Cartesian Composition in Bipolar Fuzzy Finite State Machines

S. Subramaniyan
Assistant Professor
Mathematics section,FEAT
Annamalai University
Chidambaram
Tamilnadu, India

M. Rajasekar
Assistant Professor
Mathematics section,FEAT
Annamalai University
Chidambaram
Tamilnadu, India

ABSTRACT

In this paper we introduce Cartesian composition in bipolar fuzzy finite state machines and study their properties.

Keywords:

Bipolar Fuzzy finite state machines, Retrievability, Separability, Cartesian composition, Connectivity.

1. INTRODUCTION

The theory of fuzzy set was introduced by L.A. Zadeh in 1965 [9]. The mathematical formulation of a fuzzy automaton was first proposed by W.G. Wee in 1967 [8]. E.S. Santos 1968 [7] proposed fuzzy automata as a model of pattern recognition.
J. N. Mordeson and D.S. Malik gave a detailed account of fuzzy automata and languages in their book 2002 [6]. Fuzzy sets are kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets etc. Bipolar-valued fuzzy sets, which are introduced by Lee [4, 5], are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In [2], Y.B. Jun and J. Kavikumar introduced bipolar fuzzy finite state machines, a bipolar successor, a bipolar exchange property.
In this paper, we introduced Cartesian composition in bipolar fuzzy finite state machines with example and discuss their properties.

2. BASIC DEFINITIONS

2.1 Definition [10]

Let X denote a universal set. Then a fuzzy set A in X is set of ordered pairs:
$A = \{(x,\ \mu_A(x)|x \in X\}$,
$\mu_A(x)$ is called the membership function or grade of membership of x in A which maps X to the membership space $[0, 1]$.

2.2 Definition [3]

A finite fuzzy automata is a system of 3 tuples, $M = (Q, X, f_M)$, where,
Q-set of states $\{q_1, q_2, ..., q_n\}$
Σ-alphabets (or) input symbols
f_M-function from $Q \times X \times Q \to [0, 1]$
$f_M(q_i, \sigma, q_j) = \mu$,$[0 \leq \mu \leq 1]$ means when M is in state q_i and reads the input σ will move to the state q_j with weight function μ.

2.3 Definition [2]

A bipolar fuzzy finite state machine (bffsm, for short) is a triple $M = (Q, X, \varphi)$, where Q and X are finite nonempty sets, called the set of states and the set of input symbols, respectively and $\varphi = \langle \varphi^-, \varphi^+ \rangle$ is a bipolar fuzzy set in $Q \times X \times Q$.
Let X^* denote the set of all words of elements of X of finite length. Let λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$.

2.4 Definition [2]

Let $M = (Q, X, \varphi)$ be a bffsm. Define a bipolar fuzzy set $\varphi_* = \langle {\varphi_*}^+, {\varphi_*}^- \rangle$ in $Q \times X^* \times Q$ by

$$\varphi_*^-(q, \lambda, p) = \begin{cases} -1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\varphi_*^+(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$\varphi_*^-(q, xa, p) = \inf_{r \in Q}[{\varphi_*}^-(q, x, r) \vee {\varphi_*}^-(r, a, p)]$
${\varphi_*}^+(q, xa, p) = \sup_{r \in Q}[{\varphi_*}^+(q, x, r) \wedge {\varphi_*}^+(r, a, p)]\ \forall\ p, q \in Q,\ x \in X^*$ and $a \in X$.

Result

Let $M = (Q, X, \varphi)$ be a bffsm. Then
${\varphi_*}^-(q, xy, p) = \inf_{r \in Q}[{\varphi_*}^-(q, x, r) \vee {\varphi_*}^-(r, y, p)]$
${\varphi_*}^+(q, xy, p) = \sup_{r \in Q}[{\varphi_*}^+(q, x, r) \wedge {\varphi_*}^+(r, y, p)] \quad \forall p, q \in Q$ and $x, y \in X^*$.

2.5 Definition [2]

Let $M = (Q, X, \varphi)$ be a bffsm and let $p, q \in Q$. Then p is called a immediate successor of q if the following condition holds
$\exists a \in X$ such that $\varphi_*{}^-(q, a, p) < 0$ and $\varphi_*{}^+(q, a, p) > 0$. We say that p is a successor of q if the following condition holds
$\exists x \in X^*$ such that $\varphi_*{}^-(q, x, p) < 0$ and $\varphi_*{}^+(q, x, p) > 0$.
We denote by $S(q)$ the set of all successors of q. For any subset T of Q the set of all successors of T denoted by $S(T)$ is defined to be the set $S(T) = \cup\{S(q) \backslash q \in T\}$

2.6 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. Let v be a bipolar fuzzy subset of $T \times X \times T$ and let $N = (T, X, v)$. The bipolar fuzzy finite state machine N is called a submachine of M if
(i) $v|T \times X \times T = v$
(ii)$S_Q(T) \subseteq T$.

2.7 Definition

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. M is said to be retrievable if $\forall\ q\ \in\ Q\ \forall\ y\ \in\ X^*$ if $\exists\ t\ \in\ Q$ such that $\varphi_*(q, y, t) = [\varphi_*{}^-(q, y, t) < 0, \varphi_*{}^+(q, y, t) > 0]$, then $\exists\ x\ \in\ X^*$ such that $\varphi_*(t, x, q) = [\varphi_*{}^-(t, x, q) < 0, \varphi_*{}^+(t, x, q) > 0]$.

2.8 Definition

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. M is said to be quasi-retrievable if $\forall\ q\ \in\ Q\ \ \forall\ y\ \in\ X^*$ if $\exists\ t\ \in\ Q$ such that $\varphi_*(q, y, t) = [\varphi_*{}^-(q, y, t) < 0, \varphi_*{}^+(q, y, t) > 0]$, then $\exists\ x\ \in\ X^*$ such that $\varphi_*(q, yx, q) = [\varphi_*{}^-(q, yx, q) < 0, \varphi_*{}^+(q, yx, q) > 0]$, where
$\varphi_*{}^-(q, yx, q) = \inf_{t\,\in\,Q}[\varphi_*{}^-(q,\ y,\ t) \vee \varphi_*{}^-(t,\ x,\ q)] < 0$ and
$\varphi_*{}^+(q, yx, q) = \sup_{t\,\in\,Q}[\varphi_*{}^+(q,\ y,\ t) \wedge \varphi_*{}^+(t,\ x,\ q)] > 0$.

2.9 Definition

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Let $q, r, s \in Q$. Then r and s are said to be q-related if $\exists y \in X^*$ such that $\varphi_*(q, y, r) = [\varphi_*{}^-(q, y, r) < 0, \varphi_*{}^+(q, y, r) > 0]$ and
$\varphi_*(q, y, s) = [\varphi_*{}^-(q, y, s) < 0, \varphi_*{}^+(q, y, s) > 0]$.

Note
If r and s are bipolar q-related then r and s are said to be q-twins if $S(s) = S(r)$.

2.10 Definition

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. We say that M satisfies the exchange property if the following condition holds:
Let $p, q \in Q$ and let $T \subseteq Q$. Suppose that if $p \in S(T \cup \{q\}), p \notin S(T)$, then $q \in S(T \cup \{p\})$.

2.11 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $T\ \subseteq\ Q$. Let v be a bipolar fuzzy subset of $T\ \times\ X\ \times\ T$ and let $N\ =\ (T,\ X,\ v)$. The bipolar fuzzy finite state machine N is called a submachine of M if
(i) $v\ |\ T\ \times\ X\ \times\ T\ =\ v$
(ii)$S_Q(T)\ \subseteq\ T$.

2.12 Definition

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. M is said to be connected if M has no separated proper submachine.

2.13 Definition

Let $M\ =\ (Q,\ X,\ \varphi)$ be a bffsm. Then M is called strongly connected if $\forall\ p, q \in Q, p \in S(q)$.

2.14 Definition

Let $M = (Q,\ X,\ \varphi)$ be a bffsm and let $N\ =\ (T,\ X,\ v)$ be a submachine of M. N is called proper if $T\ \neq\ Q$ and $T\ \neq\ \phi$. If M is strongly connected then M has no proper submachine.

2.15 Definition

Let $M\ =\ (Q,\ X,\ \varphi)$ be a bffsm and let $\varphi\ =\ \langle\varphi^-, \varphi^+\rangle$ be a subsystem of M. Then φ is called cyclic if
$\exists q_t^- \supseteq \varphi^-, q \in Q, t \in [-1, 0)$ and
$\exists q_t^+ \subseteq \varphi^+, q \in Q, t \in (0, 1]$.
Note

Let $M\ =\ (Q, X, \varphi)$ be a bffsm. Let $R\ \subseteq\ Q$. Then $N\ =\ (S(R), X, \varphi_R)$ is a submachine of M, where $\varphi_R = \left\langle\varphi_R^-, \varphi_R^+\right\rangle$, where
$\varphi_R^- = \varphi^-|_{S(R)\,\times\,X\,\times\,S(R)} < 0$
$\varphi_R^+ = \varphi^+|_{S(R)\,\times\,X\,\times\,S(R)} > 0$

2.16 Definition

Let $M\ =\ (Q, X, \varphi)$ be a bffsm. Let $R\ \subseteq\ Q$ and $\{N_i = (Q_i, X, \mu_i) | i \in I\}$ be the collection of all submachines of M whose state set contains R. Define $\langle R\rangle = \cap_{i \in I}\{Q_i\ |\ i\ \in\ I\}$. Then $\langle R\rangle$ is called the submachine generated by R. $\langle R\rangle$ is the smallest submachine of M whose state set contains R.

2.17 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. T is called free if $\forall t \in T,\ \ t \notin S\left(T|\,\{t\}\right)$.

2.18 Definition

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$.
Let $M_1.M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \varphi_1.\varphi_2)$, where

$$(\varphi_1 \cdot \varphi_2)^-((p_1, p_2), a, (q_1, q_2)) = \begin{cases} \varphi_1^-(p_1, a, q_1) & \text{if } a\ \in\ X_1 \text{and } p_2 = q_2 \\ \varphi_2^-(p_2, a, q_2) & \text{if } a\ \in\ X_2 \text{and } p_1 = q_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\varphi_1 \cdot \varphi_2)^+((p_1, p_2), a, (q_1, q_2)) = \begin{cases} \varphi_1^+(p_1, a, q_1) & \text{if } a\ \in\ X_1 \text{and } p_2 = q_2 \\ \varphi_2^+(p_2, a, q_2) & \text{if } a\ \in\ X_2 \text{and } p_1 = q_1 \\ 0 & \text{otherwise} \end{cases}$$

$\forall\ ((p_1, p_2), (q_1, q_2)) \in Q_1 \times Q_2,\ a \in X_1 \cup X_2$. Then $M_1 \circ M_2$ is called the Cartesian composition of $M_1 \circ M_2$.

2.19 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Then M is said to be commutative if $\forall\ a, b\ \in\ X$ and $\forall\ q, p\ \in\ Q$
$\varphi_*^-(q, ab, p) = \varphi_*^-(q, ba, p)$ and
$\varphi_*^+(q, ab, p) = \varphi_*^+(q, ba, p)$

3. PROPERTIES OF CARTESIAN COMPOSITION IN BIPOLAR FUZZY FINITE STATE MACHINES

3.1 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Let $M_1 \circ M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \varphi_1 \circ \varphi_2)$ be the Cartesian composition of $M_1 \circ M_2$. Then $\forall\, x \in X_1^* \cup X_2^*,\ x \neq \lambda$.

$$(\varphi_1 \cdot \varphi_2)_*^-((p_1,p_2),\, x,\, (q_1,q_2)) = \begin{cases} \varphi_1^-(p_1,\, x,\, q_1) & \text{if } x \in X_1^* \text{and } p_2 = q_2 \\ \varphi_2^-(p_2,\, x,\, q_2) & \text{if } x \in X_2^* \text{and } p_1 = q_1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\varphi_1 \cdot \varphi_2)_*^+((p_1,p_2),\, x,\, (q_1,q_2)) = \begin{cases} \varphi_1^+(p_1,\, x,\, q_1) & \text{if } x \in X_1^* \text{and } p_2 = q_2 \\ \varphi_2^+(p_2,\, x,\, q_2) & \text{if } x \in X_2^* \text{and } p_1 = q_1 \\ 0 & \text{otherwise} \end{cases}$$

Proof.
Let $x \in X_1^* \cup X_2^*,\ x \neq \lambda$ and let $|x| = n$. Suppose that $x \in X_1^*$. Clearly the result is true if $n = 1$. Suppose the result is true $\forall\, y \in X_1^*,\ |y| = n-1,\ n > 1$. Let $x = ay$ where $a \in X_1$ and $y \in X_1^*$. Now, $(\varphi_1 . \varphi_2)_*^-((p_1,p_2), ay, (q_1,q_2)) = \wedge_{(r_1,r_2) \in Q_1 \times Q_2} \left\{ (\varphi_1 . \varphi_2)^-((p_1,p_2), a, (r_1,r_2)) \vee (\varphi_1 . \varphi_2)_*^-((r_1,r_2), y, (q_1,q_2)) \right\}$
$= \wedge_{r_1 \in Q_1} \{(\varphi_1^-(p_1, a, r_1) \vee (\varphi_1 . \varphi_2)_*^-((r_1,p_2), y, (q_1,q_2))\}$

$$= \begin{cases} \wedge_{r_1 \in Q_1} \{(\varphi_1^-(p_1,a,r_1),\ \vee\ \varphi_1 *^- (r_1,y,q_1))\} & \text{if } p_2 = q_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\varphi_1 *^- (p_1,ay,q_1)) & \text{if } p_2 = q_2 \\ 0 & \text{otherwise} \end{cases}$$

Now,
$(\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), ay, (q_1,q_2)) = \vee_{(r_1,r_2) \in Q_1 \times Q_2} \left\{ (\varphi_1 \circ \varphi_2)^+((p_1,p_2), a, (r_1,r_2)) \wedge (\varphi_1 \circ \varphi_2)_*^+((r_1,r_2), y, (q_1,q_2)) \right\}$
$= \vee_{r_1 \in Q_1} \left\{ (\varphi_1^+(p_1, a, r_1) \wedge (\varphi_1 \circ \varphi_2)_*^+((r_1,p_2), y, (q_1,q_2)) \right\}$

$$= \begin{cases} \vee_{r_1 \in Q_1} \left\{ (\varphi_1^+(p_1,a,r_1),\ \wedge\ \varphi_1 *^+ (r_1,y,q_1)) \right\} & \text{if } p_2 = q_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\varphi_1 *^+ (p_1,ay,q_1)) & \text{if } p_2 = q_2 \\ 0 & \text{otherwise} \end{cases}$$

The result now follows by induction. The proof is similar if $x \in X_2^*$.

3.2 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then $\forall\, x \in X_1^*,\ \forall\, y \in X_2^*$.
$(\varphi_1 \circ \varphi_2)_*^-((p_1,p_2), xy, (q_1,q_2)) = \varphi_{1*}^-(p_1,x,q_1) \vee \varphi_{2*}^-(p_1,y,q_2)$
$=(\varphi_1 \circ \varphi_2)_*^-((p_1,p_2), yx, (q_1,q_2))$ and
$(\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), xy, (q_1,q_2)) = \varphi_{1*}^+(p_1,x,q_1) \wedge \varphi_{2*}^+(p_1,y,q_2)$
$=(\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), yx, (q_1,q_2))$
$\forall (p_1,p_2), (q_1,q_2) \in Q_1 \times Q_2$.

Proof. Let$x \in X_1^*,\ y \in X_2^*\ (p_1,p_2), (q_1,q_2) \in Q_1 \times Q_2$. If $x = \lambda = y$ then $xy = \lambda$. Suppose $(p_1,p_2) = (q_1,q_2)$ then $p_1 = q_1$ and $p_2 = q_2$. Hence,
$(\varphi_1 \circ \varphi_2)_*^-((p_1,p_2), xy, (q_1,q_2)) = -1 = -1 \vee -1 = \varphi_{1*}^-(p_1,x,q_1) \vee \varphi_{2*}^-(p_1,y,q_2)$
$(\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), xy, (q_1,q_2)) = 1 = 1 \wedge 1 = \varphi_{1*}^+(p_1,x,q_1) \wedge \varphi_{2*}^+(p_1,y,q_2)$.
If $(p_1,p_2) \neq (q_1,q_2)$ then either $p_1 \neq q_1$ or $p_2 \neq q_2$.
Thus $\varphi_{1*}^-(p_1,x,q_1) \vee \varphi_{2*}^-(p_1,y,q_2) = 0 \vee 0 = 0$ and $\varphi_{1*}^+(p_1,x,q_1) \wedge \varphi_{2*}^+(p_1,y,q_2) = 0 \wedge 0 = 0$
Hence $(\varphi_1 \circ \varphi_2)_*^-((p_1,p_2), xy, (q_1,q_2)) = 0 = \varphi_{1*}^-(p_1,x,q_1) \vee \varphi_{2*}^-(p_1,y,q_2)$
$(\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), xy, (q_1,q_2)) = 0 = \varphi_{1*}^+(p_1,x,q_1) \wedge \varphi_{2*}^+(p_1,y,q_2)$.
If $x = \lambda$ and $y \neq \lambda$ or $x \neq \lambda$ and $y = \lambda$ then the result follows by above Theorem. Suppose $x \neq \lambda$ and $y \neq \lambda$. Now
$(\varphi_1 \circ \varphi_2)_*^-((p_1,p_2), xy, (q_1,q_2)) = \wedge_{r_1,r_2 \in Q_1 \times Q_2} \left\{ (\varphi_1 \circ \varphi_2)_*^-((p_1,p_2), x, (r_1,r_2)) \vee (\varphi_1 . \varphi_2)_*^-((r_1,r_2), y, (q_1,q_2)) \right\}$
$= \wedge_{r_1 \in Q_1} \left\{ (\varphi_1 . \varphi_2)_*^-((p_1,p_2), x, (r_1,p_2)) \vee (\varphi_1 \circ \varphi_2)_*^-((r_1,p_2), y, (q_1,q_2)) \right\}$
$= \varphi_{1*}^-(p_1,x,q_1) \vee \varphi_{2*}^-(p_1,y,q_2)$

Now, $(\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), xy, (q_1,q_2)) = \vee_{r_1,r_2 \in Q_1 \times Q_2} \left\{ (\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), x, (r_1,r_2)) \wedge (\varphi_1 \circ \varphi_2)_*^+((r_1,r_2), y, (q_1,q_2)) \right\}$
$=$
$\vee_{r_1 \in Q_1} \left\{ (\varphi_1 \circ \varphi_2)_*^+((p_1,p_2), x, (r_1,p_2)) \wedge (\varphi_1 . \varphi_2)_*^+((r_1,p_2), y, (q_1,q_2)) \right\}$
$= \varphi_{1*}^+(p_1,x,q_1) \vee \varphi_{2*}^+(p_1,y,q_2)$.

Similarly,
$(\varphi_1 . \varphi_2)_*^-((p_1,p_2), yx, (q_1,q_2)) = \varphi_{2*}^-(p_1,y,q_2) \vee \varphi_{1*}^-(p_1,x,q_1)$
$(\varphi_1 . \varphi_2)_*^+((p_1,p_2), yx, (q_1,q_2)) = \varphi_{2*}^+(p_1,y,q_2) \vee \varphi_{1*}^+(p_1,x,q_1)$

3.3 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then $\forall w \in (X_1 \cup X_2)^*, \exists u \in X_1^*, v \in X_2^*$ such that
$(\varphi_1 . \varphi_2)_*^-((p_1,p_2), w, (q_1,q_2)) = \varphi_1 . \varphi_2)_*^-((p_1,p_2), uv, (q_1,q_2))$ and
$(\varphi_1 . \varphi_2)_*^+((p_1,p_2), w, (q_1,q_2)) = (\varphi_1 . \varphi_2)_*^+((p_1,p_2), uv, (q_1,q_2)) \forall ((p_1,p_2), (q_1,q_2)) \in Q_1 \times Q_2$.

Proof.
Let $w \in (X_1 \cup X_2)^*$ and $((p_1,p_2), (q_1,q_2)) \in Q_1 \times Q_2$. If $w = \lambda$ then we can choose $u = \lambda = v$. In this case the result is trivially true. Suppose $w \neq \lambda$. If $w \in X_1^*$ or $w \in X_2^*$ then again the result is trivially true. Suppose $w \notin X_1^*$ and $w \notin X_2^*$
case 1: If $w = xy,\ x \in X_1^+,\ y \in X_2^+$ then the result follows by above Theorem.
case 2: Suppose $w = x_1 y x_2,\ x_1, x_2 \in X_1^*$ and $y \in X_2^*$, x_i and y are non empty strings, $i = 1, 2$
Let $u = x_1, x_2 \in X_1^*$ and $v = y$. Now by above Theorem
$(\varphi_1 . \varphi_2)_*^-((p_1,p_2), x_2 y, (q_1,q_2)) = (\varphi_1 . \varphi_2)_*^-((p_1,p_2), y x_2, (q_1,q_2))$ and
$(\varphi_1 . \varphi_2)_*^+((p_1,p_2), x_2 y, (q_1,q_2)) = (\varphi_1 . \varphi_2)_*^+((p_1,p_2), y x_2, (q_1,q_2))$
$\forall (p_1,p_2), (q_1,q_2) \in Q_1 \times Q_2$. Thus
$(\varphi_1 . \varphi_2)_*^-((p_1,p_2), x_1 y x_2, (q_1,q_2)) =$
$\wedge_{r_1,r_2 \in Q_1 \times Q_2}$
$\{(\varphi_1 . \varphi_2)_*^-((p_1,p_2), x_1, (r_1,r_2)) \vee (\varphi_1 . \varphi_2)_*^-((r_1,r_2), y x_2, (q_1,q_2))\}$
$(\varphi_1 . \varphi_2)_*^-((p_1,p_2), x_1 y x_2, (q_1,q_2)) =$
$\wedge_{r_1,r_2 \in Q_1 \times Q_2}$
$\{(\varphi_1 . \varphi_2)_*^-((p_1,p_2), x_1, (r_1,r_2)) \vee (\varphi_1 . \varphi_2)_*^-((r_1,r_2), x_2 y, (q_1,q_2))\}$
$= \{(\varphi_1 . \varphi_2)_*^-((p_1,p_2), x_1 x_2 y, (q_1,p_2))\}$
Now, $(\varphi_1 . \varphi_2)_*^+((p_1,p_2), x_1 y x_2, (q_1,q_2)) =$
$\wedge_{r_1,r_2 \in Q_1 \times Q_2}$
$\{(\varphi_1 . \varphi_2)_*^+((p_1,p_2), x_1, (r_1,r_2)) \vee (\varphi_1 . \varphi_2)_*^+((r_1,r_2), y x_2, (q_1,q_2))\}$
$(\varphi_1 . \varphi_2)_*^+((p_1,p_2), x_1 y x_2, (q_1,q_2)) =$

$\wedge_{r_1,r_2\in Q_1\times Q_2}$
$\{(\varphi_1.\varphi_2)^+_*((p_1,p_2),x_1,(r_1,r_2)) \vee (\varphi_1.\varphi_2)^+_*((r_1,r_2),x_2y,(q_1,q_2))\}$
$= \{(\varphi_1.\varphi_2)^+_*((p_1,p_2),x_1x_2y,(q_1,p_2))\}$

case 3:
Suppose $w = y_1xy_2,\ y_1,y_2 \in X_2^*$ and $x \in X_1^*$, y_i and x are non empty strings, $i = 1,2$
Let $v = y_1,y_2 \in X_2^*$ and $u = x$. The proof is similar to case 2.

case 4:
Suppose $w = x_1y_1x_2y_2,\ x_1,x_2 \in X_1^*\ y_1,y_2 \in X_2^*$ and x_i and y_i are non empty strings, $i = 1,2$
Let $u = x_1x_2 \in X_1^*$ and $v = y_1y_2 \in X_2^*$. Then
$(\varphi_1.\varphi_2)^-_*((p_1,p_2),x_1y_1x_2y_2,(q_1,q_2)) = \wedge_{r_1,r_2\in Q_1\times Q_2}$
$\{(\varphi_1\circ\varphi_2)*^-((p_1,p_2),x_1,(r_1,r_2)) \vee (\varphi_1.\varphi_2)^-_*((r_1,r_2),y_1x_2y_2,(q_1,q_2))\}$

$= \wedge_{r_1,r_2\in Q_1\times Q_2}\{(\varphi_1\circ\varphi_2)^-_*((p_1,p_2),x_1,(r_1,r_2))\vee$
$(\varphi_1\circ\varphi_2)^-_*((r_1,r_2),x_2y_1y_2,(q_1,q_2))\}$
$= (\varphi_1\circ\varphi_2)^-_*((p_1,p_2),x_1x_2y_1y_2,(q_1,q_2))$
$= (\varphi_1\circ\varphi_2)^-_*((p_1,p_2),uv,(q_1,q_2))$
Now,
$(\varphi_1\circ\varphi_2)^+_*((p_1,p_2),x_1y_1x_2y_2,(q_1,q_2)) =$
$\vee_{r_1,r_2\in Q_1\times Q_2}$
$\left\{(\varphi_1\circ\varphi_2)*^+((p_1,p_2),x_1,(r_1,r_2)) \wedge (\varphi_1\circ\varphi_2)^+_*((r_1,r_2),y_1x_2y_2,(q_1,q_2))\right\}$
$= \vee_{r_1,r_2\in Q_1\times Q_2}$
$\left\{(\varphi_1\circ\varphi_2)^+_*((p_1,p_2),x_1,(r_1,r_2)) \wedge (\varphi_1\circ\varphi_2)^+_*((r_1,r_2),x_2y_1y_2,(q_1,q_2))\right\}$

$= (\varphi_1\circ\varphi_2)^+_*((p_1,p_2),x_1x_2y_1y_2,(q_1,q_2))$
$= (\varphi_1\circ\varphi_2)^+_*((p_1,p_2),uv,(q_1,q_2))$

case 5:
Suppose $w = y_1x_1y_2x_2,\ x_1x_2 \in X_1^*, y_1,y_2 \in X_2^*$. Let $u = x_1x_2 \in X_1^*$ and $v = y_1y_2 \in X_2^*$.
The proof is similar to case 4.

case 6:
Let $w \in (X_1\cup X_2)^*$. Then $w = x_1y_1x_2y_2.......x_ny_n$ or $w = y_1x_1y_2x_2......y_nx_n$, $x_i \in X_1^*$, $y_i \in X_2^*$, x_i and y_i are non-empty strings $i = 1,2,....n-2$.
Let $w = x_1y_1x_2y_2.......x_ny_n$. If $n = 0,1$ or 2 the result is true by the previous cases. Now assume that the result is true for $n-1$.
Suppose $z = x_1y_1x_2y_2....x_{n-1}y_{n-1} \in (X_1\cup X_2)^*, n \geq 2$.
Let $u_1 = x_1x_2.....x_{n-1},\ \ v_1 = y_1y_2......y_{n-1}$
$u = u_1x_n\ \ v = v_1y_n$.
$(\varphi_1.\varphi_2)^-_*((p_1,p_2),x_1y_1x_2y_2..x_ny_n,(q_1,q_2)) = \wedge_{r_1,r_2\in Q_1\times Q_2}$
$\{(\varphi_1\circ\varphi_2)*^-((p_1,p_2),x_1y_1..x_{n-1}y_{n-1},(r_1,r_2))\vee$
$(\varphi_1\circ\varphi_2)^-_*((r_1,r_2),x_ny_n,(q_1,q_2))\}$
$= \wedge_{r_1,r_2\in Q_1\times Q_2}\{(\varphi_1\circ\varphi_2)^-_*((p_1,p_2),u_1v_1,(r_1,r_2))\vee$
$(\varphi_1\circ\varphi_2)^-_*((r_1,r_2),x_ny_n,(q_1,q_2))\}$
$= (\varphi_1\circ\varphi_2)^-_*((p_1,p_2),u_1v_1x_ny_n,(q_1,q_2))$
$= (\varphi_1\circ\varphi_2)^-_*((p_1,p_2),u_1x_nv_1y_n,(q_1,q_2))$
Now,
$(\varphi_1\circ\varphi_2)^+_*((p_1,p_2),x_1y_1x_2y_2..x_ny_n,(q_1,q_2)) = \vee_{r_1,r_2\in Q_1\times Q_2}$
$\{(\varphi_1\circ\varphi_2)*^+((p_1,p_2),x_1y_1..x_{n-1}y_{n-1},(r_1,r_2))\wedge$
$(\varphi_1\circ\varphi_2)^+_*((r_1,r_2),x_ny_n,(q_1,q_2))\}$
$= \vee_{r_1,r_2\in Q_1\times Q_2}\{(\varphi_1\circ\varphi_2)^+_*((p_1,p_2),u_1v_1,(r_1,r_2))$
$\wedge(\varphi_1\circ\varphi_2)^+_*((r_1,r_2),x_ny_n,(q_1,q_2))\}$
$= (\varphi_1\circ\varphi_2)^+_*((p_1,p_2),u_1v_1x_ny_n,(q_1,q_2))$
$= (\varphi_1\circ\varphi_2)^+_*((p_1,p_2),u_1x_nv_1y_n,(q_1,q_2))$.
The result now follows by induction.

3.4 Theorem

Let $M_i = (Q_i,X_i,\varphi_i)$ be a bffsm, $i = 1,2$ and let $X_1\cap X_2 = \phi$. Then the Cartesian composition $M_1\circ M_2$ is cyclic if and only if M_1 and M_2 are cyclic.

Proof.
Suppose M_1 and M_2 are cyclic, say $Q_1 = S(q_0)$ and $Q_2 = S(p_0)$ for some $q_0 \in Q_1,\ p_0 \in Q_2$. Let $(q,p) \in Q_1\times Q_2$. Then $\exists x \in X_1^*$ and $y \in X_2^*$ such that $\varphi^-_{1*}(q_0,x,q) < 0$ and $\varphi^-_{2*}(p_0,y,p) < 0$
$\varphi^+_{1*}(q_0,x,q) > 0$ and $\varphi^+_{2*}(p_0,y,p) > 0$. Thus
$(\varphi_1\circ\varphi_2)^-_*((q_0,p_0),xy,(q,p)) = \varphi^-_{1*}(q_0,x,q)\vee\varphi^-_{2*}(p_0,y,p) < 0$ and
$(\varphi_1\circ\varphi_2)^+_*((q_0,p_0),xy,(q,p)) = \varphi^+_{1*}(q_0,x,q)\wedge\varphi^-_{2*}(p_0,y,p) > 0$.
Hence $(q,p) \in S(q_0,p_0)$. Thus $Q_1\times Q_2 = S(q_0,p_0)$. Hence $M_1\circ M_2$ is cyclic.
Conversely,
Suppose $M_1\circ M_2$ is cyclic. Let $Q_1\times Q_2 = S(q_0,p_0)$ for some $(q_0,p_0) \in Q_1\times Q_2$. Let $q \in Q_1$ and $p \in Q_2$. Then $\exists w \in (X_1\cup X_2)^*$ such that $(\varphi_1\circ\varphi_2)^-_*((q_0,p_0),w,(q,p)) < 0$ and
$(\varphi_1\circ\varphi_2)^+_*((q_0,p_0),w,(q,p)) > 0$. Then by above Theorem, $\exists u \in X_1^*$ and $v \in X_2^*$ such that $\varphi^-_{1*}(q_0,u,q)\vee\varphi^-_{2*}(p_0,v,p) = (\varphi_1\circ\varphi_2)^-_*((q_0,p_0),w,(q,p)) < 0$ and
$\varphi^+_{1*}(q_0,u,q)\wedge\varphi^-_{2*}(p_0,v,p) = (\varphi_1\circ\varphi_2)^+_*((q_0,p_0),w,(q,p)) > 0$. Hence $\exists uv \in X_1^*$ and $v \in X_2^*$ such that $\varphi^-_{1*}(q_0,u,q) < 0$ and $\varphi^-_{2*}(p_0,v,p) < 0$.
$\varphi^+_{1*}(q_0,u,q) > 0$ and $\varphi^+_{2*}(p_0,v,p) > 0$. Thus $q \in S(q_0)$ and $p \in S(p_0)$. Hence $Q_1 = S(q_0)$ and $Q_2 = S(p_0)$. Therefore M_1 and M_2 are cyclic.

3.5 Theorem

Let $M_i = (Q_i,X_i,\varphi_i)$ be a bffsm, $i = 1,2$ and let $X_1\cap X_2 = \phi$. Then the Cartesian composition $M_1\circ M_2$ is retrievable if and only if M_1 and M_2 are retrievable.

Proof.
Suppose that M_1 and M_2 are retrievable.
Let $(q,p),(t,s) \in Q_1\times Q_2$ and $w \in (X_1\cup X_2)^*$ be such that $(\varphi_1.\varphi_2)^-_*((q,p),w,(t,s)) < 0$ and $(\varphi_1.\varphi_2)^+_*((q,p),w,(t,s)) > 0$. Let $w^* = uv$ standard form of w $u \in X_1^*,\ v \in X_2^*$. Then
$(\varphi_1\circ\varphi_2)^-_*((q,p),w,(t,s)) = (\varphi_1.\varphi_2)^-_*((q,p),uv,(t,s))$
$= \varphi^-_{1*}(q,u,t)\ \vee\ \varphi^-_{2*}(p,v,s)$
$(\varphi_1\circ\varphi_2)^+_*((q,p),w,(t,s)) = (\varphi_1.\varphi_2)^+_*((q,p),uv,(t,s))$
$= \varphi^+_{1*}(q,u,t)\ \wedge\ \varphi^+_{2*}(p,v,s)$. Thus
$\varphi^-_{1*}(q,u,t) < 0$ and $\varphi^-_{2*}(p,v,s) < 0$
$\varphi^+_{1*}(q,u,t) > 0$ and $\varphi^+_{2*}(p,v,s) > 0$. Since M_1 and M_2 are retrievable, $\exists u' \in X_1^*,\ v' \in X_2^*$ such that $\varphi^-_{1*}(t,u',q) < 0$ and $\varphi^-_{2*}(s,v',p) < 0$
$\varphi^+_{1*}(t,u',q) > 0$ and $\varphi^+_{2*}(s,v',p) > 0$. Thus
$(\varphi_1\circ\varphi_2)^-_*((t,s),u'v',(q,p)) < 0$
$(\varphi_1\circ\varphi_2)^+_*((t,s),u'v',(q,p)) > 0$. Hence $M_1\circ M_2$ is retrievable.
Conversely,
Suppose that $M_1\circ M_2$ is retrievable. Let $q,t \in Q_1$ and $y \in X_1^*$ be such that $\varphi^-_{1*}(q,y,t) < 0$ and $\varphi^+_{1*}(q,y,t) < 0$. Then $\forall\ s \in Q_2$
$(\varphi_1\circ\varphi_2)^-_*((q,s),y,(t,s)) = \varphi^-_{1*}(q,y,t) < 0$ and
$(\varphi_1\circ\varphi_2)^+_*((q,s),y,(t,s)) = \varphi^+_{1*}(q,y,t) > 0$. Thus $w \in (X_1\cup X_2)^*$ such that $(\varphi_1\circ\varphi_2)^-_*((t,s),w,(q,s)) < 0$ and $(\varphi_1\circ\varphi_2)^+_*((t,s),w,(q,s)) > 0$.
Let $w^* = uv$ standard form of w $u \in X_1^*,\ v \in X_2^*$. Then
$0 > (\varphi_1\circ\varphi_2)^-_*((t,s),w,(q,s)) = (\varphi^-_{1*}(t,u,q)) \vee (\varphi^-_{2*}(s,v,s))$ and
$0 < (\varphi_1\circ\varphi_2)^+_*((t,s),w,(q,s)) = (\varphi^+_{1*}(t,u,q)) \wedge (\varphi^-_{2*}(s,v,s))$.
Hence M_1 is retrievable. Similarly, M_2 is retrievable.

3.6 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then the Cartesian composition $M_1 \circ M_2$ is connected if and only if M_1 and M_2 are connected.

Proof.
Suppose that M_1 and M_2 are connected.
Let $(q, q'), (p, p') \in Q_1 \times Q_2$, where $q, p \in Q_1$ and $q', p' \in Q_2$. Now $\exists q_0, q_1, ...q_n \in Q_1$, $q = q_0, p = q_n$ and $\exists\ a_1, a_2, ...a_n \in X_1$ such that $\forall i = 1, 2, ...n$ either $\varphi_1^-(q_{i-1}, a_i, q_i) < 0$ and $\varphi_1^+(q_{i-1}, a_i, q_i) > 0$ or
$\varphi_1^-(q_i, a_i, q_{i-1}) < 0$ and $\varphi_1^+(q_i, a_i, q_{i-1}) > 0$ and $\exists\ q'_0, q'_1, ...q'_m \in Q_2$, $q' = q'_0 p' = q'_m$ and $\exists b_1, b_2, ...b_m \in X_2$ such that $\forall i = 1, 2, ..m$ either
$\varphi_2^-(q_{i-1}, b_i, q_i) < 0$ and $\varphi_2^+(q_{i-1}, b_i, q_i) > 0$ or
$\varphi_2^-(q_i, b_i, q_{i-1}) < 0$ and $\varphi_2^+(q_i, a_i, q_{i-1}) > 0$.
Consider the sequence of states $(q, q') = (q_0, q'_0), (q_1, q'_0), ...(q_n, q'_0), (q_n, q'_1), ...(q_n, q'_m) = (p, p') \in Q_1 \times Q_2$ and the sequence $a_1, a_2, ...a_n, b_1, b_2, ...b_m \in X_1 \cup X_2$.
Then $\forall\ i = 1, 2, ...n$ either
$(\varphi_1 \circ \varphi_2)^-((q_i, q'_0), a_i, q_{i-1}, q'_0)) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_i, q'_0), a_i, q_{i-1}, q'_0)) > 0$ or
$(\varphi_1 \circ \varphi_2)^-((q_{i-1}, q'_0), a_i, q_i, q'_0)) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_{i-1}, q'_0), a_i, q_i, q'_0)) > 0$
$\forall\ j = 1, 2, ...m$ either
$(\varphi_1 \circ \varphi_2)^-((q_n, q'_{j-1}), b_j, q_n, q'_j)) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_n, q'_{j-1}), b_j, q_n, q'_j)) > 0$ or
$(\varphi_1 \circ \varphi_2)^-((q_n, q'_j), b_j, q_n, q'_{j-1})) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_n, q'_j), b_j, q_n, q'_{j-1})) > 0$. Hence (q, q') and (p, p') are connected.
Conversely,
Suppose that $M_1 \circ M_2$ is connected. Let $q, p \in Q_1$ and let $r \in Q_2$. If $p = q$ then p and q are connected. Suppose $p \neq q$. then $\exists\ (q, r) = (q_0, p_0), (q_1, p_1)....(q_n, p_n) = (p, r) \in Q_1 \times Q_2$ and $a_1, a_2, ...a_n \in X_1 \cup X_2$ such that $\forall\ i = 1, 2, ...n$ either
$(\varphi_1 \circ \varphi_2)^-((q_{i-1}, p_{i-1}), a_i, (q_i, p_i)) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_{i-1}, p_{i-1}), a_i, (q_i, p_i)) > 0$ or
$(\varphi_1 \circ \varphi_2)^-((q_i, p_i), a_i, (q_{i-1}, p_{i-1})) < 0$
$(\varphi_1 \circ \varphi_2)^+((q_i, p_i), a_i, (q_{i-1}, p_{i-1})) > 0$. Clearly, if $q_{i-1} \neq q_i$, then $p_{i-1} = p_i$ and if $p_{i-1} \neq p_i$, then $q_{i-1} = q_i\ \forall i = 1, 2...n$.
Let $\{q = q_{i1}, q_{i2}, ...q_{ik} = p\}$ be the set of all distinct $q_i \in \{q_0, q_1, ...q_n\}$ and let $q_{i1}, q_{i2}, ...q_{ik}$ be the corresponding $q_i's$. Then $q_{i1}, q_{i2}, ...q_{ik} \in X_1$ and $\forall\ j = 1, 2, ...k$ either
$(\varphi_1 \circ \varphi_2)^-((q_{ij-1}, a_{ij}, q_{ij}) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_{ij-1}, a_{ij}, q_{ij}) > 0$ or
$(\varphi_1 \circ \varphi_2)^-((q_{ij}, a_{ij}, q_{ij-1}) < 0$ and
$(\varphi_1 \circ \varphi_2)^+((q_{ij}, a_{ij}, q_{ij-1}) > 0$. Thus q and p are connected. Hence M_1 is connected. Similarly, M_2 is connected.

Note
Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then the Cartesian composition $M_1 \circ M_2$ is strongly connected if and only if M_1 and M_2 are strongly connected.

3.7 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then the Cartesian composition $M_1 \circ M_2$ is commutative if and only if M_1 and M_2 are commutative.

Proof.
The proof is immediate from Theorem 4.

3.8 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. If M is commutative and strongly connected, then M is said to be perfect.

3.9 Definition.

Let $M = (Q, X, \varphi)$ be a bffsm. If M is state independent if $\forall\ q, p \in Q, \forall\ x, y \in X^*$ then
$\varphi_*^-(q, x, p) < 0 \Leftrightarrow \varphi_*^-(q, y, p) < 0$ and
$\varphi_*^+(q, x, p) > 0 \Leftrightarrow \varphi_*^+(q, y, p) > 0$

3.10 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then the Cartesian composition $M_1 \circ M_2$ is perfect if and only if M_1 and M_2 are perfect.

Proof.
The proof of the Theorem is obvious.

3.11 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Then the Cartesian composition $M_1 \circ M_2$ is state independent if and only if M_1 and M_2 are state independent.

Proof.
Suppose that M_1 and M_2 are state independent. Suppose that
$(\varphi_1 \circ \varphi_2)_*^-((q'_1, q'_2), w_1, (p'_1, p'_2)) < 0$
$(\varphi_1 \circ \varphi_2)_*^+((q'_1, q'_2), w_1, (p'_1, p'_2)) > 0$ and
$(\varphi_1 \circ \varphi_2)_*^-((q'_1, q'_2), w_2, (p'_1, p'_2)) < 0$
$(\varphi_1 \circ \varphi_2)_*^+((q'_1, q'_2), w_2, (p'_1, p'_2)) > 0$,
where$(q'_1, q'_2), (p'_1, p'_2) \in Q_1 \times Q_2$ and $w_1, w_2 \in (X_1 \cup X_2)^*$.
Now, $\exists\ u_1, u_2 \in X_1^+$ and $v_1, v_2 \in X_2^+$ such that
$(\varphi_1 \circ \varphi_2)_*^-((q'_1, q'_2), w_1, (p'_1, p'_2)) = \varphi_{1*}^-(q'_1, u_1, p'_1) \vee \varphi_{2*}^-(q'_2, v_1, p'_2)$
$(\varphi_1 \circ \varphi_2)_*^+((q'_1, q'_2), w_1, (p'_1, p'_2)) = \varphi_{1*}^+(q'_1, u_1, p'_1) \wedge \varphi_{2*}^+(q'_2, v_1, p'_2)$ and
$(\varphi_1\ circ\varphi_2)_*^-((q'_1, q'_2), w_2, (p'_1, p'_2)) = \varphi_{1*}^-(q'_1, u_2, p'_1) \vee \varphi_{2*}^-(q'_2, v_2, p'_2)$
$(\varphi_1 \circ \varphi_2)_*^+((q'_1, q'_2), w_2, (p'_1, p'_2)) = \varphi_{1*}^+(q'_1, u_2, p'_1) \vee \varphi_{2*}^+(q'_2, v_2, p'_2)$.
Thus
$\varphi_{1*}^-(q'_1, u_1, p'_1) < 0$ and $\varphi_{1*}^+(q'_1, u_1, p'_1) > 0$
$\varphi_{2*}^-(q'_1, v_1, p'_2) < 0$ and $\varphi_{2*}^+(q'_1, v_1, p'_2) > 0$
$\varphi_{1*}^-(q'_1, u_2, p'_1) < 0$ and $\varphi_{1*}^+(q'_1, u_2, p'_1) > 0$
$\varphi_{2*}^-(q'_2, v_2, p'_2) < 0$ and $\varphi_{2*}^+(q'_2, v_2, p'_2) > 0$.
Hence $\forall\ q_1, p_1 \in Q_1$,
$\varphi_{1*}^-(q_1, u_1, p_1) < 0 \Leftrightarrow \varphi_{1*}^-(q_1, u_2, p_1) < 0$
$\varphi_{1*}^+(q_1, u_1, p_1) > 0 \Leftrightarrow \varphi_{1*}^+(q_1, u_2, p_1) > 0$ and
$\forall\ q_1, p_1 \in Q_1$,
$\varphi_{2*}^-(q_2, v_1, p_2) < 0 \Leftrightarrow \varphi_{2*}^-(q_2, v_2, p_2) < 0$
$\varphi_{1*}^+(q_2, v_1, p_2) > 0 \Leftrightarrow \varphi_{2*}^+(q_2, v_2, p_2) > 0$. Hence
$\varphi_{1*}^-(q_1, u_1, p_1) \vee \varphi_{2*}^-(q_2, v_1, p_2) < 0 \Leftrightarrow \varphi_{1*}^-(q_1, u_2, p_1) \vee \varphi_{1*}^-(q_2, v_2, p_2) < 0$ and
$\varphi_{1*}^+(q_1, u_1, p_1) \wedge \varphi_{2*}^+(q_2, v_1, p_2) > 0 \Leftrightarrow \varphi_{1*}^+(q_1, u_2, p_1) \wedge \varphi_{2*}^+(q_2, v_2, p_2) > 0$. $\forall\ q_1, p_1 \in Q_1$ and $q_2, p_2 \in Q_2$. Thus
$(\varphi_1 \circ \varphi_2)_*^-((q_1, q_2), w_1, (p_1, p_2)) < 0 \Leftrightarrow$
$(\varphi_1 . \varphi_2)_*^-((q_1, q_2), w_2, (p_1, p_2)) < 0$
$(\varphi_1 \circ \varphi_2)_*^+((q_1, q_2), w_1, (p_1, p_2)) > 0 \Leftrightarrow$
$(\varphi_1 \circ \varphi_2)_*^+((q_1, q_2), w_2, (p_1, p_2)) > 0$,
$\forall\ q_1, p_1 \in Q_1$ and $q_2, p_2 \in Q_2$.
Hence $M_1 \circ M_2$ is state independent.
Conversely,
Suppose that $M_1 \circ M_2$ is state independent. Suppose
$\varphi_{1*}^-(q'_1, u_1, p'_1) < 0$ and $\varphi_{1*}^+(q'_1, u_1, p'_1) > 0$

$\varphi^-_{1*}(q'_1, u_2, p'_1) < 0$ and $\varphi^+_{1*}(q'_1, u_2, p'_1) > 0$ where $u_1, u_2 \in X_1^+$ and $q'_1, p'_1 \in Q_1$. Then $\forall\, s \in Q_2$
$(\varphi_1.\varphi_2)^-_*((q'_1, s), u_1, (p'_1, s)) = \varphi^-_{1*}(q'_1, u_1, p'_1) < 0$ and
$(\varphi_1.\varphi_2)^+_*((q'_1, s), u_1, (p'_1, s)) = \varphi^+_{1*}(q'_1, u_1, p'_1) > 0$
$(\varphi_1.\varphi_2)^-_*((q'_1, s), u_2, (p'_1, s)) = \varphi^-_{1*}(q'_1, u_2, p'_1) < 0$ and
$(\varphi_1.\varphi_2)^+_*((q'_1, s), u_2, (p'_1, s)) = \varphi^+_{1*}(q'_1, u_2, p'_1) > 0$. Thus $\forall\, q, p \in Q_1, s \in Q_2$
$(\varphi_1.\varphi_2)^-_*((q, s), v_1, (p, s)) < 0 \Leftrightarrow (\varphi_1.\varphi_2)^-_*((q, s), u_2, (p, s)) < 0$
$(\varphi_1.\varphi_2)^+_*((q, s), v_1, (p, s)) > 0 \Leftrightarrow (\varphi_1.\varphi_2)^+_*((q, s), u_2, (p, s)) > 0$.
Hence $\forall\, q, p \in Q_1$,
$\varphi^-_{1*}(q, u_1, p) < 0 \Leftrightarrow \varphi^-_{1*}(q, u_2, p) < 0$
$\varphi^+_{1*}(q, u_1, p) > 0 \Leftrightarrow \varphi^+_{1*}(q, u_2, p) > 0$. Thus M_1 is state independent. Similarly, M_2 is state independent.

3.12 Theorem

Let $M_i = (Q_i, X_i, \varphi_i)$ be a bffsm, $i = 1, 2$ and let $X_1 \cap X_2 = \phi$. Let $N_i = (T_i, X_i, \nu_i)$ be a submachine of M, $i = 1, 2$. Then $N_1 \circ N_2$ is a submachine of $M_1 \circ M_2$.
Conversely, if $N = (T_1 \times T_2, X_1 \cup X_2, \nu)$is a submachines N_1 of M_1 and N_2 of M_2 such that $N = N_1 \circ N_2$.

Proof.
Let $N_i = (T_i, X_i, \nu_i)$ be a submachine of $M_i, i = 1, 2$. Now, $N_1 \circ N_2 = (T_1 \times T_2, X_1 \cup X_2, \nu_1 \circ \nu_2)$ Let $(r, s) \in S(T_1 \times T_2)$. Then $\exists w \in (X_1 \cup X_2)^*$ $(p, q) \in T_1 \times T_2$ such that
$(\varphi_1.\varphi_2)^-_*((p, q), w, (r, s)) < 0$ and
$(\varphi_1.\varphi_2)^+_*((p, q), w, (r, s)) > 0$
Let $w^* = uv$ be the standard form of w, $u \in X_1^*, v \in X_2^*$. Now, $\varphi^-_{1*}(p, u, r) \vee \varphi^-_{2*}(q, v, s) = (\varphi_1.\varphi_2)^-_*((p, q), w, (r, s)) < 0$ and $\varphi^+_{1*}(p, u, r) \vee \varphi^+_{2*}(q, v, s) = (\varphi_1.\varphi_2)^+_*((p, q), w, (r, s)) > 0$. Hence, $r \in S(p) \subseteq S(T_1) = T_1$ and $s \in S(q) \subseteq S(T_2) = T_2$. Thus, $(r, s) \in T_1 \times T_2$. Hence, $S(T_1 \times T_2) \subseteq T_1 \times T_2$. Let $(p, q), (r, s) \in T_1 \times T_2, a \in X_1 \cup X_2$. Now,

$$(\nu_1 \circ \nu_2)^-((p, q), a, (r, s)) = \begin{cases} \nu_1^-(p, a, r) & \text{if } a \in X_1\ q = s \\ \nu_2^-(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$$(\nu_1 \circ \nu_2)^+((p, q), a, (r, s)) = \begin{cases} \nu_1^+(p, a, r) & \text{if } a \in X_1\ q = s \\ \nu_2^+(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \varphi_1^-(p, a, r) & \text{if } a \in X_1\ q = s \\ \varphi_2^-(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \varphi_1^+(p, a, r) & \text{if } a \in X_1\ q = s \\ \varphi_2^+(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$= (\varphi_1 \circ \varphi_2)^-((p, q), a, (r, s))$ and
$= (\varphi_1 \circ \varphi_2)^+((p, q), a, (r, s))$.
Hence, $(\varphi_1 \circ \varphi_2)^-|_{(T_1 \times T_2) \times (X_1 \cup X_2) \times (T_1 \times T_2)} = (\nu_1 \circ \nu_2)^-$ and $(\varphi_1 \circ \varphi_2)^+|_{(T_1 \times T_2) \times (X_1 \cup X_2) \times (T_1 \times T_2)} = (\nu_1 \circ \nu_2)^+$. Thus $(\nu_1 \circ \nu_2)$ is a submachine of $M_1 \circ M_2$.
Conversely, let $N = (T_1 \times T_2, X_1 \cup X_2, \nu)$ be a submachine of $M_1 \circ M_2$. Let $\nu_1^- = \varphi_1^-|_{T_1 \times X_1 \times T_1}$, $\nu_2^+ = \varphi_1^+|_{T_2 \times X_2 \times T_2}$, $N_1 = (T_1, X_1, \nu_1)$, and$N_2 = (T_2, X_2, \nu_2)$.

Let $p \in T_1$, $x \in X_1^*$, $r \in Q_1$ such that $\varphi_1^- * (p, x, r) < 0$ and $\varphi_1^+ * (p, x, r) > 0$. Let $t \in T_2$. Then,
$(\varphi_1 \circ \varphi_2)^- * ((p, t), x, (r, t)) = \varphi_1^- * (p, x, r) < 0$ and
$(\varphi_1 \circ \varphi_2)^+ * ((p, t), x, (r, t)) = \varphi_1^+ * (p, x, r) > 0$.
Thus, $(r, t) \in S(T_1 \times T_2) = T_1 \times T_2$. Hence, $r \in T_1$ and so $S(T_1) \subseteq T_1$. Thus, N_1 is a submachine of M_1. Similarly, N_2 is a submachine of M_2. Let $(p, q), (r, s) \in T_1 \times T_2$, and $a \in X_1 \cup X_2$. Now,
$\nu^-((p, q), a, (r, s)) = (\varphi_1 \circ \varphi_2)^-((p, q), a, (r, s))$

$$= \begin{cases} \varphi_1^-(p, a, r) & \text{if } a \in X_1\ q = s \\ \varphi_2^-(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \varphi_1^+(p, a, r) & \text{if } a \in X_1\ q = s \\ \varphi_2^+(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \nu_1^-(p, a, r) & \text{if } a \in X_1\ q = s \\ \nu_2^-(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \nu_1^+(p, a, r) & \text{if } a \in X_1\ q = s \\ \nu_2^+(p, a, s) & \text{if } a \in X_2\ p = r \\ 0 & \text{otherwise} \end{cases}$$

$= (\nu_1 \circ \nu_2)^-((p, q), a, (r, s))$ and
$= (\nu_1 \circ \nu_2)^+((p, q), a, (r, s))$.

4. CONCLUSION

In this paper, we introduce Cartesian composition in bipolar fuzzy finite state machines and discuss their properties.

5. REFERENCES

[1] Dubois. D., Prade. H. 1980. *Fuzzy Sets and Systems.* Mathematics in Science and Engineering, Academic Press, New York, 144.

[2] Jun. Y. B., Kavikumar. J. 2011. *Bipolar Fuzzy Finite State Machines.* Bull.Malays.Sci.Soc. 34 no. 1: 181-188.

[3] Kandel. A., Lee. S. C. *Fuzzy Switching and Automata Theory Applications.* Edward Arnold Publishers Ltd, London.

[4] Lee. K. J. 2000. *Bipolar-valued fuzzy sets and their operations.* Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, 307-312.

[5] Lee. K. J. 2004.*Comparison of interval valued fuzzy sets, Intuitionistic fuzzy sets, and bipolar-valued fuzzy sets.* J. Fuzzy Logic Intelligent Systems, no.14: 125-129.

[6] Mordeson. J. N., Malik. D. S. 2002. *Fuzzy automata and Languages.*-Theory and Applications, Chapman & Hall/ CRC Press.

[7] Santos. E. S. 1968.*General formulation of sequential machines.* Information and control, no.12: 5-10.

[8] Wee. W. G. 1967. *On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification.* Ph.D. Thesis Purude University.

[9] Zadeh. L. A. 1965. *Fuzzy sets.* Information and control, no. 8: 338-353.

[10] Zimmermann. H. J. 1985. *Fuzzy set theory and its applications.* International Series in Management Science/Operations research, Kluwer-Nijhoff, Boston, MA.