Co-complete $k-$partite Graphs

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ABSTRACT

A co-complete bipartite graph is a bipartite graph $G = (V_1, V_2, E)$ such that for any two vertices $u, v \in V_i$, $i = 1, 2$, there exists P_3 containing them. A co-complete k-partite graph $G = (V_1, V_2, ..., V_k, E)$, $k \geq 2$ is a graph with smallest number k of disjoint parts in which any pair of vertices in the same part are at distance two. The number of parts in co-complete k-partite graph G is denoted by $k(G)$. In this paper, we initiate a study of this class in graphs and we obtain a characterization for such graphs. Each set in the partition has subpartitions such that each set in the subpartition induces K_1 or any two vertices in this subpartition are joined by P_3 and this result has significance in providing a stable network.

General Terms:

1st General Term, 2nd General Term

Keywords:

Bipartite graph, Co-complete bipartite graph, Complete k-partite graph, Chromatic number, Co-complete k-partite graph

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. For graph theoretic terminology, we refer to [5].

In the theory of graphs, the friendship theorem [4] given below has a greater significance.

Theorem 1.1 (Friendship Theorem). Given a group of people, if each pair of individuals has a unique common friend, then there is an individual in the group who is a friend of everyone in the group.

Graph theoretically the friendship theorem in stated as below.

Theorem 1.2. A graph G, in which between each pair of vertices there exists a unique path of length two, has a vertex v adjacent with all the remaining vertices in the graph.

The study of existence of such a vertex v, as in the above theorem, in a graph or in a component of a graph, has importance and has applications in the theory of networks and communications [4]. Understandably, the vertex v is very important in having contact with the remaining vertices but, at the same time the removal of the vertex v shall cause disconnectedness of the graph. So, a network with dependency on such vertices for communication may be quite vulnerable.

A graph G is called m-partite if the set of all its vertices can be partitioned into m subsets $V_1, V_2, ..., V_m$, in such a way that any edge of graph G connects vertices from different subsets. The terms bipartite graph and tripartite graph are used to describe m-partite graphs for m equal to 2 and 3, respectively. A m-partite graph is called complete if any vertex $v \in V$ is adjacent to all vertices not belonging to the same partition as v. The symbol $K(n_1, n_2, ..., n_m)$ is used to describe a complete m-partite graph, with partition sizes equal to $|V_i| = n_i$ for $i = 1, 2, ..., m$. Moreover if $n_i = 1$ for all values of i, then the complete m-partite graph is denoted as K_m [6].

A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\beta_0(G)$ or β_0. Analogously, an independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the edge independence number $\beta_1(G)$ or β_1.

A r-coloring of a graph G is a vertex coloring of G that uses at most r colors. The smallest number r for which there exists a r-coloring of graph G is called the chromatic number of graph G and is denoted by $\chi(G)$. Such a graph G is called r-chromatic. Analogously, a graph G for which there exists an edge-coloring which requires r colors is called r-edge colorable, while such a coloring is called a r-edge coloring. The smallest number r for which there exists a r-edge-coloring of graph G is called the chromatic index of graph G and is denoted by $\chi'(G)$ [6].

Definition 1.3[7]. A co-complete bipartite graph is a bipartite graph $G = (V_1, V_2, E)$ such that for any two vertices $u, v \in V_i$, $i = 1, 2$ there exists P_3 containing them.

Since in this paper, we are interested in a graph having partition of set of vertices such that any pair of vertices in the same set contains P_3, we call the above co-complete bipartite graph as co-complete

2-partite graph. From this definition, we generalize the partition of set of vertices into k-partite graph. Analogous to co-complete 2-partite graph, we shall now define co-complete k-partite graph in the following.

Definition 1.4. A co-complete k-partite graph $G = (V_1, V_2, ..., V_k, E)$, $k \geq 2$ is a graph with smallest number k of disjoint parts in which any pair of vertices in the same part are at distance two. The number of parts in co-complete k-partite graph G is denoted by $k(G)$.

In this paper, we initiate a study of this class in graphs and we obtain a characterization for such graphs.

For example, the following graph is co-complete 4-partite graph.

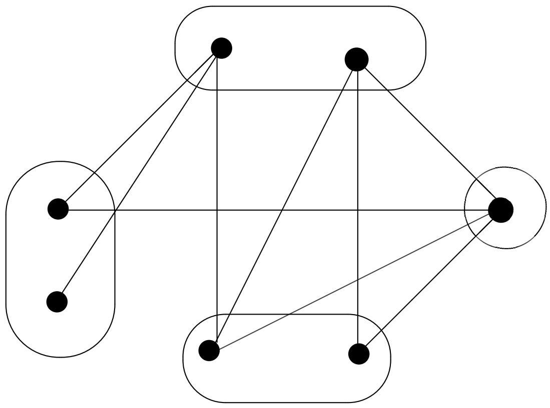

Figure 1

Remark 1.5. A graph G is a co-complete 2-partite graph if and only if G is co-complete bipartite graph.
We need the following theorem to prove some main results.

Theorem 1.6[3]. If $G = C_n$ is a cycle graph, then

$$\chi(G) = \begin{cases} 2, & \text{if } n \; is \; even \;; \\ 3, & \text{if } n \; is \; odd \;. \end{cases}$$

.

2. PRELIMINARY RESULTS

The relation between the co-complete k-partite graph G and the chromatic number χ of a graph G is $\chi(G) \leq k(G)$.

THEOREM 1. *Let G be a graph such that $\chi(G) = r$. Then G is a co-complete r-partite graph if and only if any pair of vertices in the same part in G are at distance two.*

PROOF. Let G be a graph such that $\chi(G) = r$. By the definition of chromatic number, r is the smallest number such that there exists a r- coloring of a graph G, so that r is the smallest size of partition of color classes of G. If any pair of vertices in the same part in G are at distance two, then by the definition of co-complete k-partite graph, G is co-complete r-partite graph.
Conversely, let G be a graph such that $\chi(G) = r$. Suppose that G is a co-complete r-partite graph, it follows that $\chi(G) = k(G)$. Thus any pair of vertices in the same part of G are at distance two.

□

In the following, we proceed to compute $k(G)$ for some standard graphs.

PROPOSITION 1. *For any complete m-partite graph $K(n_1, n_2, ..., n_m)$, $k(K(n_1, n_2, ..., n_m)) = m$.*

The converse of Proposition 1 need not be true, is as shown in Fig 1.

PROPOSITION 2. *For any complete graph K_n, $k(K_n) = n$.*

THEOREM 2. *For any cycle graph C_n, if $n \leq 6$, then $k(C_n) \leq 3$ and if $n \geq 7$, then*

$$k(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 1 \; (mod \; 2); \\ \frac{n}{2}, & n \equiv 0 \; (mod \; 4); \\ \frac{n}{2} + 1, & n \equiv 2 \; (mod \; 4). \end{cases}$$

PROOF. For $n = 3, 4, 5, 6$, options of a co-complete k-partite graph of C_n are as shown in Fig. 2.

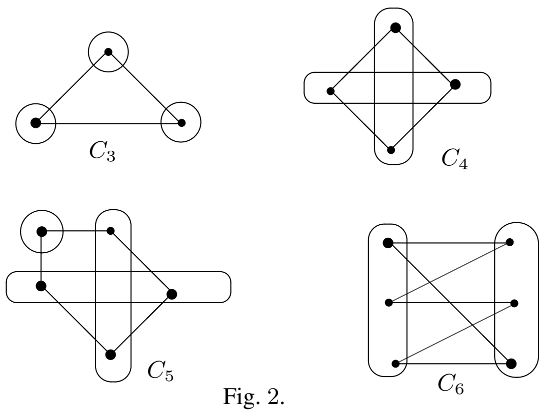

Fig. 2.

Further, let $V(C_n) = \{v_1, v_2, ..., v_n\}$, $n \geq 7$ and let P be a collection of disjoint parts of vertices of a co-complete k-partite graph. We consider the following cases.
Case 1 : n is odd. We consider the following subcases.
Subcase 1.1 : $n \equiv 1 \; (mod \; 4)$. Then
$P = \{\{v_i, v_{i+2}\} : i = 2s + 1, s = 0, 2, ..., \frac{n-5}{2}\} \cup \{\{v_i, v_{i+2}\} : i = 2s, s = 1, 3, ..., \frac{n-3}{2}\} \cup \{v_n\}$.
Subcase 1.2 : $n \equiv 3 \; (mod \; 4)$. Then
$P = \{\{v_i, v_{i+2}\} : i = 2s + 1, s = 0, 2, ..., \frac{n-3}{2}\} \cup \{\{v_i, v_{i+2}\} : i = 2s, s = 1, 3, ..., \frac{n-5}{2}\} \cup \{v_{n-1}\}$.
Therefore from the above subcases, there are $\frac{n-1}{2}$ parts, each having two vertices and one part with one vertex. Hence the number of parts of odd cycle is $\frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$.
Case 2 : n is even. We consider the following subcases.
Subcase 2.1 : $n \equiv 0 \; (mod \; 4)$. Then
$P = \{\{v_i, v_{i+2}\} : i = 2s + 1, s = 0, 2, ..., \frac{n-4}{2}\} \cup \{\{v_i, v_{i+2}\} : i = 2s, s = 1, 3, ..., \frac{n-2}{2}\}$. Hence there are $\frac{n}{2}$ parts each having two vertices.
Subcase 2.2 : $n \equiv 2 \; (mod \; 4)$. Then
$P = \{\{v_i, v_{i+2}\} : i = 2s + 1, s = 0, 2, ..., \frac{n-6}{2}\} \cup \{v_{n-1}\} \cup \{\{v_i, v_{i+2}\} : i = 2s, s = 1, 3, ..., \frac{n-4}{2}\} \cup \{v_n\}$. Therefore, there are $\frac{n-2}{2}$ parts, each having two vertices and two parts, each having one vertex. Hence the number of parts in this case is $\frac{n-2}{2} + 2 =$

$\frac{n}{2}+1$. Thus

$$k(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 1\ (mod\ 2); \\ \frac{n}{2}, & n \equiv 0\ (mod\ 4); \\ \frac{n}{2}+1, & n \equiv 2\ (mod\ 4). \end{cases}$$

□

PROPOSITION 3. *For any path P_n,*

$$k(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 1\ (mod\ 2); \\ \frac{n}{2}, & n \equiv 0\ (mod\ 4); \\ \frac{n}{2}+1, & n \equiv 2\ (mod\ 4). \end{cases}$$

PROOF. Proof follows by Theorem 2. □

THEOREM 3. *For any wheel $W_{1,n}$,*

$$k(W_{1,n}) = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

.

PROOF. Proof follows by Theorem 1.6 and Theorem 2. □

PROPOSITION 4. *For a generalized wheel graph $W_{m,n}$,*

$$k(W_{m,n}) = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 4, & \text{if } n \text{ is odd.} \end{cases}$$

.

THEOREM 4. *Let G be a graph of order 3. Then G is connected co-complete tripartite graph if and only if G is triangle.*

THEOREM 5. *Let G be a graph of order n. Then G is co-complete n-partite graph if and only if G is one of the following graphs: K_n, $\overline{K}_n$ or $\cup[K_{r_i} \cup \overline{K}_{s_j}]$, $0 \leq r_i \leq n$, $0 \leq s_j \leq n$ and $\sum(r_i + s_j) = n$.*

PROOF. Let $G = (V, E)$ be a graph of order n. Suppose that G is a co-complete n-partite graph. Then any part of G contains single vertex, so that any two vertices of any different parts are not at distance two. Therefore any two vertices in the different parts are either adjacent or they are not connected by any path. Thus G is one of the following graphs: K_n, $\overline{K}_n$ or $\cup[K_{r_i} \cup \overline{K}_{s_j}]$, $0 \leq r_i \leq n$, $0 \leq s_j \leq n$ and $\sum(r_i + s_j) = n$.
Conversely, let G be a graph of order n and suppose that G is one of the following graphs: K_n, $\overline{K}_n$ or $\cup[K_{r_i} \cup \overline{K}_{s_j}]$, $0 \leq r_i \leq n$, $0 \leq s_j \leq n$ and $\sum(r_i + s_j) = n$. Then any two vertices in G are either adjacent or they are not connected by any path. So, there are no two vertices in G at distance two. Therefore by the definition of co-complete k-partite graph, G is a co-complete n-partite graph.

□

THEOREM 6. *If $G \cong K_n - F$, where F is a set of independent edges in G, then G is a co-complete $(n - |F|)$-partite graph.*

In fact, since β_1 is the edge independence number of G, we have the co-complete $(n - \beta_1)$-partite graph obtained by deleting maximum independent edges of the complete graph as in the following theorem.

THEOREM 7. *If $G \cong K_n - F$, where F is set of maximum number of independent edges in G, then G is a co-complete $(n - \beta_1)$-partite graph.*

A balanced k-partite graph is a k-partite graph in which all k sets contain the same number of vertices. We have the following theorem.

THEOREM 8. *Let $G = (V_1, V_2, ..., V_k, E)$ be a balanced k-partite graph with $|V_i| = n$, $i = 1, 2, ..., k$ such that $\delta(G) > \lceil \frac{kn}{2} \rceil$. Then G is a co-complete k-partite graph.*

Next result show that a connected spanning subgraph of a complete k-partite graph $K(n_1, n_2, ..., n_k)$, $n_i \geq 2$, for each $i \leq k$ is a co-complete k-partite graph.

THEOREM 9. *Let $G = K(n_1, n_2, ..., n_k)$, $n_i \geq 2$ for each $i \leq k$ be a complete k-partite graph and let $S \subset E(G)$ such that $|S| < k$ and $n_i \geq 2$, for each $i \leq k$. Then every connected spanning subgraph $G - S$ is a co-complete r-partite graph, $r \leq k$.*

We observe that, the condition $n_i \geq 2$ of Theorem 9 is necessary for any non balanced complete k-partite graph having at least one part containing single vertex. Because there may be a connected spanning subgraph H of a complete k-partite graph such that H is not co-complete k-partite graph as shown in Fig. 3.

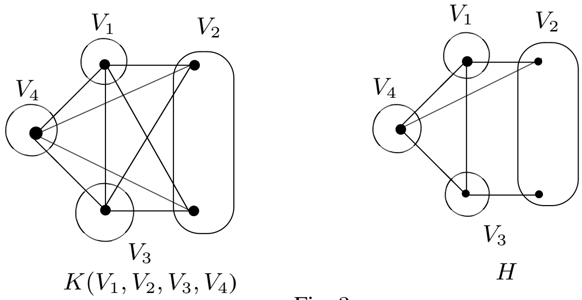

Fig. 3.

In fact, any connected spanning subgraph of a balanced complete k-partite graph satisfying the condition of Theorem 8 is a co-complete k-partite graph, as in the following theorem.

THEOREM 10. *Let $K(n_1, n_2, ..., n_k)$ be a balanced complete k-partite graph and let $H(n_1, n_2, ..., n_k)$ be a connected spanning subgraph of $K(n_1, n_2, ..., n_k)$ such that $\delta(H) > \lceil \frac{kn}{2} \rceil$. Then H is a co-complete k-partite graph.*

THEOREM 11. *Let $G = (V_1, V_2, ..., V_k, E)$ be a k-partite graph such that $|V_i| \geq 3$ for all $i \leq k$. If the subgraph $H = (V_1 - v_{i_1}, V_2 - v_{i_2}, ..., V_k - v_{i_k}, E')$ in G is a co-complete k-partite graph for each $v_{i_j} \in V_j$ and for each j, $j \leq k$, then G is a co-complete k-partite graph.*

DEFINITION 12. *A k-partite graph $G = (V_1, V_2, ..., V_k, E)$ which is not co-complete k-partite graph is said to be almost co-complete k-partite graph if and only if there exists exactly one i such that $V_{i_1} \subset V_i \subset V(G)$ and $H(V_1, V_2, ..., V_{i-1}, V_i - V_{i_1}, V_{i_1}, V_{i+1}, ..., V_k, E)$ is co-complete $(k+1)$-partite graph.*

The almost co-complete k-partite graph G is as shown in Fig. 4.

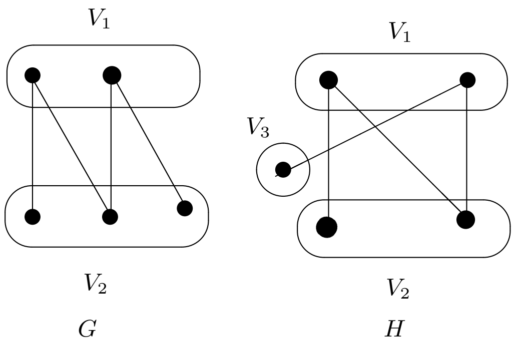

Fig.4.

THEOREM 13. *Let $G = (V_1, V_2, ..., V_k, E)$ be a k-partite graph with smallest number k. Then G is almost co-complete k-partite graph if and only if there exists exactly one i such that $V_{i_1} \subset V_i \subset V(G)$ and $d(u_{i_1}, v_{i_1}) = 2$, for any $u_{i_1}, v_{i_1} \in V_{i_1}$ and if $N(u_j) \cap N(v_j) \not\subseteq V_{i_1}$, for any $u_j, v_j \in V_j$, $j \neq i$, then $H = (V(G) - V_{i_1}, E')$, $E' \subseteq E$ is a co-complete k-partite graph.*

DEFINITION 14. *Let G be a co-complete k-partite graph. A vertex v of G is said to be critical vertex if $G - v$ is co-complete $(k-1)$-partite graph. A graph G is called a critical co-complete k-partite graph if and only if every vertex of G is a critical vertex.*

We note that, a complete graph and its complement both are critical co-complete k-partite graphs. In fact, if G is a co-complete k-partite graph such that every part in G contains single vertex, then G is a critical co-complete k-partite graph.

PROPOSITION 5. *Let $G = K(n, n, ..., n)$ be a balanced complete m-partite graph. Then $\overline{G}$ is critical co-complete nm-partite graph.*

Analogous to the friendship theorem, we have the following theorem with reference to the co-complete bipartite graph.

THEOREM 15. *Given a group of people. Then there exist two factions such that either none of them are friends or any two individuals from the same faction have at least one common friend and any two individuals from different factions are either friends or the friend of one individual is not friend with other.*

3. SOME CHARACTERIZATIONS

In this section, a characterization of a co-complete k-partite graph is given.

THEOREM 16. *A graph G is a co-complete k-partite graph if and only if the vertex set V of G can be partitioned into parts $V_1, V_2, ..., V_k$ such that $\langle V_i \rangle$ is either K_1 or each pair of vertices of V_i are at distance two and for each $i \neq j$, there exist two vertices $v \in V_i$ and $u \in V_j$ such that $d(v, u) \neq 2$.*

PROOF. Let $G = (V_1, V_2, ..., V_k, E)$ be a graph. Suppose that $\langle V_i \rangle$ is either K_1 or each pair of vertices of V_i are at distance two and for each $i \neq j$, there exist two vertices $v \in V_i$ and $u \in V_j$ such that $d(v, u) \neq 2$. It follows that for any $u, w \in V_i$, there exists $v \in V_j$ such that (u, v, w) is a path. Since for each $i \neq j$, there exist two vertices $v \in V_i$ and $u \in V_j$ such that $d(v, u) \neq 2$, k is minimum number. Therefore G is a co-complete k-partite graph.
Conversely, suppose that $G = (V_1, V_2, ..., V_k, E)$ is a co-complete k-partite graph, then G is with minimum number of parts such that $V_i = \{u\}$, for some i, $i \leq k$, or for any $u, v \in V_i$, there exists $w \in V_j$, $j \neq i$ such that (u, w, v) is a path. Therefore each part V_i induces either K_1 or each pair of vertices of V_i are at distance two and for each $i \neq j$, there exist two vertices $v \in V_i$ and $u \in V_j$ such that $d(v, u) \neq 2$. □

4. CONCLUSION

The concept of a co-complete k-partite graph can be extended to other graph valued functions, namely, co-complete k-partite line graph, co-complete k-partite middle graph, co-complete k-partite total graph, etc.

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