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Numerical Solution of Eighth Order Boundary Value Problems by Galerkin Method with Quintic B-splines

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ABSTRACT
In this paper, we present a finite element method involving Galerkin method with quintic B-splines as basis functions to solve a general eighth order two point boundary value problem.The basis functions are redefined into a new set of basis functions which vanish on the boundary where Dirichlet type of boundary conditions, Neumann boundary conditions, second order derivative boundary conditions and third order derivative type of boundary conditions are prescribed. The proposed method was applied to solve several examples of the eighth order linear and nonlinear boundary value problems. The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solution of linear boundary value problems generated by quasilinearization technique. The obtained numerical results are compared with exact solutions available in the literature.

Keywords
Galerkin method; Quintic B-spline; Basis function; Eighth order boundary value problem; Absolute error.

1. INTRODUCTION
In this paper, we consider a general eighth order linear boundary value problem given by

$$\begin{aligned} & a_0(x)y^{(8)}(x)+a_1(x)y^{(7)}(x)+a_2(x)y^{(6)}(x) \\ & +a_3(x)y^{(5)}(x)+a_4(x)y^{(4)}(x)+a_5(x)y'''(x) \\ & +a_6(x)y''(x)+a_7(x)y'(x)+a_8(x)y \\ & =b(x), \qquad c<x<d \end{aligned} \tag{1}$$

subject to the boundary conditions

$$\begin{aligned} & y(c)=A_0, y(d)=C_0, y'(c)=A_1, y'(d)=C_1, \\ & y''(c)=A_2, y''(d)=C_2, y'''(c)=A_3, y'''(d)=C_3 \end{aligned} \tag{2}$$

where A_0, C_0, A_1, C_1, A_2, C_2, A_3, C_3 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$,$a_6(x)$, $a_7(x)$, $a_8(x)$, $b(x)$ are all continuous functions defined on the interval $[c, d]$.

Generally, this type of eighth order boundary value problems arises in the study of astrophysics, hydrodynamics and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory, applied mathematics, engineering and applied physics. The boundary value problems of higher order differential equations have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. The literature on the numerical solutions of eighth order boundary value problems is very scarce. Chandra Sekhar [1] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in, when this instability is an ordinary convection the ordinary differential equation is sixth order, when the instability sets in as overstability, it is modeled by an eight order ordinary differential equation.

An eighth order differential equation derived from governing bending and axial vibrations by Shen [2], Paliwal and Pande [3] derived equations for the equilibrium in terms of displacement components for an orthotropic thin circular cylindrical shell subjected to a load that is not symmetric about the shell, which resulted in eighth order differential equations. The text book by Agarwal [4] contains theory which deals with the conditions for the existence and uniqueness of solutions of eighth order boundary value problems, though no numerical methods are given in for solving such problems. Solving such boundary value problems analytically is possible only in very rare cases. So, many numerical methods have been developed overs the years to approximate the solution for these types of boundary value problems. An eighth order differential equation occurs in torsional vibration of uniform beams was investigated by Bishop [5], Boutayes and Twizell [6] developed finite difference methods for the special case solution of the eighth order boundary value problems, Twizell et. al. [7] developed numerical methods for eighth, tenth, twelfth order eigenvalue problems arising in thermal instability, Inc and Evans [8] presented the solution of special case of eighth order boundary value problems using Adomain decomposition method, Siddiqi et. al. [9] presented solution of special case of eighth order boundary value problems using variational iterational technique, Ghazala Akram and Hamood Ur Rehman [10] presented the solution of special case of eighth order boundary value problems using kernel space method there were used searching least square value method investigated for nonlinear eighth order boundary value problems, Liu and Wu [11] presented the solution of special case of eighth order boundary value problems using generalized differential quadrature rule, Koonprasert and Torvattanabum [12] presented variational iterational method for solving eighth order boundary value problems, Javidi and Golbai [13] presented HPM for solution of eighth order boundary value problems, Prorshouhi at. al. [14] presented variatonal iterational method for solution of special case of eighth order boundary value problems.

In the following, we mainly pay attention to the spline functions technique have been developed to solve these type of boundary value problems. Siddiqi and Ghazala [15,16] presented solution of special case of eighth order boundary value problems using nonic non polynomial spline functions and nonic polynomial spline methods, Siddiqi and Twizell [17] presented the solution of special case of eight order boundary value problems using octic splines, Kasi Viswanadham and Showri raju [18] developed quintic B-splines Collocation method to solve a general eight order boundary value problem.

In this paper, we try to present a simple finite element method which involves Gelerkin approach with quintic B-splines as basis functions to solve the eighth order two point boundary value problems of the type (1)-(2). This paper is organized as follows. Section 2, deals with the justification for using Galerkin method. In Section 3, a description of Galerkin method with quintic B-splines as basis functions is explained. In particular, we first introduce the basic concept of quintic B-splines and followed by the proposed method. In Section 4, the procedure to solve the nodal parameters has been presented. In section 5, the proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [19]. Finally, in the last section, the conclusions are presented.

2. JUSTIFICATION FOR USING GALERKIN METHOD

For the few decades, the finite element method has become very powerful, useful tool to solve the boundary value problems in the complex dynamical systems. In finite element method (FEM) the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz, Galerkin, Petrov-Galerkin, Least Squares and Collocation etc.

In Galerkin method, the residual of approximation is made orthogonal to the basis functions. When one uses Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [20,21] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to boundary conditions [22]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. Hence in this paper we employed the use of Galerkin method with quintic B-splines as basis functions to approximate the solution of eighth order boundary value problems.

3. DESCRIPTION OF THE METHOD

Definition of quintic B-spline: The quintic B-splines are defined in [23-25]. The existence of quintic spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) of a partition $c = x_0 < x_1 < ... < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the quintic B-splines. Introduce ten additional knots x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4} and x_{n+5} in such a way that

$x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}$.

Now the quintic B-splines $B_i(x)'s$ are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-3}^{i+3} \dfrac{(x_r - x)_+^5}{\pi'(x_r)}, & x \in [x_{i-3},\, x_{i+3}] \\ 0, & otherwise \end{cases}$$

where

$$(x_r - x)_+^5 = \begin{cases} (x_r - x)^5, & if\ x_r \geq x \\ 0, & if\ x_r \leq x \end{cases}$$

and $$\pi(x) = \prod_{r=i-3}^{i+3} (x - x_r)$$

where $\{B_{-2}(x), B_{-1}(x), B_0(x), B_1(x), B_2(x), B_3(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_5(\pi)$ of quintic polynomial splines. Schoenberg [25] has proved that quintic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-5}<<x_{-4}<\ldots<x_0<x_1<\ldots<x_{n-1}<x_n<x_{n+1}<x_{n+2}<\ldots\ldots<x_{n+5}.$

To solve the boundary value problem (1) and (2) by the Galerkin method with quintic B-splines as basis functions, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{3}$$

where α_j's are the nodal parameters to be determined. In Galerkin method the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of quintic B-splines$\{B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$the basis functions $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. Since, we are approximating the eighth order boundary value problem by quintic B-splines polynomial, we redefine the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type boundary conditions, Neumann boundary conditions, second order derivative boundary conditions and third order derivative type of boundary condiotions are prescribed. The procedure for redefining the basis functions is as follows.

Using the definition of quintic B-splines and the Dirichlet boundary conditions of (2), the approximate solution at the boundary points can be written as

$$\begin{aligned} A_0 = y(c) = y(x_0) &= \alpha_{-2}B_{-2}(x_0) + \alpha_{-1}B_{-1}(x_0) \\ &+ \alpha_0 B_0(x_0) + \alpha_1 B_1(x_0) + \alpha_2 B_2(x_0) \end{aligned} \tag{4}$$

$$\begin{aligned} C_0 = y(d) = y(x_n) &= \alpha_{n-2}B_{n-2}(x_n) + \alpha_{n-1}B_{n-1}(x_n) \\ &+ \alpha_n B_n(x_n) + \alpha_{n+1}B_{n+1}(x_n) + \alpha_{n+2}B_{n+2}(x_n) \end{aligned} \tag{5}$$

Eliminating α_{-2} and α_{n+2} from the equations (3), (4) and (5), we get

$$y(x) = w_1(x) + \sum_{j=-1}^{n+1} \alpha_j P_j(x) \tag{6}$$

where

$$w_1(x) = \frac{A_0}{B_{-2}(x_0)} B_{-2}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x) \tag{7}$$

and

$$P_j(x)=\begin{cases} B_j(x)-\dfrac{B_j(x_0)}{B_{-2}(x_0)}B_{-2}(x), & j=-1,0,1,2 \\ B_j(x), & j=3,...,n-3 \\ B_j(x)-\dfrac{B_j(x_n)}{B_{n+2}(x_n)}B_{n+2}(x), & j=n-2,n-1,n,n+1. \end{cases} \quad (8)$$

Using the Neumann boundary conditions of (2) to the approximate solution *y(x)* in (6), we get

$$A_1=y'(c)=y'(x_0)=w_1'(x_0)+\alpha_{-1}P_{-1}'(x_0) + \alpha_0P_0'(x_0)+\alpha_1P_1'(x_0)+\alpha_2P_2'(x_0) \quad (9)$$

$$C_1=y'(d)=y'(x_n)=w_1'(x_n)+\alpha_{n-2}P_{n-2}'(x_n) + \alpha_{n-1}P_{n-1}'(x_n)+\alpha_nP_n'(x_n)+\alpha_{n+1}P_{n+1}'(x_n) \quad (10)$$

Eliminating α_{-1} and α_{n+1} from the equations (6), (9) and (10), the approximation for *y(x)* can be written as

$$y(x)=w_2(x)+\sum_{j=0}^{n}\alpha_jQ_j(x) \quad (11)$$

where

$$w_2(x)=w_1(x)+\frac{A_1-w_1'(x_0)}{P_{-1}'(x_0)}P_{-1}(x)+\frac{C_1-w_1'(x_n)}{P_{n+1}'(x_n)}P_{n+1}(x) \quad (12)$$

and

$$Q_j(x)=\begin{cases} P_j(x)-\dfrac{P_j'(x_0)}{P_{-1}'(x_0)}P_{-1}(x), & j=0,1,2 \\ P_j(x) & j=3,\ldots,n-3 \\ P_j(x)-\dfrac{P_j'(x_n)}{P_{n+1}'(x_n)}P_{n+1}(x), & j=n-2,n-1,n \end{cases} \quad (13)$$

Using the second order derivative boundary conditions of (2) to the approximate solution *y(x)* in (11), we get

$$A_2=y''(c)=y''(x_0)=w_2''(x_0)+\alpha_0Q_0''(x_0) + \alpha_1Q_1''(x_0)+\alpha_2Q_2''(x_0) \quad (14)$$

$$C_2=y''(d)=y''(x_n)=w_2''(x_n)+\alpha_{n-2}Q_{n-2}''(x_n) + \alpha_{n-1}Q_{n-1}''(x_n)+\alpha_nQ_n''(x_n) \quad (15)$$

Eliminating α_0 and α_n from the approximations (11), (14) and (15), the approximation for *y(x)* can be written as

$$y(x)=w_3(x)+\sum_{j=1}^{n-1}\alpha_jR_j(x) \quad (16)$$

where

$$w_3(x)=w_2(x)+\frac{A_2-w_2''(x_0)}{Q_0''(x_0)}Q_0(x)+\frac{C_2-w_2''(x_n)}{Q_n''(x_n)}Q_n(x) \quad (17)$$

and

$$R_j(x)=\begin{cases} Q_j(x)-\dfrac{Q_j''(x_0)}{Q_0''(x_0)}Q_0(x), & j=1,2 \\ Q_j(x), & j=3,...,n-3 \\ Q_j(x)-\dfrac{Q_j''(x_n)}{Q_n''(x_n)}Q_n(x), & j=n-2,n-1. \end{cases} \quad (18)$$

Using the third order derivative boundary conditions of (2) to the approximate solution *y(x)* in (16), we get

$$A_3=y'''(c)=y'''(x_0)=w_3'''(x_0)+\alpha_1R_1'''(x_0) + \alpha_2R_2'''(x_0) \quad (19)$$

$$C_3=y'''(d)=y'''(x_n)=w_3'''(x_n) + \alpha_{n-2}R_{n-2}'''(x_n)+\alpha_{n-1}R_{n-1}'''(x_n) \quad (20)$$

Eliminating α_1 and α_{n-1} from the equations (11), (14) and (15), the approximation for $y(x)$ can be written as

$$y(x)=w(x)+\sum_{j=2}^{n-2}\alpha_j\tilde{B}_j(x) \quad (21)$$

where

$$w(x)=w_3(x)+\frac{A_3-w_3'''(x_0)}{R_1'''(x_0)}R_1(x) + \frac{C_3-w_3'''(x_n)}{R_{n-1}'''(x_n)}R_{n-1}(x) \quad (22)$$

and

$$\tilde{B}_j(x)=\begin{cases} R_j(x)-\dfrac{R_j'''(x_0)}{R_1'''(x_0)}R_1(x), & j=2 \\ Rj(x), & j=3,\ldots,n-3 \\ R_j(x)-\dfrac{R_j'''(x_n)}{R_{n-1}'''(x_n)}R_{n-1}(x), & j=n-2 \end{cases} \quad (23)$$

Now the new set of basis functions for the approximation *y(x)* is $\{\tilde{B}_j(x), j=2,...,n-2\}$. Applying the Galerkin method to (1) with a new set of basis functions, we get

$$\int_{x_0}^{x_n}\Big[a_0(x)y^{(8)}(x)+a_1(x)y^{(7)}(x)+a_2(x)y^{(6)}(x) + a_3(x)y^{(5)}(x)+a_4(x)y^{(4)}(x)+a_5(x)y'''(x) + a_6(x)y''(x)+a_7(x)y'(x)+a_8(x)y(x)\Big]\tilde{B}_i(x)\,dx = \int_{x_0}^{x_n}b(x)\tilde{B}_i(x)dx \quad for\, i=2,3,...n-2 \quad (24)$$

Integrating by parts the first four terms on the left hand side of (24) and after applying the boundary conditions prescribed in (2), we get

$$\int_{x_0}^{x_n}a_0(x)\tilde{B}_i(x)y^{(8)}(x)\,dx=\int_{x_0}^{x_n}\frac{d^4}{dx^4}\big[a_0(x)\tilde{B}_i(x)\big]y^{(4)}(x)\,dx$$

$$\int_{x_0}^{x_n} a_1(x)\tilde{B}_i(x)y^{(7)}(x)\,dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3}\left[a_1(x)\tilde{B}_i(x)\right]y^{(4)}(x)\,dx \tag{25}$$

$$\int_{x_0}^{x_n} a_2(x)\tilde{B}_i(x)y^{(6)}(x)\,dx = \int_{x_0}^{x_n} \frac{d^2}{dx^2}\left[a_2(x)\tilde{B}_i(x)\right]y^{(4)}(x)\,dx \tag{26}$$

$$\int_{x_0}^{x_n} a_3(x)\tilde{B}_i(x)y^{(5)}(x)\,dx = -\int_{x_0}^{x_n} \frac{d}{dx}\left[a_3(x)\tilde{B}_i(x)\right]y^{(4)}(x)\,dx \tag{27}$$

(28)

Substituting (25), (26), (27), (28) in (24) and using the approximation for $y(x)$ given in (21), and after rearranging the terms for resulting equations, the resulting system of equations can be written in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{29}$$

where $\mathbf{A} = [a_{ij}]$;

$$a_{ij} = \int_{x_0}^{x_n} \{[\frac{d^4}{dx^4}(a_0(x)\tilde{B}_i(x)) - \frac{d^3}{dx^3}(a_1(x)\tilde{B}_i(x)) + \frac{d^2}{dx^2}(a_2(x)\tilde{B}_i(x)) - \frac{d}{dx}(a_3(x)\tilde{B}_i(x)) + a_4(x)\tilde{B}_i(x)]\,\tilde{B}_j^{(4)}(x) + a_5(x)\tilde{B}_i(x)\tilde{B}_j'''(x) + a_6(x)\tilde{B}_i(x)\tilde{B}_j''(x) + a_7(x)\tilde{B}_i(x)\tilde{B}_j'(x) + a_8(x)\tilde{B}_i(x))\tilde{B}_j(x)\}dx$$

for i =2,3,…n-2; j =2,3,…,n-2 (30)

$\mathbf{B} = [b_i]$;

$$b_i = \int_{x_0}^{x_n} \{b(x)\tilde{B}_i(x) + [-\frac{d^4}{dx^4}(a_0(x)\tilde{B}_i(x)) + \frac{d^3}{dx^3}(a_1(x)\tilde{B}_i(x)) - \frac{d^2}{dx^2}(a_2(x)\tilde{B}_i(x)) + \frac{d}{dx}(a_3(x)\tilde{B}_i(x)) - a_4(x)\tilde{B}_i(x)]\,w^{(4)}(x) - a_5(x)\tilde{B}_i(x)w'''(x) - a_6(x)\tilde{B}_i(x)w''(x) - a_7(x)\tilde{B}_i(x)w'(x) - a_8(x)\tilde{B}_i(x))w(x)\}dx$$

for i = 2,3,…,n-2 (31)

and $\alpha = [\alpha_2\ \alpha_3\ \ldots\alpha_{n-2}]^T$

4. PROCEDURE TO FIND A SOLUTION FOR NODAL PARAMETERS

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} r_i(x)r_j(x)Z(x)dx$ and $r_i(x)$, $r_j(x)$ are the quintic B-spline basis functions or their derivatives. It may be noted that $I_m = 0$ if $(x_{i-3}, x_{i+3}) \cap (x_{j-3}, x_{j+3}) \cap (x_m, x_{m+1}) = \varnothing$. To evaluate each I_m, we employed 6-point Gauss-Legendre quadrature formula. Thus the stiffness matrix $\mathbf{A}$ is an eleven diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ by using a band matrix solution package. We have used the FORTRAN-90 program to solve the boundary value problems (1)-(2) by the proposed method.

5. NUMERICAL RESULTS

To demonstrate the applicability of the proposed method for solving the eighth order boundary value problems of the types (1) and (2), we considered three linear boundary value problems and two nonlinear boundary value problems. Numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$y^{(8)} - 16y = -4, \qquad -1 \le x \le 1 \tag{32}$$

subject to $y(-1) = y(1) = 0$,

$$y'(-1) = -y'(1) = .25\frac{\sinh 2 - \sin 2}{\cosh 2 + \cos 2},$$

$$y''(-1) = y''(1) = 0,$$

$$y'''(-1) = -y'''(1) = -\frac{\sin 1\cos 1 + \cosh 1 \sinh 1}{\cos 2 + \cosh 2}.$$

The exact solution for the above problem is

y(x)=25[1-2(sin 1 sinh1 sin x sinh x + cos 1 cosh1 cos x cosh x)/(cos 2 + cosh x)]

The proposed method is tested for this problem where the domain [-1, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is 7.852912×10^{-6}.

Table 1. Numerical results for Example 1

x	Exact Solution	Absolute Error by proposed method
-0.8	3.976926E-02	6.705523E-08
-0.6	7.498498E-02	1.654029E-06
-0.4	1.023106E-01	4.455447E-06
-0.2	1.195382E-01	6.906688E-06
0.0	1.254157E-01	7.852912E-06
0.2	1.195382E-01	6.698072E-06
0.4	1.023106E-01	3.896654E-06
0.6	7.498498E-02	1.467764E-06
0.8	3.976926E-02	3.725290E-07

Example 2: Consider the linear boundary value problem

$$y^{(8)} + xy = -(48+15x+x^3)e^x, \qquad 0<x<1 \quad (33)$$

subject to $y(0) = y(1) = 0,\ \ y'(0)=1,\ \ y'(1) = -e,$
$y''(0)=0,\ \ y''(1) = -4e,$
$y'''(0) = -3,\ \ y'''(1) = -9e.$

The exact solution for the above problem is $y = x(1-x)e^x$. The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is 1.227856x10^{-5}.

Table 2. Numerical results for Example 2

x	Exact Solution	Absolute Error by proposed method
0.1	9.946539E-02	5.215406E-08
0.2	1.954244E-01	2.220273E-06
0.3	2.834704E-01	7.003546E-06
0.4	3.580379E-01	1.114607E-05
0.5	4.121803E-01	1.227856E-05
0.6	4.373085E-01	8.881092E-06
0.7	4.228881E-01	2.533197E-06
0.8	3.560865E-01	1.817942E-06
0.9	2.213642E-01	2.041459E-06

Example 3: Consider the linear boundary value problem

$$y^{(8)} + y^{(7)} + 2y^{(6)} + 2y^{(5)} + 2y^{(4)} + 2y''' + 2y'' + y' + y = 14\cos x - 16\sin x - 4x\sin x, \qquad 0<x<1 \quad (34)$$

subject to $y(0)=0,\ y(1)=0,$
$y'(0) = -1,\ \ y'(1) = 2\sin 1,$
$y''(0)=0,\ y''(1) = 4\cos 1 + 2\sin 1,\qquad y'''(0) = 7,$
$y'''(1) = 6\cos 1 - 6\sin 1.$

The exact solution for the above problem is $y(x) = (x^2-1)\sin x.$ The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3. The maximum absolute error obtained by the proposed method is 7.688999x10^{-6}.

Table 3. Numerical results for Example 3

x	Exact Solution	Absolute Error by proposed method
0.1	-9.883508E-02	3.799796E-07
0.2	-1.907226E-01	2.145767E-06
0.3	-2.689234E-01	5.632639E-06
0.4	-3.271114E-01	9.745359E-06
0.5	-3.595692E-01	1.138449E-05
0.6	-3.613712E-01	1.013279E-05
0.7	-3.285510E-01	7.271767E-06
0.8	-2.582482E-01	3.874302E-06
0.9	-1.488321E-01	1.430511E-06

Example 4: Consider the nonlinear boundary value problem

$$y^{(8)} + e^{-x}y^2 = e^{-x} + e^{-3x}, \qquad 0<x<1 \quad (35)$$

subject to $y(0)=1,\ \ y(1)=e^{-1},\ \ y'(0)=-1,$
$y'(1) = -e^{-1},\ \ y''(0)=1,\ \ y''(1)=e^{-1},$
$y'''(0)=-1, y'''(1) = -e^{-1}.$

The exact solution for the above problem is $y=e^{-x}$. The nonlinear boundary value problem (35) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [19] as

$$y^{(8)}_{(n+1)} + [2y_{(n)}e^{-x}]y_{(n+1)} = [y_{(n)}]^2 e^{-x} + e^{-x} + e^{-3x}$$

for n=0,1,2,… (36)

subject to $y_{(n+1)}(0) = 1,\ \ y_{(n+1)}(1) = \dfrac{1}{e},$

$y'_{(n+1)}(0) = -1, y'_{(n+1)}(1) = \dfrac{-1}{e},$

$y''_{(n+1)}(0) = 1,\ \ y''_{(n+1)}(1) = \dfrac{1}{e},$

$y'''_{(n+1)}(0) = -1,\ \ y'''_{(n+1)}(1) = \dfrac{-1}{e}.$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of a linear problems (36). Numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is 3.641844x10^{-5}.

Table 4. Numerical results for Example 4

x	Exact Solution	Absolute Error by proposed method
0.1	9.048374E-01	3.576279E-07
0.2	8.187308E-01	6.318092E-06
0.3	7.408182E-01	1.895428E-05
0.4	6.703200E-01	3.099442E-05
0.5	6.065307E-01	3.641844E-05
0.6	5.488116E-01	3.170967E-05
0.7	4.965853E-01	1.925230E-05
0.8	4.493290E-01	7.182360E-06
0.9	4.065697E-01	1.460314E-06

Example 5: Consider the nonlinear boundary value problem

$$y^{(8)} = 7!(e^{-8y} - \frac{2}{(1+x)^8}), \qquad 0 \le x \le e^{\frac{1}{2}} - 1 \qquad (37)$$

subject to $y(0) = 0,\ y(e^{\frac{1}{2}} - 1) = \frac{1}{2},\ y'(0) = 1,$

$$y'(e^{\frac{1}{2}} - 1) = e^{\frac{-1}{2}},\ y''(0) = -1,\ y''(e^{\frac{1}{2}} - 1) = -e^{-1},$$

$$y'''(0) = 2,\ y'''(e^{\frac{1}{2}} - 1) = 2e^{\frac{-3}{2}}.$$

The exact solution for the above problem is $y(x) = ln(1+x).$ The nonlinear boundary value problem (37) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [19] as

$$y_{(n+1)}^{(8)} + (8!e^{-8y_{(n)}})y_{(n+1)} = (8!y_{(n)} + 7!)e^{-8y_{(n)}} - \frac{2\times 7!}{(1+x)^8}$$

for n = 0,1,2,… (38)

subject to $y_{(n+1)}(0) = 0,\quad y_{(n+1)}(e^{\frac{1}{2}} - 1) = \frac{1}{2},$

$$y'_{(n+1)}(0) = 1, y'_{(n+1)}(e^{\frac{1}{2}} - 1) = e^{\frac{-1}{2}},$$

$$y''_{(n+1)}(0) = -1,\quad y''_{(n+1)}(e^{\frac{1}{2}} - 1) = -e^{-1},$$

$$y'''_{(n+1)}(0) = 2,\quad y'''_{(n+1)}(e^{\frac{1}{2}} - 1) = 2e^{\frac{-3}{2}}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x).$ The domain [0,1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of a linear problems (39). Numerical results for this problem are presented in table 5.The maximum absolute error obtained by the proposed method is $1.00135\text{x}10^{-5}$.

Table 5. Numerical results for Example 5

x	Exact Solution	Absolute Error by proposed method
0.1	6.285473E-02	2.011657E-07
0.2	1.219913E-01	4.544854E-07
0.3	1.778251E-01	1.519918E-06
0.4	2.307057E-01	4.068017E-06
0.5	2.809298E-01	6.705523E-06
0.6	3.287517E-01	9.059906E-06
0.7	3.743905E-01	1.001358E-05
0.8	4.180371E-01	5.453825E-06
0.9	4.598581E-01	2.592802E-06

6. CONCLUSIONS

In this paper, a Galerkin method with quintic B-splines as basis functions to solve a general eighth order boundary value problem has been developed. The quintic B-splines basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet boundary conditions, Neumann boundary conditions, secondary order derivative boundary conditions and third order derivative boundary conditions are prescribed. The proposed method has been tested on three linear and two nonlinear eighth order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple and accurate method to solve a general eighth order boundary value problem.

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