Hyers-Ulam-Rassias Stability of Orthogonal Pexiderized Quadratic Functional Equation

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ABSTRACT
The Hyers-Ulam-Rassias stability of the conditional quadratic functional equation of Pexider type $f(x+y)+f(x-y)=2g(x)-2h(y)$ is established where $\perp$ is a symmetric orthogonality in the sense of Rätz.

Keywords
Hyers-Ulam-Rassias stability, Orthogonal spaces, Pexiderized Quadratic functional equations.

1. INTRODUCTION
S.M. Ulam [24] is the pioneer of the stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of the University of Wisconsin, he discussed a number of unsolved problems. Among them was the following question concerning the stability of homomorphisms:

Let G be a group and H be a metric group with metric d(. , .). Given $\varepsilon > 0$, does there exist a $\theta > 0$ such that if a function $g:G \to H$ satisfies $d(g(xy), g(x)g(y)) < \theta$ for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with $d(g(x), a(x)) < \varepsilon$ for all $x \in G$? In 1941, D. H. Hyers [11] was the first mathematician to present the result concerning the stability of functional equations on Banach spaces. This result of Hyers [11] is stated as follows:

Let $f : X \to Y$ satisfies $\| f(x+y) - f(x) - f(y) \| \le \varepsilon$ for all $x, y \in X$ and $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that $\| f(x) - T(x) \| \le \varepsilon$, for all $x \in X$. The generalized version of D. H. Hyers [11] result was given by famous Greece mathematician Th. M. Rassias [18] in 1978, where $f : X \to Y$ satisfies the inequality $\| f(x+y) - f(x) - f(y) \| \le \delta(\| x \|^p + \| y \|^p)$

for all $x, y \in X$ and $\delta \ge 0$ and $0 \le p < 1$. The stability paper [19] given by Th. M. Rassias has significantly influenced in the development of stability of functional equations and hence named as Hyers-Ulam-Rassias stability of functional equations.

Let us denote the the sets of real, nonnegative real numbers and positive integers by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$, respectively.

Suppose that X is a real vector space with $\dim X \ge 2$ and $\perp$ is a binary relation on with the following properties:

(O1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;

(O2) independence: if $x, y \in X - \{0\}$, $x \perp y$ then x, y are linearly independent;

(O3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: Let P is a 2-dimensional subspace of X. If $x \in P$ and $\lambda \in \mathbb{R}^+$, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space equipped with a norm.

Some examples of special interest are (i) The trivial orthogonality on a vector space X defined by (O1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \| . \|)$ defined by $x \perp y$ if and only if $\| \lambda x + y \| \ge \| x \|$ for all $\lambda \in \mathbb{R}^+$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. However, it is remarkable to note, that a real normed space of dimension greater than or equal to 3 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

Let X be a vector space (an orthogonality space) and (Y,+) be an abelian group. A mapping $f : X \to Y$ is called (orthogonally) additive if it satisfies the so-called (orthogonal) additive functional equation $f(x+y) = f(y) + f(x)$ for all $x, y \in X$ (with $x \perp y$). A mapping $f : X \to Y$ is said to be (orthogonally) quadratic if it satisfies the so-called (orthogonally) Jordan-von-Neumann quadratic function equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all $x, y \in X$ (with $x \perp y$). For example, a function $f: X \to Y$ between real vector spaces is quadratic if and only if there exists a (unique) symmetric bi-additive mapping $B: X \times X \to Y$ such that $f(x)=B(x,x)$ for all $x \in X$. In fact, $B(x,x)=\frac{1}{4}(f(x+y)-f(x-y))$, cf. [2]. In the recent decades, stability of functional equations have been investigated by many mathematicians. They have so many applications in Information Theory, Economic Theory and Social and Behaviour Sciences; cf. [1]. The first author treating the stability of the quadratic equation was F. Skof [22] by proving that if f is a maping from a normed space X into a Banach space Y satisfying $\| f(x+y)+f(x-y)-2f(x)-2f(y) \| \le \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic function $g: X \to Y$ such that $\| f(x)-g(x) \| \le \frac{\varepsilon}{2}$. P. W. Cholewa [3] extended Skofs theorem by replacing X by an abelian group G. Skof's result was later generalized by S. Czerwik [4] in the spirit of Hyers–Ulam–Rassias. K.W. Jun and Y. H. Lee [13] proved the stability of quadratic equation of Pexider type. The stability problem of the quadratic equation has been extensively investigated by some mathematicians [17], [5], [6].

The orthogonal quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y),\ x \perp y$$

was first investigated by F. Vajzović [25] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later H. Drljević [7], M. Fochi [9] and G. Szabó [23] generalized this result.

One of the significant conditional equations is the so-called orthogonally quadratic functional equation of Pexider type

$$f(x+y)+f(x-y)=2g(x)+2h(y),\ x \perp y \quad (1).$$

M. S. Moslehian [16] proved the Hyers-Ulam stability of the Pexiderized Quadratic Equations (1) in orthogonality spaces.

Throughout the paper, we denote $(X, \perp)$ an orthogonality normed space and $(Y, \|.\|)$ is a real Banach space. In order to avoid some definitional problems, we also assume for the sake of this paper that $0^0 = 1$.

2. MAIN RESULTS

In this section we show the Hyers-Ulam-Rassias stability of the orthogonally Pexiderized Quadratic functional Equations (1) in orthogonality spaces.

Lemma 2.1 (see [16]) If $a: X \to Y$ satisfying

$$a(x+y)+a(x-y)=2a(x)$$

for all $x, y \in X$ with $x \perp y$ and $\perp$ is symmetric, then $a(x)-a(0)$ is orthogonally additive.

Theorem 2.2 Let $f, g, h: X \to Y$ are mappings satisfying

$$\| f(x+y)+f(x-y)-2g(x)-2h(y) \| \le \varepsilon(\|x\|^p + \|y\|^p) \quad (2)$$

for some $\varepsilon \ge 0$, $p \in R^+ \setminus [1,2]$ and for all $x, y \in X$ with $x \perp y$. Suppose that $\perp$ is symmetric on X. If f is odd then there exists a unique additive mapping $T: X \to Y$ and a unique quadratic mapping $Q: X \to Y$ such that

$$\| f(x)-T(x)-Q(x) \| \le \frac{\varepsilon\alpha\, \mathrm{sgn}(p-1)}{2^{p-1}-1} \|x\|^p$$

$$\| g(x)-g(0)-T(x)-Q(x) \| \le \left(\frac{\varepsilon}{2}+\frac{\varepsilon\alpha\, \mathrm{sgn}(p-1)}{2^{p-1}-1}\right) \|x\|^p$$

for all $x \in X$, where

$$\alpha := 1+3^p+2^{p+1}.$$

Proof. Define $F(x)=f(x)-f(0)=f(x)$, $G(x)=g(x)-g(0)$, and $H(x)=h(x)-h(0)$. Then $F(0)=G(0)=H(0)=0$

Use (O1) and put $x=y=0$ in (2) with $p \in R^+ \setminus [1,2]$ and subtract the argument of the norm of the resulting inequality. We get

$$\| F(x+y)+F(x-y)-2G(x)-2H(y) \| \le \varepsilon(\|x\|^p + \|y\|^p) \quad (3)$$

for all $x, y \in X$.

Put $x=0$ in (3). We can do this because of (O1). Then

$$\| 2H(y) \| \le \varepsilon \|y\|^p \quad (4)$$

for all $y \in X$. Similarly, by putting $y=0$ in (3) we get

$$\| 2F(x)-2G(x) \| \le \varepsilon \|x\|^p \quad (5)$$

for all $x \in X$.

Whence

$$\begin{aligned} \| F(x+y)+F(x-y)-2F(x) \| &\le \| F(x+y)+F(x-y) \\ &-2G(x)-2H(y) \| + \| 2F(x)-2G(x) \| + \| 2H(y) \| \\ &\le 2\varepsilon(\|x\|^p + \|y\|^p) \end{aligned} \quad (6)$$

for all $x, y \in X$ with $x \perp y$.

Fix $x \in X$. By (O4), there exists $y_0 \in X$ such that $x \perp y_0$ and $x+y_0 \perp x-y_0$. Since $\perp$ is symmetric, $x-y_0 \perp x+y_0$ too. Using inequality (6) and the oddness of we get

$$|| F(x+y_0)+F(x-y_0)-2F(x)|| \le 2\varepsilon(||x||^p + ||y||^p)$$
$$|| F(2x)+F(2y_0)-2F(x+y_0)|| \le 2\varepsilon(||x+y_0||^p + ||x-y_0||^p)$$
$$|| F(2x)-F(2y_0)-2F(x-y_0)|| \le 2\varepsilon(||x+y_0||^p + ||x-y_0||^p)$$

Then

$$|| F(2x)-2F(x)|| \le || F(x+y_0)+F(x-y_0)-2F(x)|| + \frac{1}{2}|| F(2x)+F(2y_0)-2F(x+y_0)|| + \frac{1}{2}|| F(2x)-F(2y_0) -2F(x-y_0)|| \le 2\varepsilon\{||x||^p + ||y_0||^p + ||x+y_0||^p + ||x-y_0||^p\}$$

From the definition of the orthogonality, since $x \perp y_0$, we derive $||x|| \le ||x+y_0||$ and $||x|| \le ||x-y_0||$ (for $\lambda = 1$ and $\lambda = -1$, respectively), and, analogously, from $x+y_0 \perp x-y_0$ and $x-y_0 \perp x+y_0$ we derive $||x+y_0|| \le ||2x||$ and $||x-y_0|| \le ||2x||$. From these relations and the triangle inequality we have

$$||y_0|| \le ||x-y_0|| + ||x|| \le 3||x||$$

As p is a nonnegative real number, we have the approximations

$$||y_0||^p \le 3^p ||x||^p$$
$$||x+y_0||^p \le 2^p ||x||^p, \; ||x-y_0||^p \le 2^p ||x||^p$$

Then we obtain

$$|| F(2x)-2F(x)|| \le 2\varepsilon\alpha ||x||^p \tag{7}$$

with

$$\alpha := 1+3^p+2^{p+1}.$$

Assume first that $0 \le p < 1$. Then from (7) we have

$$|| \frac{F(2x)}{2} - F(x)|| \le \varepsilon\alpha ||x||^p \tag{8}$$

for all $x \in X$.

Replacing x by $2^k x$ $(k \in N)$ in (8) we get

$$|| \frac{F(2^{k+1}x)}{2} - F(2^k x)|| \le \varepsilon\alpha ||2^k x||^p \tag{9}$$

Then

$$|| \frac{F(2^{k+1}x)}{2^{k+1}} - \frac{F(2^k x)}{2^k}|| \le 2^{k(p-1)}\varepsilon\alpha ||x||^p \tag{10}$$

Whence

$$|| \sum_{k=0}^{n-1}\left\{\frac{F(2^{k+1}x)}{2^{k+1}} - \frac{F(2^k x)}{2^k}\right\}|| \le \sum_{k=0}^{n-1} || \frac{F(2^{k+1}x)}{2^{k+1}} - \frac{F(2^k x)}{2^k}|| \le \varepsilon\alpha ||x||^p \sum_{k=0}^{n-1} 2^{k(p-1)} \tag{11}$$

It follow from (11) that

$$|| \frac{F(2^n x)}{2^n} - F(x)|| \le \varepsilon\alpha(\frac{1-2^{n(p-1)}}{1-2^{p-1}})||x||^p \tag{12}$$

Replacing x by $2^m x$ $(m \in N)$ in (12) we obtain

$$|| \frac{F(2^{n+m}x)}{2^n} - F(2^m x)|| \le \varepsilon\alpha 2^{mp}(\frac{1-2^{n(p-1)}}{1-2^{p-1}})||x||^p \tag{13}$$

Then

$$|| \frac{F(2^{n+m}x)}{2^{n+m}} - \frac{F(2^m x)}{2^m}|| \le 2^{m(p-1)}\varepsilon\alpha(\frac{1-2^{n(p-1)}}{1-2^{p-1}})||x||^p \tag{14}$$

As $\lim_{m,n \to +\infty}\left\{2^{m(p-1)}\varepsilon\alpha(\frac{1-2^{n(p-1)}}{1-2^{p-1}})||x||^p\right\} = 0$

Then $\left\{\frac{F(2^n x)}{2^n}\right\}$ is Cauchy sequence in Banach space $(Y, ||.||)$. Hence $\lim_{n\to+\infty}\frac{F(2^n x)}{2^n}$ exists and we well defines the odd mapping $a(x) = \lim_{n\to+\infty}\frac{F(2^n x)}{2^n}$ from X into Y satisfying

$$||a(x)-F(x)|| \le \frac{\varepsilon\alpha}{1-2^{p-1}}||x||^p \tag{15}$$

for all $x \in X$.

For all $x, y \in X$ with $x \perp y$, by applying inequality (6) and (O3) we obtain

$$\begin{aligned} &|| 2^{-n}F(2^n(x+y)) + 2^{-n}F(2^n(x-y)) - 2^{-n+1}F(2^n x)|| \\ &\le 2^{-n+1}\varepsilon(||2^n x||^p + ||2^n y||^p) \\ &= 2\varepsilon 2^{n(p-1)}(||x||^p + ||y||^p) \end{aligned} \tag{16}$$

If $n \to +\infty$ then we deduce that

$$a(x+y)+a(x-y)-2a(x) = 0$$

for all $x, y \in X$ with $x \perp y$ Moreover $a(0) = \lim_{n\to+\infty} 2^{-n}F(2^n.0) = 0$. Using Lemma 2.1 we conclude that a is an orthogonally additive mapping.

In the case $p > 2$ we start from the inequality

$$||F(x)-2F(\frac{x}{2})|| \le 2^{1-p}\varepsilon\alpha ||x||^p \tag{17}$$

where $\alpha = 1+3^p+2^{p+1}$.

Replacing x by $2^{-k}x$ $(k \in N)$ in (17) we get

$$\|F(\frac{x}{2^k}) - 2F(\frac{x}{2^{k+1}})\| \le \frac{2\varepsilon\alpha}{2^p} 2^{-kp} \|x\|^p, \quad (18)$$

then

$$\|2^k F(\frac{x}{2^k}) - 2^{k+1} F(\frac{x}{2^{k+1}})\| \le \frac{2\varepsilon\alpha}{2^p} 2^{k(1-p)} \|x\|^p. \quad (19)$$

Whence

$$\|\sum_{k=0}^{n-1}\left\{2^k F(\frac{x}{2^k}) - 2^{k+1} F(\frac{x}{2^{k+1}})\right\}\| \le \sum_{k=0}^{n-1} \|2^k F(\frac{x}{2^k}) - 2^{k+1} F(\frac{x}{2^{k+1}})\| \le \frac{2\varepsilon\alpha}{2^p} \|x\|^p \sum_{k=0}^{n-1} 2^{k(1-p)} \quad (20)$$

It follow from (20) that

$$\|F(x) - 2^n F(\frac{x}{2^n})\| \le \frac{\varepsilon\alpha}{2^{p-1}-1}(1 - \frac{1}{2^{n(p-1)}})\|x\|^p \quad (21)$$

Same as the first case ($0 \le p < 1$) we find, for each $x \in X$ the sequence $\{2^n F(2^{-n}x)\}$ is Cauchy sequence in Banach space $(Y, \|.\|)$. Hence $\lim_{n\to+\infty} 2^n F(2^{-n}x)$ exists and we well define the odd mapping $a(x) = \lim_{n\to+\infty} 2^n F(2^{-n}x)$ from X into Y satisfying

$$\|a(x) - F(x)\| \le \frac{\varepsilon\alpha}{2^{p-1}-1}\|x\|^p \quad (22)$$

for all $x \in X$, also we deduce that

$$a(x+y) + a(x-y) - 2a(x) = 0,\ a(0) = 0$$

for all $x, y \in X$ with $x \perp y$.

By the first case and the second case we obtain

$$\|a(x) - F(x)\| \le \frac{\varepsilon\alpha\, \mathrm{sgn}(p-1)}{2^{p-1}-1}\|x\|^p \quad (23)$$

where

$$a(x+y) + a(x-y) - 2a(x) = 0,\ a(0) = 0$$

for all $x, y \in X$ with $x \perp y$. and a is an orthogonally additive mapping . By corollary 7 of [20], a has the form $T+Q$ with T additive and Q quadratic. If there are another Q' quadratic mapping and another additive mapping T' satisfying the required in our theorem and $a' = T' + Q'$.

Then

$$\|a(x) - a'(x)\| \le \|a(x) - F(x)\| + \|F(x) - a'(x)\| \le \frac{2\varepsilon\alpha\, \mathrm{sgn}(p-1)}{2^{p-1}-1}\|x\|^p \quad (24)$$

for all $x \in X$ and $p \in R^+ \setminus [1,2]$. Using the fact that additive mapping are odd and quadratic mapping are even we obtain

$$\begin{aligned}\|T(x) - T'(x)\| &= \frac{1}{2}\|T(x) + Q(x) - T'(x) - Q'(x) + T(x) - Q(x) - T'(x) + Q'(x)\| \\ &\le \frac{1}{2}\|T(x) + Q(x) - (T'(x) + Q'(x))\| + \frac{1}{2}\|T(x) - Q(x) - (T'(x) - Q'(x))\| \\ &\le \frac{1}{2}\|a(x) - a'(x)\| + \frac{1}{2}\|a(-x) - a'(-x)\| \\ &\le \frac{2\varepsilon\alpha\, \mathrm{sgn}(p-1)}{2^{p-1}-1}\|x\|^p \end{aligned} \quad (25)$$

for all $x \in X$ and $p \in R^+ \setminus [1,2]$.

Similarly

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \frac{1}{2}\|T(x) + Q(x) - T'(x) - Q'(x) - T(x) + Q(x) + T'(x) - Q'(x)\| \\ &\le \frac{1}{2}\|a(x) - a'(x)\| + \frac{1}{2}\|a(-x) - a'(-x)\| \\ &\le \frac{2\varepsilon\alpha\, \mathrm{sgn}(p-1)}{2^{p-1}-1}\|x\|^p \end{aligned} \quad (26)$$

for all $x \in X$ and $p \in R^+ \setminus [1,2]$.

If $0 \le p < 1$ we have

$$\|T(x) - T'(x)\| = \frac{1}{n}\|T(nx) - T'(nx))\| \le \frac{2\varepsilon\alpha}{1-2^{p-1}} n^{p-1}\|x\|^p \quad (27)$$

and

$$\|Q(x) - Q'(x)\| = \frac{1}{n^2}\|Q(nx) - Q'(nx)\| \le \frac{2\varepsilon\alpha}{1-2^{p-1}} n^{p-2}\|x\|^p \quad (28)$$

for all $x \in X$.

Tending $n \to +\infty$ we infer that $T = T'$ and $Q = Q'$.

Then $a = a'$.

Otherwise if $p > 2$ we have

$$\|T(x) - T'(x)\| = n\|T(\frac{1}{n}x) - T'(\frac{1}{n}x))\| \le \frac{2\varepsilon\alpha}{2^{p-1}-1} n^{-p+1}\|x\|^p \quad (29)$$

and

$$\| Q(x) - Q'(x) \| = n^2 \| Q(\frac{1}{n}x) - Q'(\frac{1}{n}x)) \| \le \frac{2\varepsilon\alpha}{2^{p-1} - 1} n^{-p+2} \| x \|^p \quad (30)$$

for all $x \in X$. Taking the limit, we conclude that $T = T'$ and $Q = Q'$.

Then $a = a'$.

Using (5) and (23) we infer that for all $x \in X$

$$\| G(x) - a(x) \| = \| G(x) - F(x) \| + \| F(x) - a(x) \| \le (\frac{\varepsilon}{2} + \frac{\varepsilon\alpha \operatorname{sgn}(p-1)}{2^{p-1} - 1}) \| x \|^p \quad (31)$$

Finally we get

$$\| f(x) - T(x) - Q(x) \| \le \frac{\varepsilon\alpha \operatorname{sgn}(p-1)}{2^{p-1} - 1} \| x \|^p$$

and

$$\| g(x) - g(0)) - T(x) - Q(x) \| \le (\frac{\varepsilon}{2} + \frac{\varepsilon\alpha \operatorname{sgn}(p-1)}{2^{p-1} - 1}) \| x \|^p$$

for all $x \in X$. This completes the proof of theorem.

Remark 2.3 (i) If $G = k.F$ for some number $k \neq 1$, then

Inequality (5) implies that $|1 - k| \| F(x) \| \le \frac{\varepsilon}{2} \| x \|^p$

for all $x \in X$ and $p \in R^+ \setminus [1,2]$. Hence

$$|1 - k| \| 2^{-n} F(2^n x) \| \le \frac{\varepsilon}{2} 2^{n(p-1)} \| x \|^p ;\ 0 \le p < 1$$

and $|1 - k| \| 2^n F(2^{-n} x) \| \le \frac{\varepsilon}{2} 2^{n(1-p)} \| x \|^p ;\ p > 2$

so $a(x) = \begin{cases} \lim_{n\to+\infty} 2^{-n} f(2^n x) = 0 \text{ if } 0 \le p < 1 \\ \lim_{n\to+\infty} 2^n f(2^{-n} x) = 0 \text{ if } p > 2 \end{cases}$

for all $x \in X$.

(ii) Similarly, if $H = k.F$ for some number $k \neq 1$, then it follows from (4) that $a(x) = 0$.

Corollary 2.4 Let $f, g, h : X \to Y$ are mappings satisfying

$$\| f(x+y) + f(x-y) - 2g(x) - 2h(y) \| \le \varepsilon$$

for some $\varepsilon \ge 0$ and for all $x, y \in X$ with $x \perp y$. Suppose that $\perp$ is symmetric on X. If f is odd then there exists a unique additive mapping $T : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - T(x) - Q(x) \| \le 4\varepsilon$$

$$\| g(x) - g(0) - T(x) - Q(x) \| \le \frac{17\varepsilon}{4}$$

for all $x \in X$.

Corollary 2.5 Let $f, h : X \to Y$ are mappings satisfying

$$\| f(x+y) + f(x-y) - 2f(x) - 2h(y) \| \le \varepsilon(\| x \|^p + \| y \|^p)$$

for some $\varepsilon \ge 0$, $p \in R^+ \setminus [1,2]$ and for all $x, y \in X$ with $x \perp y$. Suppose that $\perp$ is symmetric on X. If f is odd then there exists a unique additive mapping $T : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - T(x) - Q(x) \| \le \frac{\varepsilon\alpha \operatorname{sgn}(p-1)}{2^{p-1} - 1} \| x \|^p$$

for all $x \in X$, where

$$\alpha := 1 + 3^p + 2^{p+1}.$$

Theorem 2.6 Let $f, g, h : X \to Y$ are mappings satisfying

$$\| f(x+y) + f(x-y) - 2g(x) - 2h(y) \| \le \varepsilon(\| x \|^p + \| y \|^p) \quad (32)$$

for some $\varepsilon \ge 0$, $p \in R^+ \setminus \{2\}$ and for all $x, y \in X$ with $x \perp y$. If f is even mapping then there exists a unique quadratic mapping $q : X \to Y$ such that

$$\| f(x) - f(0) - q(x) \| \le \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2} \| x \|^p$$

$$\| g(x) - g(0) - q(x) \| \le (\frac{\varepsilon}{2} + \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}) \| x \|^p$$

and

$$\| h(x) - h(0) - q(x) \| \le (\frac{\varepsilon}{2} + \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}) \| x \|^p$$

for all $x \in X$.

Proof. Define $F(x) = f(x) - f(0)$, $G(x) = g(x) - g(0)$, and $H(x) = h(x) - h(0)$. Then $F(0) = G(0) = H(0) = 0$ and F is even mapping.

Use (O1) and put $x = y = 0$ in (32) with $p \in R^+ \setminus \{2\}$ and subtract the argument of the norm of the resulting inequality from that of inequality (32) to get

$$\| F(x+y) + F(x-y) - 2G(x) - 2H(y) \| \le \varepsilon(\| x \|^p + \| y \|^p) \quad (33)$$

for all $x, y \in X$ with $x \perp y$.

Put $x = 0$ in (33), we can do this because of (O1). Then

$$\|2F(y) - 2H(y)\| \le \varepsilon \|y\|^p \qquad (34)$$

for all $y \in X$. Similarly, by putting $y = 0$ in (33) we get

$$\|2F(x) - 2G(x)\| \le \varepsilon \|x\|^p \qquad (35)$$

for all $x \in X$.

Hence

$$\|F(x+y) + F(x-y) - 2F(x) - 2F(y)\| \le \|F(x+y) + F(x-y) - 2G(x) - 2H(y)\| + \|2F(x) - 2G(x)\| + \|2F(y) - 2H(y)\| \le 2\varepsilon(\|x\|^p + \|y\|^p) \qquad (36)$$

for all $x, y \in X$ with $x \perp y$.

Then by Theorem 5.1 of [21] there exist a unique quadratic mapping $q : X \to Y$ such that

$$\|F(x) - q(x)\| \le \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2} \|x\|^p$$

Moreover by (34) and (35) we obtain

$$\|G(x) - q(x)\| \le \left(\frac{\varepsilon}{2} + \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}\right) \|x\|^p$$

$$\|H(x) - q(x)\| \le \left(\frac{\varepsilon}{2} + \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}\right) \|x\|^p$$

for all $x \in X$. This completes the proof of theorem.

Corollary 2.7 Let $f, g, h : X \to Y$ are mappings satisfying

$$\|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \le \varepsilon$$

for some $\varepsilon \ge 0$ and for all $x, y \in X$ with $x \perp y$. If f is even mapping then there exists a unique quadratic mapping $q : X \to Y$ such that

$$\|f(x) - f(0) - q(x)\| \le 3\varepsilon$$

$$\|g(x) - g(0) - q(x)\| \le \frac{13\varepsilon}{4}$$

and

$$\|h(x) - h(0) - q(x)\| \le \frac{13\varepsilon}{4}$$

for all $x \in X$.

Theorem 2.8 Let $f, g, h : X \to Y$ are mappings satisfying

$$\|f(x+y) + f(x-y) - 2g(x) - 2h(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for some $\varepsilon \ge 0$, $p \in R^+ \setminus [1,2]$ and for all $x, y \in X$ with $x \perp y$. Suppose that $\perp$ is symmetric on X. Then there exists a unique additive mapping $T : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - f(0) - T(x) - Q(x)\| \le \varepsilon\left(\frac{6\operatorname{sgn}(p-2)}{2^{p/2} - 2} + \frac{\alpha \operatorname{sgn}(p-1)}{2^{p-1} - 1}\right)\|x\|^p$$

$$\|g(x) - g(0) - T(x) - Q(x)\| \le \varepsilon\left(1 + \frac{6\operatorname{sgn}(p-2)}{2^{p/2} - 2} + \frac{\alpha \operatorname{sgn}(p-1)}{2^{p-1} - 1}\right)\|x\|^p$$

and

$$\|h(x) - h(0) - \frac{1}{2}Q(x)\| \le \varepsilon\left(1 + \frac{6\operatorname{sgn}(p-2)}{2^{p/2} - 2}\right)\|x\|^p$$

for all $x \in X$, where

$$\alpha := 1 + 3^p + 2^{p+1}.$$

Proof. Let f^e, g^e and h^e are the even mappings such that

$$f^e(x) = \frac{f(x) + f(-x)}{2}, \quad g^e(x) = \frac{g(x) + g(-x)}{2}$$

and $h^e(x) = \dfrac{h(x) + h(-x)}{2}$.

We have

$$\|f^e(x+y) + f^e(x-y) - 2g^e(x) - 2h^e(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

By theorem 2.6 there exists a unique quadratic mapping $q' : X \to Y$ such that

$$\|f^e(x) - f(0) - q'(x)\| \le \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}\|x\|^p$$

$$\|g^e(x) - g(0) - q'(x)\| \le \left(\frac{\varepsilon}{2} + \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}\right)\|x\|^p$$

and

$$\|h^e(x) - h(0) - q'(x)\| \le \left(\frac{\varepsilon}{2} + \frac{6\varepsilon \operatorname{sgn}(p-2)}{2^{p/2} - 2}\right)\|x\|^p$$

Also let f^o, g^o and h^o are the odd mappings such that

$$f^o(x) = \frac{f(x) - f(-x)}{2}, \quad g^o(x) = \frac{g(x) - g(-x)}{2}$$

and $h^o(x) = \dfrac{h(x) - h(-x)}{2}$. We have

$$\|f^o(x+y) + f^o(x-y) - 2g^o(x) - 2h^o(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

By Theorem 2.2 then there exists a unique additive mapping $T : X \to Y$ and a unique quadratic mapping $Q' : X \to Y$ such that

$$||f^o(x) - T(x) - Q'(x)|| \leq \frac{\varepsilon\alpha\,\mathrm{sgn}(p-1)}{2^{p-1}-1}||x||^p$$
$$||g^o(x) - T(x) - Q'(x)|| \leq (\frac{\varepsilon}{2} + \frac{\varepsilon\alpha\,\mathrm{sgn}(p-1)}{2^{p-1}-1})||x||^p$$

for all $x \in X$, with

$$\alpha := 1 + 3^p + 2^{p+1}.$$

Whence

$$||f(x) - f(0) - T(x) - Q'(x) - q'(x)|| =$$
$$||f^e(x) - f(0) - q'(x) + f^o(x) - T(x) - Q(x)||$$
$$\leq ||f^e(x) - f(0) - q'(x)|| + ||f^o(x) - T(x) - Q(x)||$$
$$\leq \varepsilon(\frac{6\,\mathrm{sgn}(p-2)}{2^{p/2}-2} + \frac{\alpha\,\mathrm{sgn}(p-1)}{2^{p-1}-1})||x||^p$$

Similarly we find

$$||g(x) - g(0) - T(x) - Q'(x) - q'(x)||$$
$$\leq \varepsilon(1 + \frac{6\,\mathrm{sgn}(p-2)}{2^{p/2}-2} + \frac{\alpha\,\mathrm{sgn}(p-1)}{2^{p-1}-1})||x||^p$$

$$||h(x) - h(0) - q'(x)|| \leq \varepsilon(1 + \frac{6\,\mathrm{sgn}(p-2)}{2^{p/2}-2})||x||^p$$

taking $Q(x) = 2Q'(x) = 2q'(x)$ for all $x \in X$, we find the complete proof of the theorem.

Corollary 2.9 Let $f, g, h : X \to Y$ are mappings satisfying

$$||f(x+y) + f(x-y) - 2g(x) - 2h(y)|| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x, y \in X$ with $x \perp y$. Suppose that $\perp$ is symmetric on X. Then there exists a unique additive mapping $T : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$||f(x) - f(0) - T(x) - Q(x)|| \leq 7\varepsilon$$

$$||g(x) - g(0) - T(x) - Q(x)|| \leq \frac{15\varepsilon}{2}$$

and $$||h(x) - h(0) - \frac{1}{2}Q(x)|| \leq \frac{7\varepsilon}{2}$$

for all $x \in X$.

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