# Subsystem in Bipolar Fuzzy Finite State Machines

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## ABSTRACT

In this paper we introduce various types of subsystem, strong subsystem in bipolar fuzzy finite state machines. We study some equivalent conditions on subsystem.

### Keywords:

Bipolar fuzzy finite state machines, Subsystem, Strong subsystem.

## 1. INTRODUCTION

The theory of fuzzy set was introduced by L.A. Zadeh in 1965 [9]. The mathematical formulation of a fuzzy automaton was first proposed by W.G. Wee in 1967 [8]. E.S. Santos 1968 [7] proposed fuzzy automata as a model of pattern recognition.
J. N. Mordeson and D.S. Malik gave a detailed account of fuzzy automata and languages in their book 2002 [6]. Fuzzy sets are kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionstic fuzzy sets, interval-valued fuzzy sets etc. Bipolar-valued fuzzy sets, which are introduced by Lee [4, 5], are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In [2], Y.B. Jun and J. Kavikumar are introduced bipolar fuzzy finite state machines. In this paper, we introduce subsystem, strong subsystem in bipolar fuzzy finite state machines with example and discuss their properties.

## 2. BASIC DEFINITIONS

### 2.1 Definition [10]

Let $X$ denote a universal set. Then a fuzzy set $A$ in $X$ is set of ordered pairs:
$A = \{(x, \mu_A(x)|x \in X\}$, $\mu_A(x)$ is called the membership function or grade of membership of $x$ in $A$ which maps $X$ to the membership space $[0, 1]$.

### 2.2 Definition [3]

A finite fuzzy automata is a system of 3 tuples, $M = (Q, X, f_M)$, where,
$Q$-set of states $\{q_1, q_2, ...., q_n\}$
$\Sigma$-alphabets (or) input symbols
$f_M$-function from $Q \times X \times Q \to [0, 1]$
$f_M(q_i, \sigma, q_j) = \mu$, $[0 \leq \mu \leq 1]$ means when $M$ is in state $q_i$ and reads the input $\sigma$ will move to the state $q_j$ with weight function $\mu$.

### 2.3 Definition [2]

A bipolar fuzzy finite state machine (bffsm, for short) is a triple $M = (Q, X, \varphi)$, where $Q$ and $X$ are finite nonempty sets, called the set of states and the set of input symbols, respectively and $\varphi = \langle \varphi^-, \varphi^+ \rangle$ is a bipolar fuzzy set in $Q \times X \times Q$.
Let $X^*$ denote the set of all words of elements of $X$ of finite length. Let $\lambda$ denote the empty word in $X^*$ and $|x|$ denote the length of $x$ for every $x \in X^*$.

### 2.4 Definition [2]

Let $M = (Q, X, \varphi)$ be a bffsm. Define a bipolar fuzzy set $\varphi_* = \langle {\varphi_*}^+, {\varphi_*}^- \rangle$ in $Q \times X^* \times Q$ by

$$\varphi_*^-(q, \lambda, p) = \begin{cases} -1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\varphi_*^+(q, \lambda, p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$\varphi_*^-(q, xa, p) = \inf_{r \in Q}[{\varphi_*}^-(q, x, r) \vee {\varphi_*}^-(r, a, p)]$
${\varphi_*}^+(q, xa, p) = \sup_{r \in Q}[{\varphi_*}^+(q, x, r) \wedge {\varphi_*}^+(r, a, p)]$ $\forall$ $p, q \in Q$, $x \in X^*$ and $a \in X$.

**Result**

Let $M = (Q, X, \varphi)$ be a bffsm. Then
${\varphi_*}^-(q, xy, p) = \inf_{r \in Q}[{\varphi_*}^-(q, x, r) \vee {\varphi_*}^-(r, y, p)]$
${\varphi_*}^+(q, xy, p) = \sup_{r \in Q}[{\varphi_*}^+(q, x, r) \wedge {\varphi_*}^+(r, y, p)] \forall p, q \in Q$ and $x, y \in X^*$.

### 2.5 Definition [2]

Let $M = (Q, X, \varphi)$ be a bffsm and let $p, q \in Q$. Then $p$ is called a immediate successor of $q$ if the following condition holds.
$\exists a \in X$ such that ${\varphi_*}^-(q, a, p) < 0$ and ${\varphi_*}^+(q, a, p) > 0$. We say that $p$ is a successor of $q$ if the following condition holds $\exists x \in X^*$ such that ${\varphi_*}^-(q, x, p) < 0$ and ${\varphi_*}^+(q, x, p) > 0$. We denote by $S(q)$ the set of all successors of $q$. For any subset $T$ of $Q$ the set

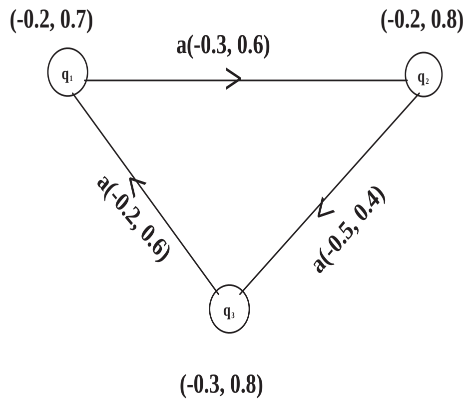


Fig- 1

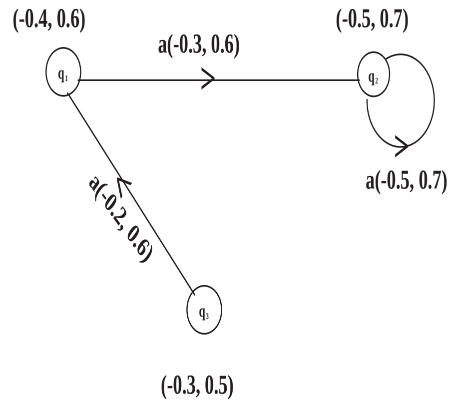


Fig- 2

of all successors of $T$ denoted by $S(T)$ is defined to be the set $S(T) = \cup\{S(q) \backslash q \in T\}$

### 2.6 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. Let $v$ be a bipolar fuzzy subset of $T \times X \times T$ and let $N = (T, X, v)$. The bipolar fuzzy finite state machine $N$ is called a submachine of $M$ if
(i) $v|T \times X \times T = v$
(ii)$S_Q(T) \subseteq T$.

## 3. SUBSYSTEM AND STRONG SUBSYSTEM IN BIPOLAR FUZZY FINITE STATE MACHINES

### 3.1 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. Then $(Q, \varphi_Q, X, \varphi)$ is called a subsystem of $M$ if for all $p, q \in Q$ and $a \in X$ such that $\varphi_Q^-(q) \leq \varphi_Q^-(p) \vee \varphi^-(p, a, q)$ and $\varphi_Q^+(q) \geq \varphi_Q^+(p) \wedge \varphi^+(p, a, q)$.

### 3.2 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. Then $(Q, \varphi_Q, X, \varphi)$ is called a strong subsystem of $M$ if and only if for all $p, q \in Q$ if $\exists a \in X$ such that $\varphi^-(p, a, q) < 0$ and $\varphi^+(p, a, q) > 0$ then $\varphi_Q^-(q) \leq \varphi_Q^-(p)$ and $\varphi_Q^+(q) \geq \varphi_Q^+(p)$.

**Note**
$(Q, \varphi_Q, X, \varphi)$ is a subsystem of $M$, then we write $\varphi_Q$ for $(Q, \varphi_Q, X, \varphi)$.

### 3.3 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. For all $x \in X^*$ define the bipolar fuzzy subset $\varphi_Q x$ of $Q$ by, $\varphi_Q^- x(q) = \text{Inf} \left\{ \varphi_Q^-(p) \vee \varphi_*^-(p, x, q) | p \in Q \right\}$, $\varphi_Q^+ x(q) = \text{Sup} \left\{ \varphi_Q^+(p) \wedge \varphi_*^+(p, x, q) | p \in Q \right\} \ \forall \ q \in Q$.

### 3.4 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $q \in Q$ and $t = (t_1, t_2)$ where $t_1 \in [-1, 0)$ and $t_2 \in (0, 1]$. Define the bipolar fuzzy subset $q_t X = \langle q_{t_1}^- X, q_{t_2}^+ X \rangle$ of $Q$ by,$(q_{t_1}^- X)(p) = \text{Inf}\{t_1 \vee \varphi^-(q, a, p) | a \in X\}$ and $(q_{t_2}^+ X)(p) = \text{Sup}\{t_2 \wedge \varphi^+(q, a, p) | a \in X\} \, \forall p \in Q$.

### 3.5 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $q \in Q$ and $t = (t_1, t_2)$ where $t_1 \in [-1, 0)$ and $t_2 \in (0, 1]$. Define the bipolar fuzzy subset $q_t X^* = \langle q_t^- X^*, q_t^+ X^* \rangle$ of $Q$ by, $(q_t^- X^*)(p) = \text{Inf}\{t_1 \vee \varphi_*^-(q, y, p) | y \in X^*\}$ and $(q_t^+ X^*)(p) = \text{Sup}\{t_2 \wedge \varphi_*^+(q, y, p) | y \in X^*\} \, \forall p \in Q$.

### 3.6 Definition

Let $M = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q + \rangle$ be a bipolar fuzzy subset of $X$. The *support* $\varphi_Q$ is defined to be the set $supp(\varphi_Q) = \left\{ x \in X | \varphi_Q^-(x) < 0 \ \& \ \varphi_Q^+(x) > 0 \right\}$.

## 4. PROPERTIES OF SUBSYSTEM IN BIPOLAR FUZZY FINITE STATE MACHINES

### 4.1 Theorem

Let $M = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. Then $\varphi_Q$ is a subsystem of $M$ if and only if $\forall p, q \in Q, \forall x \in X^*$. then $\varphi_Q^-(q) \leq \varphi_Q^-(p) \vee \varphi_*^-(p, x, q)$ and $\varphi_Q^+(q) \geq \varphi_Q^+(p) \wedge \varphi_*^+(p, x, q)$

**Proof:**
Suppose $\varphi_Q$ is a subsystem of $M$. Let $p, q \in Q$ and $x \in X^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \lambda$.
Now, if $p = q$ then $\varphi_Q^-(q) \vee \varphi_*^-(p, \lambda, q) = \varphi_Q^-(q)$ and
$\varphi_Q^+(q) \wedge \varphi_*^+(p, \lambda, q) = \varphi_Q^+(q)$
Now, if $p \neq q$ then $\varphi_Q^-(p) \vee \varphi_*^-(p, \lambda, q) = 0 \geq \varphi_Q^-(q)$ and
$\varphi_Q^+(p) \wedge \varphi_*^+(p, \lambda, q) = 0 \leq \varphi_Q^+(q)$. Thus the result is true if $n = 0$.
Suppose the result is true for all $y \in X^*$ such that $|y| = n-1, n > 0$. Let $x = ya, |y| = n - 1, y \in X^*, a \in X$. Then

$$\begin{aligned} \varphi_Q^-(p) \vee \varphi_*^-(p, x, q) &= \varphi_Q^-(p) \vee \varphi_*^-(p, ya, q) \\ &= \varphi_Q^-(q) \vee \{inf_{r \in Q} \{\varphi_*^-(p, y, r) \vee \varphi^-(r, a, q)\}\} \\ &= \wedge_{r \in Q} \left\{ \varphi_Q^-(p) \vee \varphi_*^-(p, y, r) \vee \varphi^-(r, a, q) \right\} \\ &\geq \wedge_{r \in Q} \left\{ \varphi_Q^-(r) \vee \varphi^-(r, a, q) \right\} \end{aligned}$$

$\geq \varphi_Q^-(q)$.
$\varphi_Q^-(p) \vee \varphi_*^-(p,x,q) \geq \varphi_Q^-(q)$.
$\varphi_Q^+(p) \wedge \varphi_*^+(p,x,q) = \varphi_Q^+(p) \wedge \varphi_*^+(p,ya,q)$
$= \varphi_Q^+(q) \wedge \{sup_{r\in Q} \{\varphi_*^+(p,y,r) \wedge \varphi^+(r,a,q)\}\}$
$= \vee_{r\in Q} \left\{\varphi_Q^+(p) \wedge \varphi_*^+(p,y,r) \wedge \varphi^+(r,a,q)\right\}$
$\leq \vee_{r\in Q} \left\{\varphi_Q^+(r) \wedge \varphi^+(r,a,q)\right\}$
$\leq \varphi_Q^+(q)$.
$\varphi_Q^+(p) \wedge \varphi_*^+(p,x,q) \leq \varphi_Q^+(q)$.
The converse is obvious.

### 4.2 Theorem

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Then for all bipolar fuzzy subsets $\varphi_Q$ of $Q$ and $\forall x, y \in X^*$
$(\varphi_Q^- x)y = \varphi_Q^-(xy)$ and
$(\varphi_Q^+ x)y = \varphi_Q^+(xy)$.

**Proof:**
Let $\varphi_Q$ be a bipolar fuzzy subset of $Q$ and let $x, y \in X^*$. We prove the result by induction on $|y| = n$. If $n = 0$, then $y = \lambda$. Let $q \in Q$
$((\varphi_Q^- x)\lambda)(q) = \wedge \left\{(\varphi_Q^- x)(p) \vee \varphi_*^-(p,\lambda,q) | p \in Q\right\}$
$= (\varphi_Q^- x)(q)$and
$((\varphi_Q^+ x)\lambda)(q) = \vee \left\{(\varphi_Q^+ x)(p) \wedge \varphi_*^+(p,\lambda,q) | p \in Q\right\}$
$= (\varphi_Q^+ x)(q)$.
Hence $(\varphi_Q^- x)\lambda = \varphi_Q^- x = \varphi_Q^-(x\lambda)$. Similarly, we have
$(\varphi_Q^+ x)\lambda = \varphi_Q^+ x = \varphi_Q^+(x\lambda)$.
Suppose the result is true for all $u \in X^*$ such that $|u| = n-1, n > 0$. Let $y = ua$, where $u \in X^*, a \in X$. Let $q \in Q$ then
$(\varphi_Q^-(xy))(q) = (\varphi_Q^-(xua))(q)$
$= (\varphi_Q^-((xu)a))(q)$
$= \wedge \left\{(\varphi_Q^-(xu))(r) \vee \varphi^-(r,a,q) | r \in Q\right\}$
$= \wedge \left\{\wedge \left\{(\varphi_Q^- x)(p) \vee \varphi_*^-(p,u,r) | p \in Q\right\} \vee \varphi^-(r,a,q) | r \in Q\right\} =$
$\wedge \left\{(\varphi_Q^- x)(p) \vee (\wedge \{\varphi_*^-(p,u,r) \vee \varphi^-(r,a,q) | r \in Q\}) | p \in Q\right\}$
$= \wedge \left\{(\varphi_Q^- x)(p) \vee \varphi_*^-(p,ua,q) | p \in Q\right\}$
$= ((\varphi_Q^- x)y)(q)$ and
$(\varphi_Q^+(xy))(q) = (\varphi_Q^+(xua))(q)$
$= (\varphi_Q^+((xu)a))(q)$
$= \vee \left\{(\varphi_Q^+(xu))(r) \wedge \varphi^+(r,a,q) | r \in Q\right\} =$
$\vee \left\{\vee \left\{(\varphi_Q^+ x)(p) \wedge \varphi_*^+(p,u,r) | p \in Q\right\} \wedge \varphi^+(r,a,q) | r \in Q\right\} =$
$\vee \left\{(\varphi_Q^+ x)(p) \wedge (\vee \{\varphi_*^-(p,u,r) \wedge \varphi^+(r,a,q) | r \in Q\}) | p \in Q\right\}$
$= \vee \left\{(\varphi_Q^+ x)(p) \wedge \varphi_*^+(p,ua,q) | p \in Q\right\}$
$= ((\varphi_Q^+ x)y)(q)$

### 4.3 Theorem

Let $M = (Q, X, \varphi)$ be a bffsm. Let $\varphi_Q$ be a bipolar fuzzy subset of $Q$. Then $\varphi_Q$ is a subsystem of $M$ if and only if
$\varphi_Q^- x \supseteq \varphi_Q^-$ and
$\varphi_Q^+ x \subseteq \varphi_Q^+ \quad \forall x \in X^*$.

**Proof:**
Let $\varphi_Q$ be a subsystem of $M$. Let $x \in X^*$ and $q \in Q$. Then
$(\varphi_Q^- x)(q) = \wedge \left\{\varphi_Q^-(p) \vee \varphi_*^-(p,x,q) | p \in Q\right\} \geq \varphi_Q^-(q)$
$(\varphi_Q^+ x)(q) = \vee \left\{\varphi_Q^+(p) \wedge \varphi_*^+(p,x,q) | p \in Q\right\} \leq \varphi_Q^+(q)$.
Hence $\varphi_Q^- x \supseteq \varphi_Q^-$ and
$\varphi_Q^+ x \subseteq \varphi_Q^+ \quad \forall x \in X^*$.
Conversely,
suppose $\varphi_Q^- x \supseteq \varphi_Q^-$ and
$\varphi_Q^+ x \subseteq \varphi_Q^+ \quad \forall x \in X^*$.
Now, $\varphi_Q^-(q) \leq (\varphi_Q^- x)(q) = \wedge \left\{\varphi_Q^-(q) \vee \varphi_*^-(p,x,q) | p \in Q\right\}$
$\leq \varphi_Q^-(q) \vee \varphi_*^-(p,x,q)$.
$\varphi_Q^+(q) \geq (\varphi_Q^+ x)(q) = \vee \left\{\varphi_Q^+(q) \wedge \varphi_*^+(p,x,q) | p \in Q\right\}$
$\geq \varphi_Q^+(q) \wedge \varphi_*^+(p,x,q) \forall p \in Q$.
Hence $\varphi_Q$ is a subsystem of $M$.

### 4.4 Theorem

Let $M = (Q, X, \varphi)$ be a bffsm. Let $t = (t_1, t_2)$ where $t_1 \in [-1, 0)$ and $t_2 \in (0, 1]$. Let $q \in Q$. Then
(i) $q_t X^*$ is a subsystem of $M$.
(ii) $Supp(q_t X^*) = S(q)$.

**Proof:**
Let $r, s \in Q$ and $x \in X^*$. Now,
$((q_{t_1}^- X^*)x)(r) = \wedge \left\{(q_{t_1}^- X^*)(p) \vee \varphi_*^-(p,x,r) | p \in Q\right\}$
$= \wedge \{(\wedge \{\varphi_*^-(q,y,p) \vee t_1 | y \in X^*\}) \vee \varphi_*^-(p,x,r) | p \in Q\}$
$= \wedge \{\varphi_*^-(q,y,p) \vee \varphi_*^-(p,x,r) \vee t_1 | y \in X^*, p \in Q\}$
$= \wedge \{\varphi_*^-(q,yx,r) \vee t_1 | y \in X^*\}$
$\geq \{\varphi_*^-(q,u,r) \vee t_1 | u \in X^*\}$
$= (q_{t_1}^- X^*)(r)$
Hence $(q_{t_1}^- X^*)x \supseteq q_{t_1}^- X^*$.
$((q_{t_2}^+ X^*)x)(r) = \vee \left\{(q_{t_2}^+ X^*)(p) \wedge \varphi_*^+(p,x,r) | p \in Q\right\}$
$= \vee \{(\vee \{\varphi_*^+(q,y,p) \wedge t_2 | y \in X^*\}) \wedge \varphi_*^+(p,x,r) | p \in Q\}$
$= \vee \{\varphi_*^+(q,y,p) \wedge \varphi_*^+(p,x,r) \wedge t_2 | y \in X^*, p \in Q\}$
$= \vee \{\varphi_*^+(q,yx,r) \wedge t_2 | y \in X^*\}$
$\leq \{\varphi_*^+(q,u,r) \wedge t_2 | u \in X^*\}$
$= (q_{t_2}^+ X^*)(r)$
Hence $(q_{t_2}^+ X^*)x \subseteq q_{t_2}^+ X^*$. Thus $q_{t_2} X^*$ is a subsystem of $M$ [By Theorem 4.3].

(ii) Let $p \in S(q) \Leftrightarrow \exists x \in X^*$ such that $\varphi_*^-(q,x,p) < 0$
$\Leftrightarrow \wedge \{t_1 \vee \varphi_*^-(q,x,p) | x \in X^*\} < 0$
$\Leftrightarrow ((q_{t_1}^- X^*)(p) < 0$————(1)

$p \in S(q) \Leftrightarrow \exists x \in X^*$ such that $\varphi_*^+(q,x,p) > 0$
$\Leftrightarrow \vee \{t_2 \wedge \varphi_*^+(q,x,p) | x \in X^*\} > 0$
$\Leftrightarrow ((q_{t_2}^+ X^*)(p) > 0$————(2)
From (1) and (2), $Supp(q_{t_2} X^*) = S(q)$.

### 4.5 Theorem

Let $M = (Q, X, \varphi)$ be a bffsm and let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. Then the following conditions are equivalent.
(i)$\varphi_Q$ is a subsystem of $M$.
(ii) $q_{t_1}^- X^* \supseteq \varphi_Q^-, \forall\, q_{t_1}^- \supseteq \varphi_Q^-, q \in Q, t_1 \in [-1, 0)$ and
$q_{t_2}^+ X^* \subseteq \varphi_Q^+, \forall\, q_{t_2}^+ \subseteq \varphi_Q^+, q \in Q,\ \ t_2 \in (0, 1]$
(iii) $q_{t_1}^- X \supseteq \varphi_Q^-, \forall q_{t_1}^- \supseteq \varphi_Q^-, q \in Q, t_1 \in [-1, 0)$and
$q_{t_2}^+ X \subseteq \varphi_Q^+, \forall\, q_{t_2}^+ \subseteq \varphi_Q^+, q \in Q, t_2 \in (0, 1]$.

**Proof:**
$(i) \Rightarrow (ii)$
Let $q_{t_1}^- \supseteq \varphi_Q^-,\ q \in Q, t_1 \in [-1,\ 0)$. Let $p \in Q$ and $y \in X^*$ then
$\varphi_*^-(q,y,p) \vee t_1 = \varphi_*^-(q,y,p) \vee q_{t_1}^-(q)$

$\geq \varphi_*^-(q,y,p) \vee \varphi_Q^-(q)$
$\geq \varphi_Q^-(p)$.[Since $\varphi_Q^-$ is a subsystem]
Hence $q_{t_1}^- X^* \supseteq \varphi_Q^-$.
Let $q_{t_2}^+ \subseteq \varphi_Q^+$, $q \in Q$, $t_2 \in (0,\ 1]$. Let $p \in Q$ and $y \in X^*$ then
$\varphi_*^+(q,y,p) \wedge t_2 = \varphi_*^-(q,y,p) \wedge q_{t_2}^+(q)$
$\leq \varphi_*^+(q,y,p) \wedge \varphi_Q^+(q)$
$\leq \varphi_Q^+(p)$.[Since $\varphi_Q^+$ is a subsystem]
Hence $q_{t_2}^+ X^* \subseteq \varphi_Q^+$.
$(ii) \Rightarrow (iii)$ Since the alphabet $x \in X^*$, the result (iii) follows.
$(iii) \Rightarrow (i)$
Let $p,q \in Q$ and $a \in X$. If $\varphi_Q^-(q) = 0$ or $\varphi^-(q,a,p) = 0$ then $\varphi_Q^-(p) \leq 0 = \varphi_Q^-(q) \vee \varphi^-(q,a,p)$.
Suppose $\varphi_Q^-(q) < 0$ and $\varphi^-(q,a,p) < 0$. Let $\varphi_Q^-(q) = t_1$. Then $q_{t_1}^- \supseteq \varphi_Q^-$.
Thus by hypothesis, $q_{t_1}^- X \supseteq \varphi_Q^-$.
Thus $\varphi_Q^-(p) \leq (q_{t_1}^- X)(p) = \wedge\{t_1 \vee \varphi^-(q,a,p) | a \in X\}$
$\leq t_1 \vee \varphi^-(q,a,p) = \varphi_Q^-(q) \vee \varphi^-(q,a,p)$———(1)
Let $p,q \in Q$ and $a \in X$. If $\varphi_Q^+(q) = 0$ or $\varphi^+(q,a,p) = 0$ then $\varphi_Q^+(p) \geq 0 = \varphi_Q^+(q) \wedge \varphi^+(q,a,p)$.
Suppose $\varphi_Q^+(q) > 0$ and $\varphi^+(q,a,p) > 0$. Let $\varphi_Q^+(q) = t_2$. Then $q_{t_2}^+ \subseteq \varphi_Q^+$.
Thus by hypothesis, $q_{t_2}^+ X \subseteq \varphi_Q^+$.
Thus $\varphi_Q^+(p) \geq (q_{t_2}^+ X)(p) = \vee\{t_2 \wedge \varphi^+(q,a,p) | a \in X\}$
$\geq t_2 \wedge \varphi^+(q,a,p) = \varphi_Q^+(q) \wedge \varphi^+(q,a,p)$.———(2)
From (1) and (2), $\varphi_Q$ is a subsystem of $M$.

### 4.6 Theorem

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Let $\varphi_Q$, $\varphi_{Q1}$, and $\varphi_{Q2}$, be subsystems of $M$. Then the following conditions hold.
(i) $\varphi_{Q_1} \wedge \varphi_{Q_2}$ is a subsystem of $M$.
(ii)$\varphi_{Q_1} \vee \varphi_{Q_2}$ is a subsystem of $M$.
(iii) $N = (Supp(\varphi_Q), X, \nu)$ is a submachine of $M$, where $\nu = \varphi|_{Supp(\varphi_Q)\,\times\,X\,\times\,Supp(\varphi_Q)}$.
(iv) Let $t = (t_1,t_2)$ where $t_1 \in [-1,0)$ and $t_2 \in (0,1]$. Let $N_t = (\varphi_{Q_t}, X, \nu_{(t)})$ where $\varphi_{Q_t}$ is bipolar fuzzy subset of $Q$,$\nu_{(t)} = \varphi|_{\varphi_{Q_t}\,\times\,X\,\times\,\varphi_{Q_t}}$. If $\forall t$, $N_t$ is a submachine of $M$, then $\varphi_Q$ is a subsystem of $M$.

**Proof:**
(i) Since $\varphi_{Q_1}$ and $\varphi_{Q_2}$ be a subsystem of $M$. Then $\forall\ p,q \in Q$ and $a \in X$ such that $\varphi_{Q_1}^-(q) \leq \varphi_{Q_1}^-(p) \vee \varphi^-(p,a,q)$,
$\varphi_{Q_1}^+(q) \geq \varphi_{Q_1}^+(p) \wedge \varphi^+(p,a,q)$and $\varphi_{Q_2}^-(q) \leq \varphi_{Q_2}^-(p) \vee \varphi^-(p,a,q)$,
$\varphi_{Q_2}^+(q) \leq \varphi_{Q_2}^+(p) \wedge \varphi^-(p,a,q)$.
Now to prove $\varphi_{Q1} \wedge \varphi_{Q2}$ is a subsystem of $M$, it is enough to prove
$(\varphi_{Q_1}^- \wedge \varphi_{Q_2}^-)(q) \leq (\varphi_{Q_1}^- \wedge \varphi_{Q_2}^-)(p) \vee \varphi^-(p,a,q)$ and
$(\varphi_{Q_1}^+ \wedge \varphi_{Q_2}^+)(q) \geq (\varphi_{Q_1}^+ \wedge \varphi_{Q_2}^+)(p) \wedge \varphi^+(p,a,q)$.
Now $(\varphi_{Q_1}^-(q) \wedge \varphi_{Q_2}^-(q)) \leq ((\varphi_{Q_1}^-(p) \vee \varphi^-(p,a,q)) \wedge ((\varphi_{Q_2}^-(p) \vee \varphi^-(p,a,q))$
$(\varphi_{Q_1}^-(q) \wedge \varphi_{Q_2}^-(q)) \leq (\varphi_{Q_1}^-(p) \wedge \varphi_{Q_2}^-(p) \vee \varphi^-(p,a,q))$
$(\varphi_{Q_1}^- \wedge \varphi_{Q_2}^-)(q) \leq (\varphi_{Q_1}^- \wedge \varphi_{Q_2}^-)(p) \vee \varphi^-(p,a,q)$—(1)

$(\varphi_{Q_1}^+(q) \wedge \varphi_{Q_2}^+(q)) \geq ((\varphi_{Q_1}^+(p) \wedge \varphi^+(p,a,q)) \vee (\varphi_{Q_2}^+)(p) \wedge \varphi^+(p,a,q))$
$(\varphi_{Q_1}^+(q) \wedge \varphi_{Q_2}^+(q)) \geq (\varphi_{Q1}^+(p) \vee \varphi_{Q_2}^+(p) \wedge \varphi^+(p,a,q))$
$(\varphi_{Q_1}^+ \wedge \varphi_{Q_2}^+)(q) \geq (\varphi_{Q_1}^+ \vee \varphi_{Q_2}^+)(p) \wedge \varphi^+(p,a,q)$—(2)
From (1)and(2), $\varphi_{Q_1} \wedge \varphi_{Q_2}$ is a subsystem of $M$.

(ii) Since $\varphi_{Q_1}$ and $\varphi_{Q_2}$ be a subsystem of $M$. Then $\forall\ p,q \in Q$ and $a \in X$ such that $\varphi_{Q_1}^-(q) \leq \varphi_{Q_1}^-(p) \vee \varphi^-(p,a,q)$,
$\varphi_{Q_1}^+(q) \geq \varphi_{Q_1}^+(p) \wedge \varphi^+(p,a,q)$and $\varphi_{Q_2}^-(q) \leq \varphi_{Q_2}^-(p) \vee \varphi^-(p,a,q)$,
$\varphi_{Q_2}^+(q) \leq \varphi_{Q_2}^+(p) \wedge \varphi^-(p,a,q)$.
Now to prove $\varphi_{Q_1} \vee \varphi_{Q_2}$ is a subsystem of $M$, it is enough to prove
$(\varphi_{Q_1}^- \vee \varphi_{Q_2}^-)(q) \leq (\varphi_{Q_1}^- \vee \varphi_{Q_2}^-)(p) \vee \varphi^-(p,a,q)$ and
$(\varphi_{Q_1}^+ \vee \varphi_{Q_2}^+)(q) \geq (\varphi_{Q_1}^+ \vee \varphi_{Q_2}^+)(p) \wedge \varphi^+(p,a,q)$.
Now $(\varphi_{Q_1}^-(q) \vee \varphi_{Q_2}^-(q)) \leq ((\varphi_{Q_1}^-(p) \vee \varphi^-(p,a,q)) \vee ((\varphi_{Q_2}^-(p) \vee \varphi^-(p,a,q))$
$(\varphi_{Q_1}^-(q) \vee \varphi_{Q_2}^-(q)) \leq (\varphi_{Q_1}^-(p) \vee \varphi_{Q_2}^-(p) \vee \varphi^-(p,a,q))$
$(\varphi_{Q_1}^- \vee \varphi_{Q_2}^-)(q) \leq (\varphi_{Q_1}^- \vee \varphi_{Q_2}^-)(p) \vee \varphi^-(p,a,q)$—(1)
$(\varphi_{Q_1}^+(q) \vee \varphi_{Q_2}^+(q)) \geq ((\varphi_{Q_1}^+(p) \wedge \varphi^+(p,a,q)) \vee (\varphi_{Q_2}^+)(p) \wedge \varphi^+(p,a,q))$
$(\varphi_{Q_1}^+(q) \vee \varphi_{Q_2}^+(q)) \geq (\varphi_{Q_1}^+(p) \vee \varphi_{Q_2}^+(p) \wedge \varphi^+(p,a,q))$
$(\varphi_{Q_1}^+ \vee \varphi_{Q_2}^+)(q) \geq (\varphi_{Q_1}^+ \vee \varphi_{Q_2}^+)(p) \wedge \varphi^+(p,a,q)$—(2)
From (1)and(2), $\varphi_{Q_1} \vee \varphi_{Q_2}$ is a subsystem of $M$.

(iii) Let $p \in S(Supp(\varphi_Q^-))$. Then $p \in S(q)$ for some $q \in Supp(\varphi_Q^-)$. Then $\varphi_Q^-(q) < 0$. Since $p \in S(q), \exists x \in X^*$ such that $\varphi_*^-(q,x,p) < 0$. Since $\varphi_Q$ is a subsystem $\varphi_Q^-(p) \leq \varphi_Q^-(q) \vee \varphi_*^-(q,x,p) < 0$. Thus $p \in Supp(\varphi_Q^-)$. Hence $S(Supp(\varphi_Q^-) \subseteq Supp(\varphi_Q^-)$.
Let $p \in S(Supp(\varphi_Q^+))$. Then $p \in S(q)$ for some $q \in Supp(\varphi_Q^+)$. Then $\varphi_Q^+(q) > 0$. Since $p \in S(q), \exists x \in X^*$ such that $\varphi_*^+(q,x,p) > 0$. Since $\varphi_Q$ is a subsystem $\varphi_Q^+(p) \geq \varphi_Q^+(q) \wedge \varphi_*^+(q,x,p) > 0$. Thus $p \in Supp(\varphi_Q^+)$. Hence $S(Supp(\varphi_Q^+) \subseteq Supp(\varphi_Q^+)$. Thus $N$ is a submachine of $M$.

(iv)Let $p,q \in Q$ and $x \in X^*$. If $\varphi_Q^-(p) = 0$ or $\varphi_*^-(p,x,q) = 0$, $\varphi_Q^+(p) = 0$ or $\varphi_*^+(p,x,q) = 0$, then $\varphi_Q^-(q) \leq 0 = \varphi_Q^-(p) \vee \varphi_*^-(p,x,q)$ and $\varphi_Q^+(q) \geq 0 = \varphi_Q^+(p) \wedge \varphi_*^+(p,x,q)$.
Suppose $\varphi_Q^-(p) < 0$ and $\varphi_*^-(p,x,q) < 0$, $\varphi_Q^+(p) > 0$ and $\varphi_*^+(p,x,q) > 0$. Let $\varphi_Q^-(p) \vee \varphi_*^-(p,x,q) = t_1$ and $\varphi_Q^+(p) \wedge \varphi_*^+(p,x,q) = t_2$. Then $p \in \varphi_{Q_t}$. Since $N_t$ is a submachine of $M$, $S(\varphi_{Q_t}) = \varphi_{Q_t}$. Hence $q \in S(p) \subseteq S(\varphi_{Q_t}) = \varphi_{Q_t}$.
Hence $\varphi_Q^-(q) \leq t_1 = \varphi_Q^-(p) \vee \varphi_*^-(p,x,q)$ and $\varphi_Q^+(q) \geq t_2 = \varphi_Q^+(p) \wedge \varphi_*^+(p,x,q)$.
Thus $\varphi_Q$ is a subsystem.

### 4.7 Theorem

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. Then $\varphi_Q$ is a strong subsystem of $M$ if and only if for all $p,q \in Q$ if $\exists x \in X^*$ such that ${\varphi_*}^-(p,x,q) < 0$ and ${\varphi_*}^+(p,x,q) > 0$ then
$\varphi_Q^-(q) \leq \varphi_Q^-(p)$ and $\varphi_Q^+(q) \geq \varphi_Q^+(p)$.

**Proof:**
Suppose $\varphi_Q$ is a strong subsystem. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \lambda$. Now if $p = q$, then $\varphi_*^-(q,\lambda,q) = -1$ and $\varphi_Q^-(q) = \varphi_Q^-(q)$. If $q \neq p$, then $\varphi_*^-(p,\lambda,q) = 0$. Thus the result is true if $n = 0$. Suppose the result is true $\forall y \in X^*$ such that $|y| = n-1, n > 0$. Let $x =$

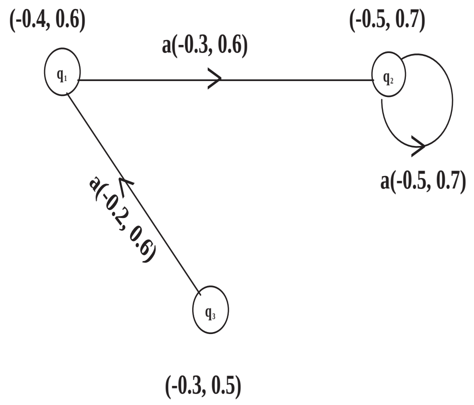


Fig- 3

$ya, |y| = n-1, y \in X^*, a \in X$. Suppose that $\varphi_*^-(p,x,q) < 0$. Then $\varphi_*^-(p,x,q) < 0 = \varphi_*^-(p,ya,q) < 0$.

$$\wedge_{r\in Q}\,\{\varphi_*{}^-(p,y,r) \vee \varphi_*{}^-(r,a,q)\} < 0.$$

Thus $\exists r \in Q$ such that $\varphi_*{}^-(p,y,r) < 0$ and $\varphi_*{}^-(r,a,q) < 0$. Hence $\varphi_Q^-(q) \le \varphi_Q^-(r)$ and $\varphi_Q^-(r) \le \varphi_Q^-(p)$. Thus $\varphi_Q^-(q) \le \varphi_Q^-(p)$.

Suppose $\varphi_Q$ is a strong subsystem. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \lambda$. Now if $p = q$, then $\varphi_*{}^+(q,\lambda,q) = 1$ and $\varphi_Q^+(q) = \varphi_Q^+(q)$. If $q \neq p$, then $\varphi_*{}^+(p,\lambda,q) = 0$. Thus the result is true if $n = 0$. Suppose the result is true $\forall y \in X^*$ such that $|y| = n-1, n > 0$. Let $x = ya, |y| = n-1, y \in X^*, a \in X$. Suppose that $\varphi_*{}^+(p,x,q) > 0$. Then $\varphi_*{}^+(p,x,q) > 0 = \varphi_*{}^+(p,ya,q) > 0$.

$$\vee_{r\in Q}\,\{\varphi_*{}^+(p,y,r) \wedge \varphi_*{}^+(r,a,q)\} > 0.$$

Thus $\exists r \in Q$ such that $\varphi_*{}^+(p,y,r) > 0$ and $\varphi_*{}^+(r,a,q) > 0$. Hence $\varphi_Q^+(q) \ge \varphi_Q^+(r)$ and $\varphi_Q^+(r) \ge \varphi_Q^+(p)$. Thus $\varphi_Q^+(q) \ge \varphi_Q^+(p)$.

The converse obvious.

### 4.8 Theorem

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Let $\varphi_Q = \langle \varphi_Q^-, \varphi_Q^+ \rangle$ be a bipolar fuzzy subset in $Q$. If $\varphi_Q$ is a strong subsystem of $M$, then $\varphi_Q$ is a subsystem of $M$.

The Converse is not true.

**Proof:**
The proof of the Theorem is obvious.

**Example**
The bipolar fuzzy finite state machine $M$ in Fig.3 is a strong subsystem of $M$, implies that $\varphi_Q$ is a subsystem of $M$.

The bipolar fuzzy finite state machine $M$ in Fig.4, $\varphi_Q$ is a subsystem of $M$ but not a strong subsystem of $M$. Since $\varphi_Q^-(q_2) \ge \varphi_Q^-(q_1)$.

### 4.9 Theorem

Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Let $\varphi_{Q_1}$, and $\varphi_{Q_2}$, be strong subsystems of $M$. Then the following conditions hold.

(i) $\varphi_{Q_1} \wedge \varphi_{Q_2}$ is a strong subsystem of $M$.

(ii)$\varphi_{Q_1} \vee \varphi_{Q_2}$ is a strong subsystem of $M$.

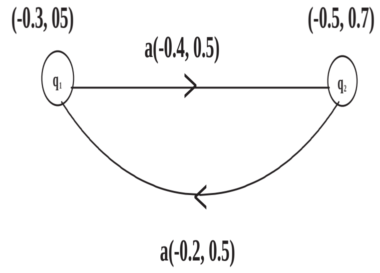


Fig- 4

(iii) Let $\varphi_Q$ be a strong subsystem of $M$. Then $N = (Supp(\varphi_Q), X, \nu)$ is a submachine of $M$, where $\nu = \varphi|_{Supp(\varphi_Q)\,\times\,X\,\times\,Supp(\varphi_Q)}$.

(iv) Let $t = (t_1, t_2)$ where $t_1 \in [-1,\ 0)$ and $t_2 \in (0, 1]$.
Let $N_t = (\varphi_{Q_t}, X, \nu_{(t)})$ where $\varphi_Q$ be a strong subsystem of $M$, $\nu_{(t)} = \varphi|_{\varphi(Q_t)\,\times\,X\,\times\,\varphi_{Q_t}}$. Then $\forall\ t$, $N_t$ is a submachine of $M$.

(v) Let $t = (t_1, t_2)$ where $t_1 \in [-1,\ 0)$ and $t_2 \in (0, 1]$. Let $N_t = (\varphi_{Q_t}, X, \nu_{(t)})$ where $\varphi_Q$ be a bipolar fuzzy subset of $Q$, $\nu_{(t)} = \varphi|_{\varphi_{Q_t}\,\times\,X\,\times\,\varphi_{Q_t}}$ If $\forall t$, $N_t$ is a submachine of $M$, then $\varphi_Q$ is a strong subsystem of $M$.

**Proof:**
The proofs of (i) and (ii) are straightforward.

(iii) Let $p \in S(Supp(\varphi_Q^-))$. Then $p \in S(q)$ for some $q \in Supp(\varphi_Q^-)$. Then $\varphi_Q^-(q) < 0$. Since $p \in S(q), \exists x \in X^*$ such that $\varphi_*^-(q,x,p) < 0$. Since $\varphi_Q$ is a strong subsystem $\varphi_Q^-(p) \le \varphi_Q^-(q) < 0$. Thus $p \in Supp(\varphi_Q^-)$. Hence $S(Supp(\varphi_Q^-)) \subseteq Supp(\varphi_Q^-)$.

Let $p \in S(Supp(\varphi_Q^+))$. Then $p \in S(q)$ for some $q \in Supp(\varphi_Q^+)$. Then $\varphi_Q^+(q) > 0$. Since $p \in S(q), \exists x \in X^*$ such that $\varphi_*^+(q,x,p) > 0$. Since $\varphi_Q$ is a strong subsystem $\varphi_Q^+(p) \ge \varphi_Q^+(q) > 0$. Thus $p \in Supp(\varphi_Q^+)$. Hence $S(Supp(\varphi_Q^+) \subseteq Supp(\varphi_Q^+)$. Thus $N$ is a submachine of $M$.

(iv) Let $q \in S(\varphi_{Q_t})$. Then $q \in S(p)$ for some $p \in \varphi_{Q_t}$. Thus $\varphi_Q^-(p) \le t_1$ and $\varphi_Q^+(p) \ge t_2$. Now $\exists x \in X^*$ such that $\varphi_*^-(p,x,q) < 0$ and $\varphi_*^+(p,x,q) > 0$. Then $\varphi_Q^-(q) \le \varphi_Q^-(p) \le t_1$ and $\varphi_Q^+(q) \ge \varphi_Q^+(p) \ge t_2$. Thus $q \in \varphi_{Q_t}$. Hence $N_t$ is a submachine of $M$.

(v)Let $p, q \in Q$, $x \in X^*$ be such that $\varphi_*^-(p,x,q) < 0$ and $\varphi_*^+(p,x,q) > 0$. Suppose $\varphi_Q^-(p) < 0$ and $\varphi_Q^+(p) > 0$. Let $\varphi_Q^-(p) = t_1$ and $\varphi_Q^+(p) = t_2$. Then $p \in \varphi_{Q_t}$. Since $N_t$ is a submachine of $M$, $S(\varphi_{Q_t}) = \varphi_{Q_t}$. Thus $q \in S(p) \subseteq S(\varphi_{Q_t}) = \varphi_{Q_t}$. Thus $\varphi_Q^-(q) \le t_1$ and $\varphi_Q^+(q) \ge t_2$. Hence, $\varphi_Q$ is a strong subsystem.

## 5. CONCLUSION

In this paper, we introduce various types of subsystem, strong subsystem in bipolar fuzzy finite state machines and discuss their properties.

## 6. REFERENCES

[1] Dubois. D., Prade. H. 1980. *Fuzzy Sets and Systems.* Mathematics in Science and Engineering, Academic Press, New York, 144.

[2] Jun. Y. B., Kavikumar. J. 2011. *Bipolar Fuzzy Finite State Machines.* Bull.Malays.Sci.Soc. 34 no. 1: 181-188.

[3] Kandel. A., Lee. S. C. *Fuzzy Switching and Automata Theory Applications.* Edward Arnold Publishers Ltd, London.

[4] Lee. K. J. 2000. *Bipolar-valued fuzzy sets and their operations.* Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand, 307-312.

[5] Lee. K. J. 2004.*Comparison of interval valued fuzzy sets, Intuitionstic fuzzy sets, and bipolar-valued fuzzy sets.* J. Fuzzy Logic Intelligent Systems, no.14: 125-129.

[6] Mordeson. J. N., Malik. D. S. 2002. *Fuzzy automata and Languages.*-Theory and Applications, Chapman & Hall/ CRC Press.

[7] Santos. E. S. 1968.*General formulation of sequential machines.* Information and control, no.12: 5-10.

[8] Wee. W. G. 1967. *On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification.* Ph.D. Thesis Purude University.

[9] Zadeh. L. A. 1965. *Fuzzy sets.* Information and control, no. 8: 338-353.

[10] Zimmermann. H. J. 1985. *Fuzzy set theory and its applications.* International Series in Management Science/Operations research, Kluwer-Nijhoff, Boston, MA.