Some Domination Parameters of Arithmetic V_n Graph

M. Manjuri
Department of Applied Mathematics
Sri Padmavati Women's University, Tirupati,
Andhra Pradesh, India.

B. Maheswari
Department of Applied Mathematics
Sri Padmavati Women's University, Tirupati,
Andhra Pradesh, India.

ABSTRACT
Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern. Nathanson [1] was the pioneer in introducing the concepts of Number Theory, particularly, the 'Theory of Congruences' in Graph Theory, and paved the way for the emergence of a new class of graphs, namely "Arithmetic Graphs". Inspired by the interplay between Number Theory and Graph Theory several researchers in recent times are carrying out extensive studies on various Arithmetic graphs in which adjacency between vertices is defined through various arithmetic functions.

Cayley Graphs are another class of graphs associated with elements of a group. If this group is associated with some Arithmetic function then the Cayley graph becomes an Arithmetic graph and in this paper we study the Efficient domination, Clique domination parameters of Arithmetic V_n Graphs.

Keywords
Arithmetic V_n Graphs, Efficient domination, Dominating clique, Cayley Graph.

Subject Classification: 68R10

1. INTRODUCTION
The concept of Arithmetic $\boldsymbol{V_n}$ graph and some of its properties are given in [2]. The definition of Arithmetic $\boldsymbol{V_n}$ graph is as follows.

Let n be a positive integer such that $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence consider Arithmetic V_n graph without vertex 1 as the contribution of this isolated vertex is nothing when domination parameters are studied.

Clearly, V_n graph is a connected graph. If n is a prime, then V_n graph consists of a single vertex. Hence it is connected. In other cases, by the definition of adjacency V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n. While studying the various domination parameters it is observed that the domination parameters of these graphs are functions of k, where k is the core of n, i.e., the number of distinct prime divisors of n.

Some results on the domination parameters like domination number, total domination number, independent domination number and connected domination number are presented in [3].

Let $G(V_n)$ denote the V_n graph throughout this paper.

2. EFFICIENT DOMINATION IN ARITHMETIC $\boldsymbol{V_n}$ GRAPH
Cockayne et al. [4] introduced the concept of efficient domination in graphs.

Definition: A set D of vertices in G is called an efficient dominating set, if every vertex u in $V - D$ is adjacent to exactly one vertex in D. The efficient domination number γ_e is the minimum cardinality of an efficient dominating set.

This section is devoted to the study of efficient dominating sets of Arithmetic Vn graph and obtained efficient domination number in various cases.

Theorem 2.1: If n is a product of two distinct primes or p^α, where α is an integer > 1, then the efficient domination number of $G(V_n)$ is 1.

Proof: Let $n = p_1 p_2$, where p_1, p_2 are distinct primes.

Consider the graph $G(V_n)$. Then $G(V_n)$ contains three vertices p_1, p_2 and $p_1 p_2$. Since $GCD(p_1, p_1 p_2) = p_1$ and $GCD(p_2, p_1 p_2) = p_2$, it follows that the vertex $p_1 p_2$ is adjacent with the vertices p_1 and p_2.

Hence if $D = \{p_1 p_2\}$, then D becomes a minimum dominating set of $G(V_n)$. Since the vertex p_1 or p_2 is adjacent with exactly one vertex of D, it follows that D is an efficient dominating set of $G(V_n)$ and this set is a minimum efficient dominating set.

Therefore $\gamma_e\big(G(V_n)\big) = 1$.

Let $n = p^\alpha$, where α is an integer > 1. Consider the graph $G(V_n)$. Then $G(V_n)$ contains the vertices $p, p^2, p^3, ., p^\alpha$. Since $GCD(p, p^2) = p, GCD(p, p^3) = p, \ldots, GCD(p, p^\alpha) = p,$ we have that the vertex p is adjacent with the vertices $p^2, p^3, \ldots,$ p^α. Further there is no edge between any pair of the vertices $p^2, p^3, \ldots p^\alpha$, because $CD(p^2, p^3) = p^2, ., GCD(p^2, p^\alpha) = p^2,$ $GCD(p^3, p^4) = p^3, \ldots, GCD(p^3, p^\alpha) = p^3$ and so on. So, all the vertices $p^2, p^3, \ldots \ldots, p^\alpha$ are adjacent to the vertex p only. That is p dominates all the vertices $p^2, p^3, \ldots \ldots, p^\alpha$.

Hence if $D = \{p\}$, then D becomes a minimum dominating set of $G(V_n)$ which is also efficient.

Therefore $\gamma_e\big(G(V_n)\big) = 1$.

Theorem 2.2: If n is neither product of two distinct primes nor p^α and $n = p_1^{\alpha_1}p_2$, where p_1, p_2 are two distinct primes and α_1, α_2 are integers > 1, then the efficient domination number of $G(V_n)$ is 2.

Proof: Let $n = p_1^{\alpha_1}p_2$, where p_1, p_2 are two distinct primes and consider the graph $G(V_n)$. Then $G(V_n)$ contains the vertices $\{p_2, p_1^r, p_1^r p_2\}, r = 1,2,.,\alpha_1$.

Now we show that $D = \{p_2, p_1p_2\}$ is a dominating set of $G(V_n)$.

Consider the vertices in $V - D$ which are $\{p_1^r, p_1^s p_2\}$,

$s = 2,3, \ldots . \alpha_1$.

Since $GCD(p_1^r, p_1p_2) = p_1$, it follows that the vertices p_1^r are adjacent with the vertex p_1p_2. Again $GCD(p_1^s p_2, p_2) = p_2$, so that the vertices $p_1^s p_2$ are adjacent to the vertex p_2.

Thus all vertices in $V - D$ are adjacent with the vertices of D and therefore D is a dominating set of $G(V_n)$. This set is also minimal, because deletion of either vertex from D makes D no more a dominating set.

From the above discussion it is clear that the vertices p_1^r are adjacent with the vertex p_1p_2 only, the vertices $p_1^s p_2$ are adjacent with the vertex p_2 only.

Hence D becomes an efficient dominating set of $G(V_n)$ which is also minimal.

Therefore $\gamma_e\big(G(V_n)\big) = 2$.

Theorem 2.3: If n is neither product of two distinct primes nor p^α nor $p_1^{\alpha_1}p_2$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where p_1, p_2 are two distinct primes and α_1, α_2 are integers > 1, then the efficient domination number of $G(V_n)$ is 2.

Proof: Let $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where p_1, p_2 are two distinct primes and α_1, α_2 are integers > 1. Consider the graph $G(V_n)$. Then $G(V_n)$ contains the vertices $\{p_1^{r_1}, p_2^{r_2}, p_1^{r_1}p_2, p_1^{r_1}p_2^2, \ldots, p_1^{r_1}p_2^{\alpha_2}\}$ where $r_1 = 1,2, \ldots \ldots, \alpha_1, r_2 = 1,2, \ldots \ldots, \alpha_2$.

Now we show that $D = \{p_1, {p_1}^{\alpha_1}p_2\}$ is a dominating set of $G(V_n)$.

Consider the vertices in $V - D$ which are $\{p_1^r, p_2^{r_2}, p_1^s p_2, p_1^{r_1}p_2^2, \ldots \ldots, p_1^{r_1}p_2^{\alpha_2}\}, r_1 = 1,2, \ldots, \alpha_1$,

$r_2 = 1,2,.., \alpha_2, r = 2,3,.., \alpha_1, s = 1,2,.., \alpha_1 - 1$.

Since $GCD(p_1^r, p_1) = p_1, GCD(p_1^s p_2, p_1) = p_1$, $GCD\big(p_1^{r_1}{p_2}^2, p_1\big) = p_1, \ldots \ldots, GCD\big(p_1^{r_1}{p_2}^{\alpha_2}, p_1\big) = p_1$, it follows that the vertices $p_1^r, p_1^s p_2, p_1^{r_1}p_2^2, .., p_1^{r_1}p_2^{\alpha_2}$ are adjacent with the vertex p_1. Again $GCD\big(p_2^{r_2}, {p_1}^{\alpha_1}p_2\big) = p_2$, so that the vertices $p_2^{r_2}$, $r_2 = 1,2, \ldots \ldots, \alpha_2$ are adjacent with the vertex ${p_1}^{\alpha_1}p_2$.

Thus all vertices in $V - D$ are adjacent with the vertices of D and therefore D is a dominating set of $G(V_n)$. This set is also minimal, because deletion of either vertex from D makes D no more a dominating set.

From the above discussion it is clear that the vertices $p_1^r, p_1^s p_2, p_1^{r_1}p_2^2, \ldots \ldots, p_1^{r_1}p_2^{\alpha_2}$ are adjacent with the vertex p_1 only, the vertices $p_2^{r_2}$ are adjacent with the vertex ${p_1}^{\alpha_1}p_2$ only.

Hence D becomes an efficient dominating set of $G(V_n)$ which is also minimal.

Therefore $\gamma_e\big(G(V_n)\big) = 2$.

Theorem 2.4: If n is neither a prime nor p_1p_2 nor p^α nor $p_1^{\alpha_1}p_2^{\alpha_2}$ and $n = p_1^{\alpha_1}p_2^{\alpha_2} \ldots \ldots \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots \ldots, \alpha_k$ are integers ≥ 1, then efficient domination number does not exist for the graph $G(V_n)$.

Proof: Suppose $n = p_1^{\alpha_1}p_2^{\alpha_2} \ldots \ldots . p_k^{\alpha_k}$. Consider $G(V_n)$ with vertex set V.

We have the following possibilities.

Case 1: Suppose $\alpha_i = 1$ for more than one i. Denote the prime divisors of n with exponent 1 by $p_1, p_2, \ldots \ldots, p_i$ and write $n = p_1 . p_2 . \ldots \ldots p_i . p_{i+1}^{\alpha_{i+1}} . \ldots . . p_k^{\alpha_k}$.

We know that $D = \{p_1, p_2, \ldots ., p_{i-2}, p_{i-1} \cdot p_i, p_{i+1}, \ldots ., p_k\}$ is a dominating set of $G(V_n)$ [3].

Now we verify whether D is an efficient dominating set or not.

Let u be any vertex in$V - D$. Suppose $u = p_1p_2 . p_k$. Then

$GCD(u, p_1) = p_1, GCD(u, p_2) = p_2$ and so on. So the vertex u is adjacent with the vertices $p_1, p_2, ., p_{i-2}, p_{i+1}, ., p_k \in D$.

Thus the set D is not an efficient dominating set of $G(V_n)$.

We consider another set D given by

$D = \{p_1, p_1p_2, p_2p_3, \ldots \ldots, p_{k-3}p_{k-2}, p_{k-1}p_k\}$. We now show that D is a dominating set of $G(V_n)$.

By the definition of $G(V_n)$ graph, it is obvious that the vertices in $G(V_n)$ are primes $p_1, p_2, \ldots \ldots, p_k$, their powers and their products.

All the vertices $u \in V - D$, for which $GCD(u, p_1) = p_1$ are adjacent to the vertex p_1 in D and the vertices p_i^r,

$r = 1,2,.., \alpha_i, i = 2,3,., k$ are adjacent to the vertices p_1p_2 and $p_2p_3; p_2p_3$ and $p_3p_4; .: p_{k-4}p_{k-3}$ and $p_{k-3}p_{k-2}; p_{k-1}p_k$ respectively, as $GCD(p_2^r, p_1p_2) = p_2, GCD(p_2^r, p_2p_3) = p_2, \ldots$,

$GCD({p_{k-1}}^r, p_{k-1}p_k) = p_{k-1}, \ldots \ldots, GCD(, p_k^r, p_{k-1}p_k) = p_k$.

Similarly we can show that any vertex in $V - D$ shares a common factor with some vertex in D.Thus every vertex in $V - D$ is adjacent to at least one vertex in D. Hence D becomes a dominating set of $G(V_n)$.

Here we observe that the vertices $p_i^r, r = 1,2,.., \alpha_i$,

$i = 2,3, \ldots, k - 3$ in $V - D$ are adjacent with more than one vertex in D.

Hence we conclude that D is a dominating set of $G(V_n)$, but not an efficient dominating set of $G(V_n)$.

In a similar way we can show that any dominating set of $G(V_n)$ constructed in any manner cannot be an efficient dominating set of $G(V_n)$. This is because the prime power decomposition that n has and the way of adjacency defined between the vertices in $G(V_n)$.

Case 2: Suppose $\alpha_i > 1$, for all i or $\alpha_i = 1$, for only one i.

We know that $D = \{p_1, p_2, \ldots \ldots, p_k\}$ is a dominating set of $G(V_n)$ [3]. The vertex $p_1p_2 \ldots . . p_k$ in $V - D$ is adjacent with all the vertices $p_1, p_2, \ldots \ldots, p_k \in D$. Thus D is not an efficient dominating set of $G(V_n)$.

We consider another set D given by

$D = \{p_k, p_k p_1, \dots, p_k p_{k-1}\}$. We now show that D is a dominating set of $G(V_n)$.

All the vertices $u \in V - D$, for which $GCD(u, p_k) = p_k$ are adjacent with the vertex p_k in D.

The vertices $p_1, p_2, , p_{k-1}$ and $p_1{}^r, p_2{}^r, \dots, p_{k-1}{}^r \in V - D$, $r > 1$ are adjacent with the vertices $p_k, p_k p_1, \dots, p_k p_{k-1}$ in D, since $GCD(p_i, p_k p_i) = p_i$ and $GCD(p_i{}^r, p_k p_i) = p_i$,

$i = 1,2, \dots, k-1$. All the vertices $v \in V - D$, for which

$GCD(v, p_k p_i) = p_i$ are adjacent with the vertex $p_k p_i$ in D.

Now we conclude that every vertex in $V - D$ is adjacent with atleast one vertex in D.Thus D is a dominating set of $G(V_n)$.

Here we observe that the vertex p_k^r, $r > 1$ is adjacent with the vertex p_k and $p_k p_j$, $j = 1,2, \dots, k-1$ in D as $GCD(p_k^r, p_k) = p_k, GCD(p_k^r, p_k p_j) = p_k$.

Hence D is a dominating set but not an efficient dominating set of $G(V_n)$.

In a similar way we can show that any dominating set of $G(V_n)$ constructed in any manner cannot be an efficient dominating set of $G(V_n)$. This is because the prime power decomposition that n has and the way of adjacency defined between the vertices in $G(V_n)$.

Hence an efficient dominating set does not exist for the graph $G(V_n)$, if n has prime power decomposition.

Note: If $n = p$, then by the definition of Arithmetic V_n graph the vertex set contains only one vertex. Obviously the efficient domination number of $G(V_n)$ is 1.

3. CLIQUE DOMINATION IN ARITHMETIC V_n GRAPH

The concept of clique domination in graphs is introduced by Cozzens and Kelleher [5]. They established a sufficient condition for a graph to have a dominating clique in terms of forbidden sub graphs. Further they gave an upper bound on the clique domination number. Also Basco [6] studied dominating cliques in P_5 – free graphs and obtained bounds for clique domination number.

Definition :A dominating set D of a graph G is called a dominating clique if the induced sub graph $< D >$ is a complete graph. The cardinality of the smallest dominating clique is called clique domination number and is denoted by γ_{cl}.

This section is devoted to the study of dominating cliques of Arithmetic V_n graph and obtained clique domination number in various cases.

Theorem 3.1: If n is the product of two distinct primes, then the clique domination number of $G(V_n)$ is 1.

Proof: Suppose $n = p_1 p_2$, where p_1 and p_2 are two distinct primes. Consider the graph $G(V_n)$. It contains three vertices p_1, p_2 and $p_1 p_2$. So, $GCD\ (p_1, p_1 p_2) = p_1$ and

$GCD\ (p_2, p_1 p_2) = p_2$. That is the vertex $p_1 p_2$ is adjacent with both the vertices p_1, p_2. Hence the vertex $p_1 p_2$ is dominates the vertices p_1, p_2.

Therefore $\gamma\big(G(V_n)\big) = 1$.

So, if $D_{cl} = \{p_1 p_2\}$, then the induced subgraph $< D_{cl} >$ is a complete graph with minimum cardinality. Hence D_{cl} is a minimum dominating clique of $G(V_n)$.

Therefore $\gamma_{cl}\big(G(V_n)\big) = 1$.

Theorem 3.2: If $n \neq p_1 p_2$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $\alpha_i \geq 1$, then the clique domination number of $G(V_n)$ is k, where k is the core of n.

Proof: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$, where $\alpha_i \geq 1$. Consider the graph $G(V_n)$. The vertices in $G(V_n)$ are primes, their powers and products.

Since $G(V_n)$ is not a complete graph for any n, it follows that any single vertex in $G(V_n)$ does not form a dominating set in $G(V_n)$.

Now we claim that $D_{cl} = \{p_k, p_k p_1, \dots, p_k p_{k-1}\}$ as a dominating set of $G(V_n)$.

All the vertices $u \in V - D$, for which $GCD(u, p_k) = p_k$ are adjacent with the vertex p_k in D. The vertices $p_1, p_2, \dots, p_{k-1}$ and $p_1{}^r, p_2{}^r, \dots, p_{k-1}{}^r \in V - D$, $r > 1$ are adjacent with the vertices $p_k, p_k p_1, \dots, p_k p_{k-1}$ in D, since $GCD(p_i, p_k p_i) = p_i$ and $GCD(p_i{}^r, p_k p_i) = p_i, i = 1, \dots, k-1$.

All the vertices $v \in V - D$, for which $GCD(v, p_k p_i) = p_i$ are adjacent with the vertex $p_k p_i$ in D.

Thus every vertex in $V - D$ is adjacent with at least one vertex in D_{cl}. Thus D_{cl} is a dominating set of $G(V_n)$.

Now we show that $D_{cl} = \{p_k, p_k p_1, \dots \dots, p_k p_{k-1}\}$is a dominating clique of $G(V_n)$, where k is the core of n. Obviously the vertex p_k is adjacent with the vertices $p_k p_1, \dots \dots, p_k p_{k-1}$ and also the vertices $p_k p_1, \dots \dots, p_k p_{k-1}$ are adjacent with each other.

Thus each pair of distinct vertices in D_{cl} are adjacent. This gives that the $< D_{cl} >$ is a complete graph. Hence D_{cl} is a dominating clique of $G(V_n)$.

Now we prove that D_{cl} is minimum. Suppose we delete the vertex p_k from D_{cl}.Then the vertex n is not adjacent with any vertex in D_{cl}. Again if we remove any vertex other than $p_k$from D_{cl}, say $p_k p_i$ where $i = 1,2, \dots, k-1$, then the vertex p_i is not adjacent with any vertex in $D_{cl} - \{p_k p_i\}$ because $GCD(p_i, p_k) = 1$ and $GCD\big(p_i, p_k p_j\big) = 1, i \neq j$.That is D_{cl} is not a dominating set, a contradiction.

Hence $D_{cl} = \{p_k, p_k p_1, \dots \dots, p_k p_{k-1}\}$ is a dominating clique of $G(V_n)$ with minimum cardinality.

Therefore $\gamma_{cl}\big(G(V_n)\big) = k$.

4. CONCLUSION

Using Number theory, it is interesting to study the various concepts of Graph theory. In particular finding the efficient dominating sets and dominating clique of the Arithmetic Vn graphs is attractive. This work gives the scope for the study of efficient dominating sets and dominating clique of product graphs of Euler totient Cayley graph with Arithmetic V_n graph. The authors have studied this aspect for direct product, Strong product, lexicographic product graphs of Euler totient Cayley graph with Arithmetic V_n graph This work further gives the scope for the study of dominating functions of various graphs.

ILLUSTRATIONS

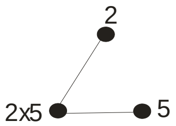

Fig.1

$G(V_{10})$

Efficient Dominating Set = {10}

Dominating Clique = {10}

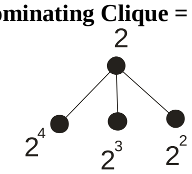

Fig.2

$G(V_{16})$

Efficient Dominating Set = {2}

Dominating Clique = {2}

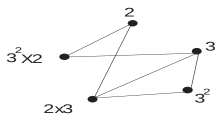

Fig.3

$G(V_{18})$

Efficient Dominating Set = {2,6}

Dominating Clique = {2,6}

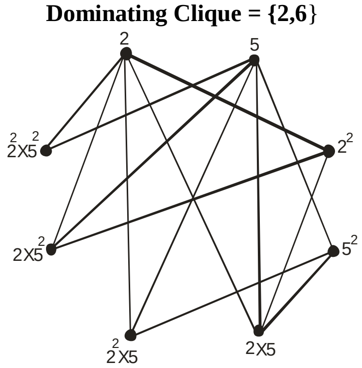

Fig.4

$G(V_{100})$

Efficient Dominating Set = {2, 20}

Dominating Clique = {5,10}

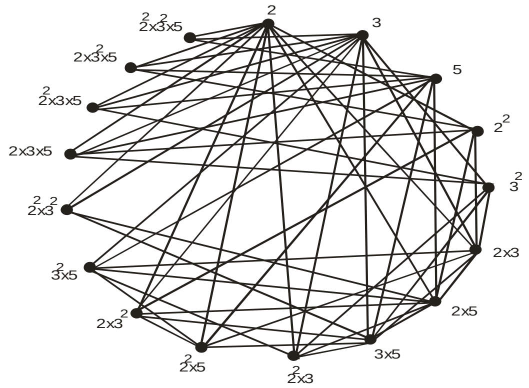

Fig.5

$G(V_{180})$

Efficient Dominating Set does not exist

Dominating Clique = {5,10,15}

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