A New Iterative Scheme for Non-expansive and Monotone Lipschitz Continuous Mappings

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ABSTRACT
The aim of paper is to prove a weak convergenceresult for finding a common of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. Using an example in C++, validity of the result will be proved. Also, we shall find a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of a monotone, Lipschitz continuous mapping.

KeyWords
Fixed Points, Hilbert Spaces, Monotone Mappings, Nonexpansive Mappings, Variational Inequalities.2000 Mathematics Subject Classification: Primary 47H05, 47J05, 47J25

1. INTRODUCTION
Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively. Let C be a closed convex subset of H. The variational inequality problem is to find u ϵ C such that $< Au, v - u > \geq 0$, $\forall$ v ϵ C.

The set of solutions of variational inequality problem VI(C, A) is denoted by Ω. The variational inequality problem has been extensively studied in literature, see, for example,[1, 2, 5, 10] and references therein.

Definitions. Let A: C → H be a mapping of C into H. A is called monotone if

$< Au - Av, u - v > \geq 0 \quad \forall$ u, v ϵ C.

A is called α-inverse-strongly-monotone [1, 5, 9] if there exists a positive real number α such that

$< Au - Av, u - v > \geq \alpha \| Au - Av \|^2 \quad \forall$ u,v ϵ C.

It is easy to see that an α-inverse-strongly mapping A is monotone and Lipschitz continuous but converse is not true. A mapping S : C→C is called non-expansive [10-11] if

$\| Su - Sv \| \leq \| u - v \| \quad \forall$ u,v ϵ C.

We denote by F(S) the set of fixed points of S. A mapping S : C→ C is called Lipschitz continuous if there exists a real number L > 0 such that $\| Su - Sv \| \leq L \| u - v \| \quad \forall$ u,v ϵ C.

Takahashi and Toyoda [12] introduced the following iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse-strongly-monotone mapping in a real Hilbert space.

Theorem 1.1**.** [12] Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that F (S) ∩ VI(C, A) ≠ ϕ. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n)$, (1.1) for every n = 0,1,2,…….. where $\{\lambda_n\} \subset [a,b]$ for some a,b ϵ (0, 2α) and $\{\alpha_n\} \subset [c,d]$ for some c,d ϵ (0,1). Then, the sequence $\{x_n\}$ converges weakly to some point z ϵ F(S) ∩ VI(C,A), where, $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,\ A)} x_n$. Further Iiduka and Takahashi [4], introduced an iterative scheme to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a Hilbert space.

Theorem 1.2**.** [4] Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that F (S) ∩ VI(C,A) ≠ ϕ. Suppose $x_1 = x \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = P_C(\alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n)), \qquad (1.2)$$

for every n = 1,2,…….. where $\{\alpha_n\}$ is a sequence in [0, 1) and $\{\lambda_n\}$ is a sequence in [0, 2α]. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \subset [a,b]$ for some a, b with 0 < a < b < 2α,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,\ A)} x$.

Motivated by above results, we shall prove the weak convergence theorem for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping in a real Hilbert space. Also we shall give a numerical example to show the existence of solution. Further, we consider the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of a monotone and Lipschitz continuous mapping.

2. PRELIMINARIES
Let H be a real Hilbert space with inner product < . , . > and norm $\|.\|$. Let C be a closed convex subset of H. We shall write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. It is well known that for any x ϵ H, there exists a unique nearest point in C, such that

$\| u - P_C x \| = \inf\{ \| u - y \| : y \in C\}$

P_C is called the metric projection of H onto C.

P_C is characterized by the properties:

$P_C x \in C$,

$< x - P_C x, y - P_C x > \leq 0$, for all x ϵ H, y ϵ C,

$\|x - y\|^2 \geq \| x - P_C x\|^2 + \| y - P_C x\|^2$, for all $x \in H$, $y \in C$. The metric projection P_C of H onto C satisfies

$< x - y, P_C x - P_C y > \geq \| P_C x - P_C y\|^2$, for every $x,y \in H$,

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see that $u \in \Omega \Leftrightarrow u = P_C(u - \lambda Au)$, for any λ

It is known that H satisfies the Opial condition [7] that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality, $\lim_{n\to\infty} inf \|x_n - x\| < \lim_{n\to\infty} inf \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. We also know that, if $\{x_n\}$ is sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there holds that $x_n \to x$.

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x,y \in H$, $f \in Tx$ and $g \in Ty$ imply $< x - y, f - g > \geq 0$. A monotone mapping $T: H \to 2^H$ is maximal if its graph G(T) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $< x - y, f - g > \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let $A : C \to H$ be a monotone, k – Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, that is $N_C v = \{w \in H : < v - u, w > \geq 0, \forall u \in C\}$

$$\text{Define, } Tv = \begin{cases} Av + N_C v, & if\ v \in C \\ \phi, & if\ v \notin C \end{cases}$$

Then, T is maximal monotone [8-9] and $0 \in Tv$ if and only if $v \in VI(C,A)$.

Now we give some lemmas, which will be used in the proof of result.

Lemma 2.1 [6] Let C be a closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself such that $F(S) \neq \phi$. Then $F(S) = F(P_C S)$.

Lemma 2.2. [12] Let H be a real Hilbert space and let D be a non empty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,
$\|x_{n+1} - u\| \leq \|x_n - u\|$,
for every n = 0,1,2,……… Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Lemma 2.3. [3] Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $T: C \to H$ be a nonexpansive mapping. Then, the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I - T)x_n \to y$ imply that $x \in C$ and $(I - T)x = y$.

3. MAIN RESULT

Now, we prove a weak convergence theorem for a nonexpansive mapping and a monotone mapping.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \phi$. Let $\{x_n\}$ be sequence generated by $x_0 = x \in C$,

$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)), \qquad (3.1.1)$$

for every n = 0,1,2,…….. where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

$\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \cap VI(C,A)$, where, $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Proof. Let $y_n = P_C(x_n - \lambda_n Ax_n)$ for every n = 0,1,2,3,…………

Let $u \in F(S) \cap VI(C,A)$.

Now,

$\|y_n - u\|^2 \leq \| x_n - \lambda_n Ax_n - u\|^2 - \| x_n - \lambda_n Ax_n - y_n\|^2$

$= \|x_n - u\|^2 + \|\lambda_n Ax_n\|^2 - 2\lambda_n < x_n - u, Ax_n > - \|x_n - y_n\|^2 - \|\lambda_n Ax_n\| + 2\lambda_n < x_n - y_n, Ax_n >$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\lambda_n < x_n - y_n - x_n + u, Ax_n >$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\lambda_n < Ax_n, u - y_n >$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2\lambda_n(< Ax_n - Au, u - x_n > + < Au, u - x_n > + < Ax_n, x_n - y_n >)$

$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2 < \lambda_n Ax_n, x_n - y_n >$

$= \|x_n - u\|^2 - \|x_n - y_n\|^2 + 2 < x_n + \lambda_n Ax_n - y_n, x_n - y_n > + < y_n - x_n, x_n - y_n >$

$\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|x_n - y_n\|^2$

$= \|x_n - u\|^2 - 2\|x_n - y_n\|^2$

$\leq \|x_n - u\|^2$ for every n = 0,1,2, ….. (1)

Now, using (1), we obtain,

$\|x_{n+1} - u\|^2$

$= \|P_C(\alpha_n x_n + (1 - \alpha_n)Sy_n) - P_C Su\|^2$

$\leq \|(\alpha_n x_n + (1 - \alpha_n)Sy_n) - Su\|^2$

$= \|\alpha_n(x_n - u) + (1 - \alpha_n)(Sy_n - Su)\|^2$

$= \|\alpha_n(x_n - u) + (1 - \alpha_n)(y_n - u)\|^2$

$\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|y_n - u\|^2$

$\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)(\|x_n - u\|^2 - 2\|x_n - y_n\|^2)$

$\leq \|x_n - u\|^2 - 2(1 - \alpha_n)\|x_n - y_n\|^2$ (2)

$\leq \|x_n - u\|^2$

So, $\|x_{n+1} - u\| \leq \|x_n - u\|$

So, there exists $c = \lim_{n\to\infty} \|x_n - u\|$ and hence the sequences $\{x_n\}$, $\{y_n\}$ are bounded. From equation (2),

$2(1 - \alpha_n)\|x_n - y_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2$

Since, $\lim_{n\to\infty} \|x_n - u\|^2 = \lim_{n\to\infty} \|x_{n+1} - u\|^2$

So we obtain, $x_n - y_n \to 0$.

Since A is lipschitz continuous, so $Ax_n - Ay_n \to 0$.

Now,

$\|y_n - y_{n-1}\| \leq \|P_C(x_n - \lambda_n Ax_n) - P_C(x_{n-1} - \lambda_{n-1}Ax_{n-1})\|$

$\leq \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_{n-1}Ax_{n-1})\|$

$= \|(x_n - x_{n-1}) - (\lambda_n Ax_n - \lambda_{n-1}Ax_{n-1})\|$

$= \|(x_n - x_{n-1}) - (\lambda_n Ax_n - \lambda_n Ax_{n-1} + \lambda_n Ax_{n-1} - \lambda_{n-1}Ax_{n-1})\|$

$\leq \|(x_n - x_{n-1})\| + \lambda_n\| Ax_n - Ax_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Ax_{n-1}\|$

$\leq (1 + \lambda_n k)\|(x_n - x_{n-1})\| + |\lambda_n - \lambda_{n-1}| \|Ax_{n-1}\|$

Using the given conditions, we obtain,

$\|y_n - y_{n-1}\| \le \|(x_n - x_{n-1})\|$. (3)

Since,

$\|x_{n+1} - x_n\| \le \|x_{n+1} - y_n\| + \|y_n - x_n\| \to 0$ as $n \to \infty$, so (3) implies,

$\|y_n - y_{n-1}\| \to 0$ as $n \to \infty$. (4)

Now, using (4) and given conditions:

$\|x_n - P_C Sy_n\| \le \|x_n - P_C Sy_{n-1}\| + \|P_C Sy_{n-1} - P_C Sy_n\|$

$\le \alpha_{n-1}\|x_{n-1} - Sy_{n-1}\| + \|y_{n-1} - y_n\| \to 0$ as $n \to \infty$.

Also, $\|y_n - P_C Sy_n\| \le \|x_n - P_C Sy_n\| + \|y_n - x_n\| \to 0$ as $n \to \infty$. (5)

As $\{x_n\}$ is bounded, we have that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to z. Then, we shall prove that $z \in F(S) \cap VI(C,A)$.

Firstly, we shall show that $z \in VI(C,A)$.

Since $x_n - y_n \to 0$, we have, $y_{n_i} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & if\ v \in C \\ \phi, & if\ v \notin C \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, so we get

$< v - y_n, w - Av > \ \ge 0$.

On the other hand, from $y_n = P_C(x_n - \lambda_n Ax_n)$, we have that

$< x_n - \lambda_n Ax_n - y_n, y_n - v > \ \ge 0$ and hence, $< v - y_n, (y_n - x_n)/\lambda_n + Ax_n > \ \ge 0$.

Therefore, we have

$< v - y_{n_i}, w > \ \ge \ < v - y_{n_i}, Av >$

$\ge \ < v - y_{n_i}, Av > - < v - y_{n_i}, (y_{n_i} - x_{n_i})/\lambda_{n_i} + Ax_{n_i} >$

$= \ < v - y_{n_i}, Av - Ax_{n_i} - (y_{n_i} - x_{n_i})/\lambda_{n_i} >$

$= \ < v - y_{n_i}, Av - Ay_{n_i} > + < v - y_{n_i}, Ay_{n_i} - Ax_{n_i} > - < v - y_{n_i}, (y_{n_i} - x_{n_i})/\lambda_{n_i} >$

$\ge \ < v - y_{n_i}, Ay_{n_i} - Ax_{n_i} > - < v - y_{n_i}, (y_{n_i} - x_{n_i})/\lambda_{n_i} >$.

Hence we get, $< v - z, w > \ \ge 0$, as $i \to \infty$.

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C,A)$.

Next we shall show that $z \in F(P_C S)$.

Since $y_{n_i} \rightharpoonup z$ and $\|y_n - P_C Sy_n\| \to 0$ as $n \to \infty$, so by demiclosedness of $I - S$, we get $z \in F(P_C S)$.

By using Lemma 2.1, we obtain $z \in F(S)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$, such that $\{x_{n_j}\} \rightharpoonup z'$. Then, $z' \in F(S) \cap VI(C,A)$. Let us show that $z = z'$. Assume that $z \neq z'$. From the Opial condition, we have

$\lim_{n\to\infty} \|x_n - z\|$

$= \lim_{i\to\infty} inf \|x_{n_i} - z\|$

$< \lim_{i\to\infty} inf \|x_{n_i} - z'\|$

$= \lim_{n\to\infty} \|x_n - z'\| = \lim_{j\to\infty} inf \|x_{n_j} - z'\| < \lim_{j\to\infty} inf \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|$.

This is a contradiction. Thus we have $z = z'$. This implies, $x_n \rightharpoonup z \in F(S) \cap VI(C,A)$.

Now, put $u_n = P_{F(S) \cap VI(C,A)} x_n$.

We show that $z = \lim_{n\to\infty} u_n$.

From, $u_n = P_{F(S)\cap VI(C,A)} x_n$ and $z \in F(S) \cap VI(C,A)$, we have, $< z - u_n, u_n - x_n > \ \ge 0$.

By lemma 2.2, $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap VI(C,A)$.

Then, we have, $< z - z_0, z_0 - z > \ \ge 0$.

And hence $z = z_0$.

Remark. We know that every inverse-strongly-monotone mapping is monotone and Lipschitz continuous, but converse is not true. So our result is the generalization of Theorem 1.1 given by Takahashi and Toyoda [12] and Theorem 1.2 given by Iiduka and Takahashi [4].

4. NUMERICAL EXAMPLE

Now we prove the validity of our main result with the help of a numerical example and a program in C++.

Example 4.1. Let H = R be a real Hilbert space with usual inner product defined on R. Let C = [0, 1] be a closed convex subset of H. Let $Ax = \frac{x^2}{1+x}$ be a monotone, Lipschitz-continuous mapping of C into H and let $Sx = \frac{x}{2}$ be a nonexpansive mapping of C into itself. Let $\{\alpha_n\} = \{\lambda_n\} = \frac{1}{n+1}$ be two sequences satisfying the conditions of theorem 3.1.

Now using the iterative scheme (3.1.1), we obtain the following data given in table 4.1. when initial approximation is taken as $x_0 = 0.1$.

Table 4.1

n	x_{n+1}	Sx_n	$P_C(x_n - \lambda_n Ax_n)$
0	0.1	0.05	0.1
1	0.045455	0.045455	0.090909
3	0.022727	0.022727	0.045455
98	2.868585e-31	2.868585e-31	5.73717e-31
99	1.434293e-31	1.434293e-31	2.868585e-31
100	7.171463e-32	7.171463e-32	1.434293e-31
279	9.359196e-86	9.359196e-86	1.871839e-85
280	4.679598e-86	4.679598e-86	9.359196e-86
770	1.463897e-233	1.463897e-233	2.927795e-233
771	7.319487e-234	7.319487e-234	1.463897e-233
1071	4.940656e-324	4.940656e-324	9.881313e-324
1072	0	0	4.940656e-324
1073	0	0	0
1074	0	0	0

Thus we see that the sequence $\{x_n\}$ converges to z = 0, which is a fixed point of S as well as a solution of variational inequality problem.

5. APPLICATION

Theorem: Let H be a real Hilbert space. Let A be a monotone k-Lipschitz continuous mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \phi$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in H$ and let

$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)S(x_n - \lambda_n Ax_n))$,

for every n = 0,1,2,…….. where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \cap A^{-1}0$, where, $z = \lim_{n\to\infty} P_{F(S) \cap A^{-1}0} x_n$.

Proof. We have $A^{-1}0 = VI(H,A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result.

Remark. Notice that $F(S) \cap A^{-1}0 \subset VI(F(S),A)$. See Yamada [13] for the case when A is strongly monotone and Lipschitz continuous mapping of H into itself.

6. REFERENCES

[1] Browder F E and Petryshyn W V (1967) Construction of Fixed Points of non-linear Mapping in Hilbert Spaces J Math Anal Appl **20** 197-228.

[2] Browder F E (1965) Fixed-point Theorems for Noncompact Mappings in Hilbert Space Proc Natl Acad Sci U S A **53** 1272-1276.

[3] Goebel K and Kirk W A (1990) Topics in Metric Fixed Point Theory vol. **28** of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK.

[4] Iiduka H and Takahashi W (2004), Stronge convergence theorems for nonexpansive nonself mappings and inverse strongly monotone mappings, J convex Anal, vol 11, 69-79.

[5] Liu F and Nashed M Z (1998) Regularization of Non-linear Ill- posed Variational inequalities and convergence rates,Set-valued Anal **6** 313-344.

[6] Matsushita S and Kuroiwa, Approximation of fixed points of nonexpansive nonself mappings, Sci Math Jpn, vol 57(2003), 171-176.

[7] Opial Z (1967) Weak convergence of the sequence of successive Approximations of Nonexpansive Mappings Bull Am Math Soc **73** 591-597.

[8] Rockafellar R T (1970) On the Maximality of Sums of nonlinear Monotone Operators Trans Amer Math Soc **149** 75-88.

[9] Schu J (1991) Weak and Strong Convergence to fixed points of Asymptotically nonexpansive Mappings Bull Aust Math Soc **43**, 153-159.

[10] Takahashi W (2000) Non-linear functional Analysis Yokohama Publisher, Yokohama Japan.

[11] Takahashi W, Tamura T (1998) Convergence Theorems for a pair of nonexpansive Mapping J Convex Anal **5** 45-56.

[12] Takahashi W and Toyoda M (2003) Weak Convergence Theorems for Non-expansive Mappings and Monotone Mappings J Optim Theory Appl **118** 417-428.

[13] Yamda I (2001) The Hybrid Steepest-Descent Method for the Variational Inequality Problem over the Intersection of fixed-point Sets of nonexpansive Mapping, inherently Parallel Algorithms in Feasibility and Optimization and their applications, Edited by D. Butnariu, y. Censor, and S. Reich, kluwer Academic Pulishers, Dordrecht, Holland, 473-504.