized central moments:

$$\tau(u_j , u_k), \;\; \tau(u_i , u_j , u_k) \qquad (1.11)$$

This is formally defined by using the Eqn. 1.10.

$$\tau(u_i , u_j) = \langle u_i , u_j \rangle - \langle u_i \rangle \langle u_j \rangle \qquad (1.12a)$$

$$\tau(u_i , u_j , u_k) = \langle u_i \, u_j \, u_k \rangle - \langle u_i \rangle \, \tau \langle u_i , u_j \rangle - \langle u_j \rangle \, \tau \langle u_k , u_i \rangle$$
$$- \langle u_k \rangle \, \tau \langle u_i , u_j \rangle - \langle u_i \rangle \langle u_j \rangle \langle u_k \rangle \qquad (1.12b)$$

$$\tau(u_i , u_j , u_k , u_l) = \ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots \, . \qquad (1.12c)$$

Here the simplicity is regained, as we see will later. A trace of this idea can be found in the papers of Lilly (1966) [10] and Deardroff (1970) [11]. In this last paper, when the Reynolds rules of the mean are assumed valid in the case of the characteristic explicit filter Eqn. 1.4, he states: ‘However, this assumption is not separately necessary and may be incorporated into later assumption if $\langle u_i' \, u_j' \rangle$ is formally replaced by $\langle u_i \, u_j \rangle - \langle u_i \rangle \langle u_j \rangle$ whenever it appears.'

In the next section we will apply this formal replacement in a rigorous way, and the evolutionary equations for the generalized central moments will be deduced for a general linear filtering operator. We expect that the result will be very simple and surprising at the same time: the evolutionary equations of the generalized central moments are exactly the Reynolds equations, and the algebraic structure of the closure is the same for same for every linear commuting filter. We consider this the averaging invariance of the turbulent equations.

The averaging invariance of the turbulent equations in terms of the generalized central moments:

Let us consider the generic linear and constant preserving operator,

$$\langle f + g \rangle = \langle f \rangle + \langle g \rangle \qquad (1.13)$$

$$\langle \alpha f \rangle = \alpha \langle f \rangle \quad \text{if} \quad \alpha = \text{constant} \qquad (1.14)$$

having only the commuting properties with space and time derivatives

$$\langle f,_l \rangle = \langle f \rangle,_l \; ;$$

$$\langle f,_k \rangle = \langle f \rangle,_k \; ; \qquad (1.15)$$

If we now consider the Navier-Stokes equations for incompressible fluids:

$$\frac{\partial u_k}{\partial x_k} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i \, u_k) = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$\sigma_{ik} = 2\nu S_{ik} \; ;$$

Where $S_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$

Then

$$\sigma_{ik,k} = \frac{\partial}{\partial x_k}\{2\nu s_{ik}\}$$
$$= \frac{\partial}{\partial x_k}\left\{2\nu \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right\}$$
$$= \nu \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

So that, the continuity and momentum equation can be rewritten as follows:

$$u_{k,k} = 0 \qquad (1.16)$$

$$u_{i,t} + (u_i \, u_k),_k = -p_{,i} + \sigma_{ik,k} \qquad (1.17)$$

Taking a moment of the Eqn. 1.17 with u_j and adding this to another moment of the Eqn. 1.17 but with the indices interchanged, we derived the following equation:

$$(u_i u_j),_t + (u_i u_j u_k)_k = -\{p u_i \delta_{jk} + p u_j \delta_{ik} - \nu (u_i u_j),_k\},_k + 2p\delta_{ij} - 2\nu u_{i,k} \qquad (1.18)$$

$$\sigma_{ij} = 2\nu S_{ij} \; ;$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (1.19)$$

Now in terms of the moments $\langle u_i \, u_j \rangle, \langle u_i \, u_j \, u_k \rangle$...... the filters equations are the same for every filter. This averaging invariance can be extended directly and without recourse to the fluctuations to the generalized central moments that are the usual quantities modelled in the sub grid-scale or closure problem. Now using some algebraic operation we can derive the following equations,

$$\frac{\partial u_k}{\partial x_k} = 0 \Rightarrow u_{k,k} = 0 \qquad (1.20)$$

$$\langle u_i \rangle,_t + (\langle u_i \rangle \langle u_k \rangle),_k = -\langle p \rangle,_i + \langle \sigma_{ik} \rangle,_k - \{\tau(u_i, u_k)\},_k \qquad (1.21)$$

Now, taking the generalized central moments of the Eqn. 2.18 and using the Eqns. 1.12a & 1.12b, we can write the following equation:

$$\begin{aligned} &[\tau(u_i, u_j)]_t + [\tau(u_i, u_j)\langle u_k \rangle]_k \\ &= -\left\{\begin{array}{l} \tau(u_i, u_j, u_k) + \tau(p, u_i)\delta_{jk} \\ + \tau(p, u_j)\delta_{ik} - \nu[\tau(u_i, u_j)]_k \end{array}\right\},_k \\ &\quad + 2\tau(p, \delta_{ij}) - 2\nu\tau(u_{i,k}, u_{j,k}) \\ &- \tau(u_i, u_k)\langle u_i \rangle,_k - \tau(u_j, u_k)\langle u_i \rangle,_k \end{aligned} \qquad (1.22)$$

$$[\tau(u_i, u_j, u_k)],_t + [\tau(u_i, u_j, u_k)\langle u_l \rangle],_t = \ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.23)$$

Where the generalized central moments $\tau(f, g), \tau(f, g, h)$ are defined as in the Eqn. **1.12a** as follows:

$$\tau(f,g)=\langle fg\rangle-\langle f\rangle\langle g\rangle \quad (1.24a)$$

$$\tau(f,g,h)=\langle fgh\rangle-\langle f\rangle\tau\langle g,h\rangle- \langle g\rangle\tau\langle h,f\rangle-\langle h\rangle\tau\langle f,g\rangle-\langle f\rangle\langle g\rangle\langle h\rangle \quad (1.24b)$$

$$\tau(f,g,h,k)=..........\ .... \quad (1.24c)$$

Now a generalized equation for the turbulent energy can be derived using the Eqn. 1.22 as follows:

$$E_{T,t}+\left(E_T\langle u_k\rangle\right)_{,k}= -\left[\frac{1}{2}\tau(u_i,u_j,u_k)+\tau(p,u_k)-\nu E_{T,k}\right]_k -\nu\tau(u_{i,k},u_{i,k})-\tau(u_i,u_k)\langle S_{ik}\rangle \quad (1.25)$$

where E_T is the generalized turbulent energy given by

$$E_T=\frac{1}{2}\tau(u_i,u_j) \quad (1.26)$$

It is easy to see that the structure of the averaging equations in terms of the generalized central moments does not depend on the particular filter and is formally equal to the structure of the well known statistical central moments expressed in terms of the fluctuations and given by the relation:

$$\tau(f,g)=\langle f'g'\rangle \quad \tau(f,g,h)=\langle f'g'h'\rangle \quad (1.27)$$

3.1 An algebraic property of the generalized central moments

3.2 The resolved turbulence

The averaging invariance of the turbulent equations is the largely unexplored. Note that in some way the equations to the implicit or explicit filter actually applied in a single-level filtered representation. If the one-level filtered equations are independent of what real filter is applied, we can explore multi-level filtering procedures in order to generate improved sub-grid models usually the multi-level filtering procedures are based on special splitting operators, and we refer to the papers of Tehen (1973)[12] and Schiestel (1987)[13] on the matter. We notice that multi-level procedures have usually been produced in terms of a multiple decomposition of the velocity field u_i in ranks or in components $u_i^{(\alpha)}$ that when summed reproduced the original field.

$$u_i=\sum_{\alpha}u_i^{(\alpha)} \quad (1.28)$$

In this study we prefer to compare what happen at different levels and there will be no recourse to any kind of decomposition. In a sense this approach arises our grid intervals and compares the results at different levels of resolution.

We see that the main problem of large –eddy simulation is to (1.24a) model the generalized turbulent stress $\tau_f(u_i,u_j)$ related to the two velocity components u_i,u_j and defined:

(1.24b)

$$\tau_f(u_i,u_j)=\langle u_i u_j\rangle-\langle u_i\rangle_f\langle u_j\rangle_f \quad (1.29)$$

(1.24c)

Now if (F) is the particular implicit or explicit filter applied and $\langle u_i\rangle_f$ the F-level filtered values Let us now introduce another filter, explicit test filter G and let us denoted by $\langle u_i\rangle_{fg}$

$$\langle u_i\rangle_{fg}=\left\langle\langle u_i\rangle_f\right\rangle_g=\left\langle\langle u_i\rangle_g\right\rangle_f \quad (1.30)$$

Where $FG=GF$ filtered values

Considering the turbulent stress $\tau_{fg}(u_i,u_j)$ (1.26) at the FG level, we get

$$\tau_{fg}(u_i,u_j)=\langle u_i u_j\rangle_{fg}-\langle u_i\rangle_{fg}\langle u_j\rangle_{fg} \quad (1.31)$$

And by $\tau_g\left(\langle u_i\rangle_f,\langle u_j\rangle_f\right)$ the resolve turbulent stress extracted from the resolved scale F,

$$\tau_g\left(\langle u_i\rangle_f,\langle u_j\rangle_f\right)=\left\langle\langle u_i\rangle_f\langle u_j\rangle_f\right\rangle_g-\langle u_i\rangle_{fg}\langle u_j\rangle_{fg} \quad (1.32)$$

The following algebraic relation holds:

$$\tau_{fg}(u_i,u_j)=\left\langle\tau_f(u_i,u_j)\right\rangle_g+\tau_g\left\langle\langle u_i\rangle_f,\langle u_j\rangle_f\right\rangle \quad (1.33)$$

The physical meaning of this algebraic relation is the turbulent stress at the FG-level is equal to the G-averaged value of the turbulent stress at the F-level plus the resolved turbulent stress $\tau_g\left(\langle u_i\rangle_f,\langle u_j\rangle_f\right)$ extracted from the resolved scale F.

Similarly we can extract from the resolved scale the resolved turbulent energy or the resolved production or the resolved dissipation or anything that we would like to test.

(1.28)

The algebraic relation of the Eqn. (1.33) applies locally in space and time, so that the resolved turbulence is composed of fluctuating terms. In the applications we will see the utility of this algebraic relation in an ensemble form.

If we denote an ensemble average with an over line, we can also write:

$$\overline{\tau_{fg}(u_i,u_j)}=\overline{\left\langle\tau_f(u_i,u_j)\right\rangle}_g+\tau_g\overline{\left(\langle u_i\rangle_f,\langle u_j\rangle_f\right)} \quad (1.34)$$

To apply the Eqn. (1.33) to the case in which the test filter G is the ensemble average $E.$

We obtain:

$$\tau_{fe}(u_i, u_j) = \overline{\tau_f(u_i, u_j)} + \tau(\langle u_i \rangle_f, \langle u_j \rangle_f) \quad (1.35)$$

and in the particular case in which $EF = E.$ we have

$$\tau_e(u_i, u_j) = \overline{\tau_e(u_i, u_j)} + \tau_e(\langle u_i \rangle_f, \langle u_j \rangle_f) \quad (1.36)$$

Where $\tau_e(u_i, u_j)$ represents the usual Reynolds stress.

$$\tau_e(u_i, u_j) = \overline{u_i u_j} - \overline{u_i}\, \overline{u}_j \quad (1.37)$$

The Eqn. (1.36) can be interpreted as follows: The Reynolds stress is equal to the ensemble value of the turbulent stress at the $F - level$ plus the resolved turbulent stress $\tau_e(\langle u_i \rangle_f, \langle u_j \rangle_f)$ can be explicitly calculated in a large-eddy simulation and in the following the possible use of this algebraic property in multilevel sub-grid modelling is discussed.

3.3 Derivation of LES equation from Navier-Stokes equations

3.3.1 Filtered Navier-Stokes Equations

This chapter describes the equations of Large-Eddy simulation such as they are obtained by applying a homogeneous filter verifying the properties of linearity, conservation of constant and commutation with derivation to the Navier-Stokes equations. These are equations that will be resolved in the numerical simulation.

The Navier-Stokes equations can be written in vector form with body force as follows:

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{F} - \bar{\nabla} \rho + \mu \nabla^2 \bar{u} + \frac{\mu}{3} \nabla (\bar{\nabla} . \bar{u})$$

$$\Rightarrow \rho \left[\frac{\partial \bar{u}}{\partial t} + (\vec{u} . \vec{\nabla}) \vec{u} \right] = \rho \vec{F} - \vec{\nabla} p + \mu \nabla^2 \vec{u} + \frac{\mu}{3} \vec{\nabla} (\vec{\nabla} . \vec{u}) \quad (1.38)$$

Again we know from the vector properties.

$$(\vec{u} . \vec{\nabla}) \vec{u} = \vec{\nabla} \left(\frac{1}{2} u^2 \right) - \vec{u} \times (\vec{\nabla} \times \vec{u})$$

And $\nabla^2 \vec{u} = \vec{\nabla} (\vec{\nabla} . \vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u})$

Using these properties, the Eqn. (1.38) yields:

$$\Rightarrow \rho \left[\frac{\partial \bar{u}}{\partial t} + \vec{\nabla} \left(\frac{1}{2} u^2 \right) - \vec{u} \times (\vec{\nabla} \times \vec{u}) \right] = \rho \vec{F} - \vec{\nabla} p + \mu \left[\vec{\nabla} (\vec{\nabla} . \vec{u}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) \right] + \frac{\mu}{3} \vec{\nabla} (\vec{\nabla} . \vec{u}) \quad (1.39)$$

For incompressible fluid, $\vec{\nabla} . \vec{u} = 0$ So the Eqn. (1.39) yields:

$$\Rightarrow \rho \left[\frac{\partial \bar{u}}{\partial t} + \vec{\nabla} \left(\frac{1}{2} u^2 \right) - \vec{u} \times (\vec{\nabla} \times \vec{u}) \right] = \rho \vec{F} - \vec{\nabla} p - \mu \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) \quad (1.40)$$

Again, using the relation

$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla} (\vec{\nabla} . \vec{u}) - \nabla^2 \vec{u}$ in Eqn. (1.40) we obtain the following equations:

$$\rho \left[\frac{\partial \bar{u}}{\partial t} + \vec{\nabla} \left(\frac{1}{2} u^2 \right) - \vec{u} \times (\vec{\nabla} \times \vec{u}) \right] = \rho \vec{F} - \vec{\nabla} p - \mu \left[\vec{\nabla} (\vec{\nabla} . \vec{u}) - \nabla^2 \vec{u} \right] \quad (1.41)$$

Therefore, in the physical space, for in compressible fluid, the velocity field $u = (u_1, u_2, u_3)$ expressed in a reference Cartesian coordinate system $X = (x_1, x_2, x_3)$ is a solution of the system comprising the momentum and continuity equations having no body force given as follows:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \right), \; i = 1,2,3 \quad (1.42)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (1.43)$$

in which $p = \frac{p}{\rho}$ *and* ν are the static pressure and the assumedly constant, uniform kinematics viscosity, respectively, and there is no body force.

3.3.2 Derivation of LES equation from Navier-Stokes

In light of the commutation with derivation property, the application of a filter to Eqn. (1.42) and (1.43) are expressed as follows:

$$\frac{\partial \vec{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}(\overline{u_i u_j}) = -\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial}{\partial x_j} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad (1.44)$$

$$\frac{\partial \overline{u_i}}{\partial x_j} = 0 \quad (1.45)$$

Where $\bar{p}$ is the filtered pressure. That is the Eqn. (1.44) and (1.45) are considered as the filtered Navier-Stokes equations.

In LES, the filtered or grid scale velocity is the convolution integral given as follows:

$$\bar{u}\,(x)=\langle u_i(x,t)\rangle_{t,\theta}=\int u_i(x',t')\xi(x-x',t-t';l,\theta)d^3x'dt' \quad (1.46)$$

where $\xi(x-x',t-t';l,\theta)$ is filter kernel function. $l\ and\ \theta$ are characteristic filter length and characteristic filter time in i-direction.

The filter momentum equation brings out the non-linear term $\overline{u_i u_j}$ which will have to be expressed as a function of $\bar{u}\ and\ u'$, which are now the only unknowns left in the problem and where $u'=u-\bar{u}$. Here $\bar{u}$ is the Grid-Scale (GS) (or large scale) component and u' is the Sub-Grid Scale (SGS) (or small-scale) component.

Again the non-linear term $\overline{u_i u_j}$ can be expressed as,

$$\begin{aligned}\overline{u_i u_j} &= \overline{(\overline{u_i}+u_i')(\overline{u_j}+u_j')} \\ &= \overline{\bar{u}_i\ \bar{u}_j}+\overline{\bar{u}\ u_j'}+\overline{\bar{u}_j\ u_i'}+\overline{u' u_j'},\end{aligned}$$

$$\begin{aligned}&where\ \bar{\bar{u}}=\bar{u}\ ,\overline{u'}=0,\ \overline{\bar{u}\ u}=\bar{u}\bar{u} \\ &=\overline{\bar{u}_i\ \bar{u}}_j+0+0+0 \\ &=\overline{\bar{u}_i\ \bar{u}}_j\end{aligned}$$

Again the Eqn. (1.42) can be written as,

$$\begin{aligned}&\frac{\partial \bar{u}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\overline{\bar{u}_i\,\bar{u}_j}\right)=-\frac{\partial \bar{p}}{\partial x_i} \\ &+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\left(\overline{u_i\,u_j}-\overline{\bar{u}_i\ \bar{u}_j}\right) \\ &=-\frac{\partial \bar{p}}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\frac{\partial \tau_{ij}}{\partial x_j}\end{aligned} \quad (1.47)$$

where $\tau_{ij}=\overline{u_i\ u_j}-\overline{\bar{u}_i\,\bar{u}_j}$ is the sub-grid tensor. The Eqn. (1.47) with continuity equation (1.45) are known as the governing equations for LES.

$$\begin{aligned}\tau_{ij}&=\overline{u_i\ u_j}-\overline{\bar{u}_i\ \bar{u}}_j \\ &=\overline{u_i\ u_j}-\bar{u}_i\,\bar{u}_j\ \sin ce\ \overline{\bar{u}\,u}=\bar{u}\,\bar{u}\end{aligned}$$

$$\begin{aligned}&\text{This sub-grid tensor}=\overline{(\bar{u}_i+u_i')(\bar{u}_j+u_j')}-\bar{u}_i\,\bar{u}_j \\ &=\left(\overline{\bar{u}_i\bar{u}_j}-\bar{u}_i\bar{u}_j\right)+\left(\overline{\bar{u}_j u_i'}+\overline{\bar{u}_i\,u_j'}\right)+\overline{u_i'\ u_j'} \\ &=L_{ij}+C_{ij}+R_{ij}\end{aligned}$$

The first one is Leonard Stress tensor L_{ij}, the second is Cross Stress tensor C_{ij} and third is Reynolds Stress R_{ij}. Leonard Stress is generated from interactions of GS vortices, which can be solved explicitly by filtered GS field. The cross term is generated from the interactions of between GS and SGS. Reynolds stress is introduced from the interactions between SGS components.

For $u_i=\bar{u}_i+u_i'$ and $p=\bar{p}+p'$ the Eqn. (1.42) can be written as:

$$\begin{aligned}&\frac{\partial}{\partial t}(\bar{u}_i+u_i')+\frac{\partial}{\partial x_j}\left((\bar{u}_i+u_i')(\bar{u}_j+u_j')\right)=-\frac{\partial}{\partial x_i}(\bar{p}+p') \\ &+\nu\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}(\bar{u}_i+u_i')+\frac{\partial}{\partial x_i}(\bar{u}_j+u_j')\right]\end{aligned} \quad (1.48)$$

The momentum equation for the small scales is obtained by subtracting the large scale Eqn. (1.47) from the unfiltered momentum Eqn. (1.48), we can obtained the following equation:

$$\begin{aligned}&\frac{\partial u_i'}{\partial t}+\frac{\partial}{\partial x_j}\left((\bar{u}_i+u_i')(\bar{u}_j+u_j')-\overline{\bar{u}_i\,\bar{u}_j}-\tau_{ij}\right)=-\frac{\partial p'}{\partial x_i} \\ &+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right) \quad (1.49) \\ &\Rightarrow\frac{\partial u_i'}{\partial t}+\frac{\partial}{\partial x_j}\left((\bar{u}_i\,\bar{u}_j+u_i'\bar{u}_j+\bar{u}_i\bar{u}_j+u_i'u_j')-\bar{u}_i u_j-\left(\overline{u_i\,u_j}-\bar{u}_i\bar{u}_j\right)\right) \\ &\qquad=-\frac{\partial p'}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right) \\ &\Rightarrow\frac{\partial \bar{u}_i}{\partial t}+\frac{\partial u_i'u_j'}{\partial x_j}=-\frac{\partial p'}{\partial x_i}+\nu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right) \\ &-\frac{\partial}{\partial x_j}\left(u_i'\bar{u}_j+\bar{u}_i\,u_j'\right)+\frac{\partial}{\partial x_j}\left(\overline{u_i\,u_j}-\bar{u}_i\bar{u}_j\right)\end{aligned}$$

which is the transport equation of SGS components u_i'.

4. CONCLUSIONS

The Large Eddy Simulation in turbulent flow is based on the separation between large and small scales in which the large-scale motion is exactly calculated and the effects of small scales motions are modeled. In practice we need to filter the turbulent motion using some filtered approach, and it is shown that the explicit or implicit filter representations like spectral cut-offs or numerical discretizations are commonly used in LES for the separation of flows. In this study we have elaborately studied on this filtering approach and analyzed some general algebraic properties of the filtered representations. It is shown that the averaged equations are the same in terms of the generalized central moments, and then we have defined the resolved turbulence using these average properties. The algebraic consistency rules related with the resolved quantities to the turbulent stresses are derived and their possible use in subgrid-scale modeling is examined.

The algebraic relation, $\tau_{fg}(u_i,u_j)=\langle\tau_f\,(u_i,u_j)\rangle_g+\tau_g\left(\langle u_i\rangle_f,\langle u_j\rangle_f\right)$ given in equation (1.29) should be interpreted as a general condition that, in some way. A multi-level filtering procedure must satisfy. With this perspective, different multi-level filtering techniques could be suggested for different subgrid models. In this study, we have also discussed about the standard Smagorinsky model for LES and then we derived an expression to determine the Smagorinsky constant dynamically, which suppose to be assured the consistency between the filter and the subgrid-scale model. The consistent decomposition of the generalized central moments in turbulence has also been discussed elaborately.

Finally, we have successfully derived the governing equations for LES by applying the filtering approach to the Navier-Stokes equations.

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