Edge Semientire Block Graph

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ABSTRACT
In this communications, the edge semientire block graph of a graph is introduced. We present a characterization of those graphs whose edge semientire block graph is planar, outer planar, Eulerian, Hamiltonian with crossing number one.

General Terms
Semientire graph, Line graph.

Keywords
Edge Semientire graph, line graph, inner vertex number,

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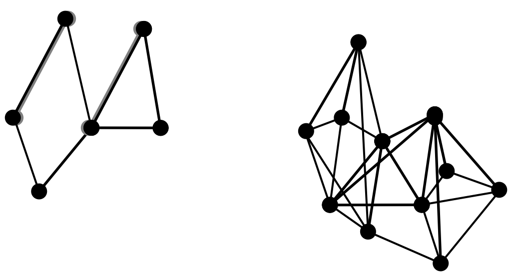
1. INTRODUCTION
The concept of block edge cut vertex graph was introduced by Venkanagouda M Goudar [5]. For the graph G(p, q), if B = u_1, u_2, ...u_r : r ≥ 2 is a block of G, then we say that the vertex u_i and the block B are incident with each other. If two blocks B_1 and B_2 are incident with a common cut vertex, then they are adjacent blocks. All undefined terminology will conform with that in Harary[2]. All graphs considered here are finite, undirected, planar and without loops or multiple edges. We now define the edge semientire block graph.

The Edge semientire block graph of a graph G denoted by E_b(G) is the graph whose vertex set is the union of set of edges, set of blocks and set of regions of G in which two vertices are adjacent if and only if the corresponding edges of G are adjacent, the corresponding edges lies on the blocks, the corresponding edges lies on the region and the corresponding blocks are incident to a cut vertex. The graph G and its edge semientire graph E_b(G) is depicted in the figure 1.



G E_b(G).

Figure 1.

The *edgedegree* of an edge {u v} is the sum of the degree of the vertices of u and v. *Blockdegree* of a cut vertex is the number of blocks incident to the cut vertex and *regionvertex* is a vertex in E_b(G) formed from the regions of G. The *Blockvertex* is a vertex in E_b(G) formed from the block of G. For the planar graph G, the inner vertex number i(G) of a graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. The crossing number Cr(G)of a graph G is the minimum number of pair wise intersections of its edges when G is drawn in the plane. A graph G has crossing number k if A graph G is outerplanar if i(G)=0 and graph G is said to be minimally non-outerplanar if i(G)=1 as was given by Kulli [4].

2. PRELIMINARY RESULTS
We need the following results to prove further results.

Theorem 2.1[2]. If G is a (p, q) graph whose vertices have degree d_i then the line graph L(G) has q vertices and q_L edges, where $q_L = -q + \frac{1}{2}\sum d_i^2$ edges.

Theorem 2.2 [2]. The line graph L(G) of a graph is planar if and only if G is planar, $\Delta \leq 4$ and if deg v = 4 for a vertex v of G, then v is a cut vertex.

Theorem 2.3[3]. A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Theorem 2.4 [4]. A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.

Theorem 2.5 [3]. A finite graph G is Eulerian if and only if all its vertex degrees are even.

3. MAIN RESULTS
Lemma 3.1: For any planar graph G, L(G) $\subseteq$ E_b(G).

Theorem 3.2. For any planar graph G, E_b(G) is always nonseparable.

Proof. We have the following cases.

Case 1. Suppose G be a tree T. Let e_1, e_2, ...e_n be the edges, b_1 = e_1, b_2 =e_2, ...b_n = e_n be the blocks and $r_1$be the region of T. By the definition of line graph L(G), e_1, e_2, ...e_n form a subgraph without isolated vertex. In E_b(G), the regionvertex is adjacent to these vertices to form a graph without isolated vertex. Since there are n blocks which are K_2, we have each b_1 = e_1, b_2 = e_2, ...b_n = e_n are adjacent to e_1, e_2, ...e_n. Further the blockvertices of E_b(G) are adjacent without cut vertex. Hence E_b(G) is nonseparable.

Case 2. Suppose a graph G is contains at least two adjacent cut vertices and is non-tree. Let e_1, e_2, ...e_n ∈ E(G), b_1 = e_1, b_2 = e_2, ...b_k = e_n be the blocks and r_1, r_2, ...r_l be the region of G. By the definition of line graph L(G),the edges incident to the cut vertices becomes a complete graph and L (G) contains at least one cut vertex. By the definition of E_b(G), the region vertices are adjacent to the vertices of L(G) and the vertices of

L(G) is adjacent to the blockvertices. Clearly $E_b(G)$ is nonseparable.

In the following theorem we obtain the number of vertices and edges of edge semientire block graph of a graph.

Theorem3.3 For any planar graph G, edge semientire block graph $E_b(G)$ whose vertices have degree d_i has (q +r+b) vertices and $-q+\frac{1}{2}\sum d_i^2+\sum q_j+\sum \frac{b_k(b_k-1)}{2}+\sum q_r$ edges where r the number of regions, b the number of blocks q_j the number edges in a block b_j, b_k be the block degree of a cut vertex C_k and q_r be the number of edges region r_1.

Proof. By the definition of $E_b(G)$, the number of vertices is the union of edges, regions and blocks of G. Hence $E_b(G)$ has (q + r + b) vertices. Further by the Theorem 2.1, number of edges in L(G) is$q_L=-q+\frac{1}{2}\sum d_i^2$. The degree of each *regionvertex* is the number of edges lies on that region in G. Thus the number of edges in $E_b(G)$, is the sum of the number of edges in L(G), the number of edges bounded by the each region which is $\sum q_r$,number of edges lies on each block which is $\sum q_j$ and the number of edges in a complete graph K_k which is obtained from each *blockdegree* and is $\frac{b_k(b_k-1)}{2}$. Henace the number of edges in $E_b(G)$ is

$$-q+\frac{1}{2}\sum d_i^2+\sum q_j+\sum \frac{b_k(b_k-1)}{2}+\sum q_r.$$

Theorem3.4 For any edge in a plane graph G with *edgedegree* n, the degree of a corresponding vertex in $E_b(G)$ is i). n+1 if edge lie on two regions and ii). n, otherwise.

Proof. Suppose an edge $e_i \in E(G)$ have degree n. We have the following cases.

Case1. Suppose the edge lie on two regions. Clearly (n-2) edges are adjacent to the edge $e_i \in E(G)$. By the definition of line graph, the degree of the vertex $e_i \in L(G)$ is (n-2). By the definition of Edge semientire block graph, this edge is lies on two regions and lies on one block, hence degree of a vertex $e_i \in E(G)$ in $E_b(G)$, is n-2+2+1=n+1.

Case2. Further, if $e_i \in E_b(G)$,lie on one region then by the case 1, the degree of a vertex e_I in $E_b(G)$, is n-2+1+1=n.

In the following theorem we obtain the condition for the planarity on edge semientire block graph of a graph.

Theorem 3.5 For any planar graph G, the $E_b(G)$ is planar if and only if G is i).a path P_n or ii). a star $K_{1,3}$.

Proof. Suppose $E_b(G)$ is planar. We have the following cases.

Case 1. Assume that a graph G is $K_{1,4}$. By the definition of line graph, e_1, e_2, e_3 , e_4 form a complete graph K_4 and it has one inner vertex. Since all edges e_1, e_2, e_3 , e_4 lies on one region r_1, which is adjacent to all vertices of K_4 in $E_b(G)$. Clearly $E_b(G)$ is non-planar, a contradiction.

Case 2. Assume that a graph G is non –path and it is a tree with at least one sub graph as $K_{1,3}$ and contains at least four edges. Clearly $L(K_{1,3})=K_3$. Also each edge is a block and all blocks are adjacent. So these blocks form a complete graph K_3. Since all four edges lies on one region r_1 and are adjacent to all vertices of K_3 in $E_b(G)$. Clearly the fourth edge is also adjacent to r_1 and it crosses the edge already drawn, which is a contradiction.

Case 3. Assume that G is any graph which contains at least one block with three edges. Let e_1, e_2, e_3 be the edges of a block b_i. These edges form one interior region r_j. By the definition of $E_b(G)$, the vertices e_1, e_2, e_3, b_i , r_j forms graph homeomorphic to $\langle K_5\rangle$, which is non-planar, a contradiction.

By the Theorem 2.3, it is non planar, a contradiction.

Conversely, suppose G is a path. By the definition of L(G), $L(P_n) = P_{n-1}$. Since all edges all edges lies on only one region and each edge is a block then by definition of $E_b(G)$, it forms a graph homeomorphic to K_4. Hence $E_b(G)$ is planar. Further if G is $K_{1,3}$ then $L(K_{1,3})=K_3$.Since all three edges lies on only one region and each edge is a block then by definition of $E_b(G)$, it forms a graph homeomorphic to K_4. Hence $E_b(G)$ is planar.

In the following theorem we obtain the condition for the outer planarity on Edge semientire block graph of a graph.

Theorem 3.6 For any planar graph G, $E_b(G)$ is outer planar if and only if G is a path P_n for $n\leq 3$.

Proof. Suppose $E_b(G)$ is outer planar. Assume that G is a path P_n $n\geq 4$. Suppose n = 4. By the definition of line graph, $L(P_4)=P_3$. Since all the edges e_1, e_2, e_3 are lies on only one region r_1, then $E_b(G)$ forms $K_4 - x$. Further each edge is a block, so each e_i is adjacent to the blockvertex b_j. Lastly each blockvertex is adjacent to the corresponding edge to form K_3. Clearly the non-pendent edge of G forms an inner vertex in $E_b(G)$, which is non- outerplanar, a contradiction. . Hence G must be a path P_n for $n\leq 3$.
Conversely, Suppose G is a path P_n for $n\leq 3$. Let e_1, e_2, e_3 be the edges in G. By the definition of line graph, $L[P_n] = P_{n-1}$. Further by the lemma 3.2, $b_1 = e_1$, $b_2 = e_2$, ..., $b_{n-1} = e_{n-1}$ becomes n-1 pendant vertices and it becomes a caterpillar. Further the region vertex r_i is adjacent to all the vertices of $L[P_n](G)$. Clearly $E_b(G)$ is outer planar.

In the following theorem we obtain the condition for the minimally non- outerplanar on Edge semientire block graph of a graph.

Theorem 3.7 For any planar graph G, $E_b(G)$ is minimally non-outer planar if and only if G is P_4.

Proof. Suppose $E_b(G)$ is minimally non-outer planar. Assume that G is a path P_n $n\geq 5$. Suppose n = 5. By the definition of line graph, $L(P_5)=P_4$. In a path, each edge is a block ,for each e_i is adjacent to b_i. Clearly the vertices of $L(P_5)$ and b_i form a graph $C_4.C_4\ldots C_4$. Also the vertices e_1, e_2, e_3, e_4 are lies on only one region r_1, then by the definition of $E_b(G)$, It contains a graph with two inner vertex number, a contradiction. Hence G is a path P_4.

Conversely, Suppose G is a path P_4. Let e_1, e_2, e_3 be the edges in G. By the definition of line graph, $L[P_4] = P_3$. In a path, each edge is a block ,for each e_i is adjacent to b_i. Clearly the vertices of $L(P_4)$ and b_i form a graph $C_4.C_4$. In a tree all edges lies on only one region r_1 and this region is adjacent to all vertices of $L(P_4)$ in $E_b(G)$. the vertex e_3 which was the inner edge in G becomes the inner vertex in $E_b(G)$. Clearly $E_b(G)$ is minimally non- outerplanar.

In the following theorem we obtain the condition for the crossing number one on Edge semientire block graph of a graph.

Theorem 3.8 For any planar graph G, $E_b(G)$ has crossing number one if and only if G is a tree with at least one vertex of degree two and an unique vertex of degree three.

Proof. Suppose $E_b(G)$ has crossing number one. We have the following cases.

Case 1. Assume that a graph G is a tree with two vertices v_i and v_j are of degree 3. Let e_{i1}, e_{i2}, e_{i3} be the edges incident to the cut vertex v_i and e_{j1}, e_{j2}, e_{j3} be the edges incident to v_j. By the definition of line graph, each star v_1 and v_2 form induced subgraph as K_4. In $E_b(G)$, the region vertex r_1 is adjacent to all vertices of L(G) to form a planar graph. Further the blockvertex is adjacent to the vertices of L(G) those are the edges of block in G form graph homeomorphic to K_5. Clearly $C[E_b(G)] \geq 2$, a contradiction.

Case 2. Assume that a graph G is a tree with one cut vertex of degree 4. By the definition of line graph, L(G) form K_4 as induced subgraph. Since each edge is a block, the vertices e_i and b_i for i= 1,2,3,4 form K_4oK_1. Further the regionvertex is adjacent to all vertices of L(G) such that $C[E_b(G)] \geq 2$, a contradiction. G is a tree with at least one vertex of degree two and an unique vertex of degree three.

Conversely suppose G is a tree with at least one vertex of degree two and an unique vertex of degree three. The line graph of a graph with a vertex of degree three becomes K_3 as subgraph. Since the regionvertex is adjacent to all the vertices of a line graph, in $E_b(G)$, the line graph L(G) and a regionvertex forms a complete graph K_4. Clearly it contains one inner vertex. By the definition block graph B(G), all block vertices b_i, b_j, b_k are adjacent to form K_3. Hence in $E_b(G)$ all block vertices becomes the inner vertices. By the condition G contains at least one vertex of degree two. The block vertex of this edge b_l is adjacent to at least one vertex of B(G). Clearly it crosses the edge we already drawn. Hence G has crossing number one.

In the following theorem we obtain the condition for the non-

Eulerian on Edge semientire block graph of a graph.

Theorem 3.8 For any planar graph G, $E_b(G)$ is Eulerian if and only if the following conditions hold

i). Edgedegree of and edge is odd if the edge lies on two regions ii). Edgedegree of an edge is even if the edge lies on one region. iii). Each region contains even number of edges.

Proof. Suppose G is Eulerian. We have the following cases.

Case 1. Assume that edge lies on two regions with edge degree even. By the Theorem 3.4, the degree of the corresponding vertex in $E_b(G)$ becomes odd. By the Theorem2.5, the graph $E_b(G)$ is non-Eulerian, a contradiction.

Case 2. Assume that the edge lie on only one region with odd degree. By the Theorem 3.4, the degree of the corresponding vertex in $E_b(G)$ becomes odd. By the Theorem2.5, the graph $E_b(G)$ is non-Eulerian, a contradiction.

Case 3. Assume that each region contains odd number of edges. By the definition of $E_b(G)$, each regionvertex is adjacent to all vertices those are the edges covered by the region in G. Since each region contains odd number of edges, the corresponding regionvertex of degree is odd. By the Theorem2.5, the graph $E_b(G)$ is non-Eulerian, a contradiction.

Hence G must satisfies all the above a conditions.

Conversely suppose a graph G satisfies all the conditions. Suppose edgedegree is odd if it lies in two regions. By the Theorem 3.4, the corresponding vertex in $E_b(G)$ becomes even. Further edgedegree of edge is even if the edge lies on one region. Again by the Theorem 3.4, the degree of the corresponding vertex in $E_b(G)$ becomes even. Lastly each region contains even number of edges. By the definition of $E_b(G)$, the degree of the corresponding vertex is even. Thus each vertex is of degree even Hence $E_b(G)$ is Eulerian.

In the following theorem we obtain the condition for the Hamiltonian on Edge semientire block graph of a graph.

Theorem 3.9 For any graph G , $E_b(G)$ is always Hamiltonian.

Proof. Suppose G is any graph. Let e_1, e_2, ...e_{n-1} be the edge set, b_1, b_2, ...b_i be the blocks and r_1, r_2, ...r_k be the regions of G such that e_1, e_2, ... $e_\ell \in V(b_1)$, $e_{\ell+1}, e_{\ell+2}$ …,...$e_m \in V(b_2)$, ... e_{m+1}, e_{m+2}, ...$e_{n-1} \in V(b_i)$. By the Theorem 3.3, $V[E_b(G)] = e_1, e_2, ...e_{n-1} \cup b_1, b_2, \ldots b_i \cup r_1, r_2, ...r_k$. By theorem 3.2 $E_b(G)$ is always non -separable. By the definition, b_1e_1, e_2, ... $e_{\ell-1}$ r_1b_2... $r_2e_mb_3$...e_{k+1}, e_{k+2},... $e_{n-1}b_kr_k e_\ell$ b_1 form a cycle which contains all the vertices of $E_b(G)$. Hence $E_b(G)$ is Hamiltonian.

4. CONCLUSION

In this paper the relation between the line graph and the edge semientire block graph is introduced. Further the conditions of planarity, Hamiltonian and Eulerian are established.

5. REFERENCES

[1] Chartrand G, Geller, D and Hedetniemi, S. J. Combinatorial Theory (1971)10:12.

[2] Harary F., Graph theory, Addition - Weseley Reading .Mass.(1969), pp 34 and 107.

[3] Harary, F., Annals of New York Academy of Science,(1975),175 :198.

[4] Kulli V R., On minimally non-outer planar graphs, Proceeding of the Indian National Science Academy, Vol.41, Part A, No.3 (1975), pp 275 -280.

[5] Y.B. Maralabhavi and Venkanagouda. M. Goudar."On Block edge cut vertex graph", in the Journal of ACTA CIENCIA INDICA, Vol. XXXIII M.No.2, (2007) pp. 493-497.