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Coupled Fixed Points Theorems for (ϕ, ψ)-Contractive Operators on G-Metric Spaces without Mixed Monotone

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ABSTRACT

The aim of this paper is to establish some coupled fixed point theorems in G-metric spaces using (ϕ, ψ) contractions for a mapping $F : X \times X \to X$. The result of this paper is conversion of the result of Phakdi Charoensawan [5] into G-metric space.

General Terms:

47H10, 54H25

Keywords:

Coupaed fixed point, G-metric space, mixed monotone property

1. INTRODUCTION

Banach contraction principle is one of the core subject that has been studied. One of the remarkable generalizations, known as ϕ-contraction, was given by Boyd and Wong [2] in 1969. In 2006, Gnana-Bhaskar and Lakshmikantham [6] introduced the notion of coupled fixed point and proved some fixed point theorems under certain conditions. After this many author worked on coupled fixed point theorems and gave very useful results in the arena of fixed point theory. Mustafa and Sims [8] introduced the notion of generalized metric space or simply G-metric space as a generalization of the concept of metric space.

2. PRELIMINARIES

DEFINITION 2.1 [8]. *Let X be a nonempty set, and let $G : X \times X \times X \to R^+$, be a function satisfying:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$
(G2) $0 < G(x, x, y)$, *for all* $x, y \in X$; *with* $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ *(symmetry in all three variables), and*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$, *(rectangle inequality),*

then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is a G-metric space.

DEFINITION 2.2 [8]. *A G-metric space (X, G) is symmetric if*

(G6) $G(x, y, y) = G(x, x, y)$, *for all* $x, y \in X$.

DEFINITION 2.3 [6]. *An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

DEFINITION 2.4. *Let (X, G) be a G-metric space and $F : X \times X \to X$ be a given mapping. Let M be a nonempty subset of X^4. We say that M is an F-invariant subset of X^4 if and only if,*

$$\text{for all } x, y, u, v \in X,\ (x, y, u, v) \in M$$
$$\Rightarrow \quad (F(x, y), F(y, x), F(u, v), F(v, u)) \in M.$$

DEFINITION 2.5. *Let (X, G) be a G-metric space and M be a subset of X^4. We say that M satisfies the transitive property if and only if,*

$$\text{for all } x, y, u, v, a, b \in X,$$
$$(x, y, u, v) \in M \text{ and } (u, v, a, b) \in M \quad \Rightarrow \quad (x, y, a, b) \in M.$$

DEFINITION 2.6 [6]. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1 \leq x_2 \ \Rightarrow\ F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X \text{ and}$$
$$y_1 \leq y_2 \ \Rightarrow\ F(x, y_2) \leq F(x, y_1), \quad \text{for } y_1, y_2 \in X.$$

By following, Matkowski [10]
Let Φ denote the set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

(i_ϕ) ϕ is continuous and non-decreasing,
(ii_ϕ) $\phi(t) = 0$ if and only if $t = 0$ and,
(iii_ϕ) $\phi(t + s) \leq \phi(t) + \phi(s)$ for all $t, s \in [0, \infty)$

and Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy

(i_ψ) $\lim\limits_{t \to 0^+} \psi(t) > 0$ for all $r > 0$, and
(ii_ψ) $\lim\limits_{t \to r} \psi(t) = 0$.

3. MAIN RESULTS

THEOREM 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric G on X such that (X, G) is a complete G-metric space. Let$F : X \times X \to X$ be a mapping for which there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ with $(x, y, u, v) \in M$.*

$$\phi \frac{\begin{pmatrix} G(F(x,y), F(u,v), F(u,v)) \\ +G(F(y,x), F(v,u), F(v,u)) \end{pmatrix}}{2} \leq \phi \frac{G(x,u,u) + G(y,v,v)}{2} - \psi \frac{G(x,u,u) + G(y,v,v)}{2} \quad (1)$$

suppose either

(a) *F is continuous or*
(b) *for any two sequences $\{x_n\}$ and $\{y_n\}$ with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$, $\{x_n\} \to x$ and $\{y_n\} \to y$ for all $n \geq 1$ implies $(x, y, x_n, y_n) \in M$ for all $n \geq 1$.*

If there exists $(x_0, y_0) \in X \times X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$ and M is an F-invariant set which satisfies the transitive property. then there exists $x, y \in X$ such that $x = F(x, y)$, $y = F(y, x)$.

PROOF. Let $(x_0, y_0) \in X \times X$. Since $F(X \times X) \subseteq X$, we can choose $x, y \in X$ such that $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Again from $F(X \times X) \subseteq X$, we can choose $x_2, y_2 \in X$ such that $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$.
Continuing like this we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_n = F(x_{n-1}, y_{n-1}) \text{ and } y_n = F(y_{n-1}, x_{n-1}) \quad \text{for all } n \geq 1 \quad (2)$$

If there exist $k \in N$ such that $x_k = x_{k-1}$ and $y_k = y_{k-1}$ then $x_k = x_{k-1} = F(x_{k-1}, y_{k-1})$ and $y_k = y_{k-1} = F(y_{k-1}, x_{k-1})$. Thus (x_{k-1}, y_{k-1}) is a coupled fixed point of F. Then our result is proved. There we may assume that $x_k \neq x_{k-1}$ and $y_k \neq y_{k-1}$ for all $n \geq 1$.
Since

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) = (x_1, y_1, x_0, y_0) \in M$$

and M is an F-invariant set, we have

$$(F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) = (x_2, y_2, x_1, y_1) \in M$$

Continuing like this, we obtain

$$(F(x_n, y_n), F(y_n, x_n), F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})) = (x_{n+1}, y_{n+1}, x_n, y_n) \in M \text{ for all } n \geq 1.$$

Now, let the sequence of non negative real numbers, $\{\delta_n\}_{n=1}^{\infty}$ given by

$$\delta_{n+1} = \frac{G(x_{n+1}, x_n, x_n) + G(y_{n+1}, y_n, y_n)}{2}, \quad n \geq 0\,. \quad (3)$$

Since $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ then

$$\frac{\begin{pmatrix} G(F(x_n, y_n), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\ +G(F(y_n, x_n), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) \end{pmatrix}}{2} = \frac{G(x_{n+1}, x_n, x_n) + G(y_{n+1}, y_n, y_n)}{2} = \delta_{n+1}$$

So, from the right hand side of (1), we have

$$\phi\left(\frac{G(x_n, x_{n-1}, x_{n-1}) + G(y_n, y_{n-1}, y_{n-1})}{2}\right) - \psi\left(\frac{G(x_n, x_{n-1}, x_{n-1}) + G(y_n, y_{n-1}, y_{n-1})}{2}\right) = \phi(\delta_n) - \psi(\delta_n).$$

Therefore the sequence $\{\delta_n\}_{n=1}^{\infty}$ satisfies

$$\phi(\delta_{n+1}) \leq \phi(\delta_n) - \psi(\delta_n) \leq \phi(\delta_n), \quad \text{for all } n \geq 0. \quad (4)$$

From (4) and $(\mathrm{i})_\phi$ it follows that the sequence $\{\delta_n\}_{n=1}^{\infty}$ is non increasing. So, there is some $\delta \geq 0$, such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[\frac{G(x_n, x_{n-1}, x_{n-1}) + G(y_n, y_{n-1}, y_{n-1})}{2}\right] = \delta. \quad (5)$$

Now, we shall show that $\delta = 0$. Let if possible $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) of both sides of (4) and by the property (i_ψ) and (i_ϕ), we get

$$\begin{aligned} \phi(\delta) &= \lim_{n\to\infty} \phi(\delta_n) \\ &\leq \lim_{n\to\infty} [\phi(\delta_{n-1} - \psi(\delta_{n-1})] \\ &= \phi(\delta) - \lim_{\delta_{n-1}\to\delta} \psi(\delta_{n-1}) \\ &< \phi(\delta) \end{aligned}$$

which is a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[\frac{G(x_n, x_{n-1}, x_{n-1}) + G(y_n, y_{n-1}, y_{n-1})}{2}\right] = 0. \quad (6)$$

Now, we will show that $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in X. Let if possible at least one of $\{x_n\}$ and $\{y_n\}$ is not Cauchy sequence, then there exist an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of sequence $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$\frac{1}{2}[G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)})] \geq \varepsilon. \quad (7)$$

Now, corresponding to $m(k)$, we can take $n(k)$ in such a way that is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (7). Then

$$\frac{1}{2}[G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)})] < \varepsilon. \quad (8)$$

Let $\frac{1}{2}[G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)})] = r_k$.
Using (7) and (8), and the triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq r_k \\ &= \frac{1}{2}[G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)})] \\ &\leq \frac{1}{2}[G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)}, x_{m(k)})] \\ &\quad + \frac{1}{2}[G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)})] \\ &\leq \frac{1}{2}[G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1})] + \varepsilon. \end{aligned}$$

Taking $k \to \infty$ and using (6), we obtain

$$\begin{aligned}\varepsilon &\leq \lim_{k\to\infty} r_k \\ &\leq \lim_{k\to\infty}\left[\frac{1}{2}[G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1})\right] \\ &\qquad + \lim_{k\to\infty}\varepsilon = 0+\varepsilon=\varepsilon,\end{aligned}$$

that is

$$\begin{aligned}\lim_{k\to\infty} r_k &\leq \lim_{k\to\infty}\left[\frac{1}{2}[G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)})\right] \\ &= \varepsilon\,. \end{aligned} \tag{9}$$

By triangle inequality, we get

$$\begin{aligned}&G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \\ &\leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) \\ &\quad + G(x_{m(k)+1}, x_{m(k)}, x_{m(k)})\end{aligned}$$

Similarly,

$$\begin{aligned}&G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\ &\leq G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}) + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) \\ &\quad + G(y_{m(k)+1}, y_{m(k)}, y_{m(k)})\end{aligned}$$

This shows that

$$\begin{aligned}r_k &= \frac{1}{2}[G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}, y_{m(k)})] \\ &\leq \frac{1}{2}\Big[G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) \\ &\quad + G(x_{m(k)+1}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}) \\ &\quad + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) + G(y_{m(k)+1}, y_{m(k)}, y_{m(k)}) \\ &= \delta_{n(k)} + \delta_{m(k)} + \frac{1}{2}[G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) \\ &\qquad + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1})]\end{aligned} \tag{10}$$

Since $n(k) > m(k)$ and M satisfies the transitive property from

$$(x_{n(k)}, y_{n(k)}, x_{n(k)-1}, y_{n(k)-1}) \in M$$

and

$$(x_{m(k)+1}, y_{m(k)+1}, x_{m(k)}, y_{m(k)}) \in M,$$

We have $(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) \in M$, by (1), we get

$$\begin{aligned}&\phi\left[\frac{1}{2}(G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) + G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}))\right] \\ &= \phi\left[\frac{1}{2}(G(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}), F(x_{m(k)}, y_{m(k)})) \right. \\ &\quad \left. + \frac{1}{2}(G(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}))\right] \\ &\leq \phi\left[\frac{1}{2}(G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}), y_{m(k)}) \right. \\ &\quad \left. - \psi\,\frac{1}{2}(G(x_{n(k)}, x_{m(k)}, x_{m(k)}) + G(y_{n(k)}, y_{m(k)}), y_{m(k)})\right] \\ &= \phi(r_k) - \psi(r_k)\,.\end{aligned} \tag{11}$$

Now, using (10) and property (iii$_\phi$), we obtain

$$\begin{aligned}\phi(r_k) &\leq \phi(\delta_{n(k)} + \delta_{m(k)}) \\ &\quad + \phi\left[\frac{\begin{pmatrix} G(x_{n(k)+1}, x_{m(k)+1}, x_{m(k)+1}) \\ +G(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1})\end{pmatrix}}{2}\right]\end{aligned} \tag{12}$$

From (11) and (12), we get

$$\phi(r_k) \leq \phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(r_k) - \psi(r_k) \tag{13}$$

Taking $k \to \infty$ in (13), and using (6) and (9), and property of ϕ and ψ, we have

$$\begin{aligned}\phi(\varepsilon) &= \lim_{k\to\infty}\phi(r_k) \\ &= \psi\left(\lim_{k\to\infty} r_k\right) \\ &\leq \lim_{k\to\infty}[\phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(r_k) - \psi(r_k)] \\ &= \phi\left[\lim_{n\to\infty}(\delta_{n(k)} + \delta_{m(k)})\right] + \phi\left(\lim_{k\to\infty}(r_k)\right) - \lim_{k\to\infty}\psi(r_k) \\ &= \phi(0) + \phi(\varepsilon) - \lim_{k\to\infty}\psi(r_k) < \phi(\varepsilon)\end{aligned}$$

which is a contradiction. This shows that $\{x_n\}$ and $\{y_n\}$ are cauchy sequences. Since X is complete G-metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y\,.$$

Now, condition (a) holds. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x).$$

Suppose condition (b), holds, we get that a sequence $\{x_n\} \to x$ and $\{y_n\} \to y$ by the assumption, we have $(x, y, x_n, y_n) \in M$ for all $n \geq 1$.
Since ϕ is non decreasing and (1), so

$$\begin{aligned}&\phi\left[\frac{1}{2}(G(x, F(x,y), F(x,y)) - G(x, x_{n+1}, x_{n+1}) \right. \\ &\quad \left. +G(y, F(y,x), F(y,x)) - G(y, y_{n+1}, y_{n+1}))\right] \\ &\leq \phi\left[\frac{1}{2}(G(F(x_n, y_n), F(x,y), F(x,y)) \right. \\ &\quad \left. +G(F(y_n, x_n), F(y,x), F(y,x)))\right] \\ &\leq \phi\left[\frac{1}{2}(G(x_n, x, x) + G(y_n, y, y))\right] \\ &\quad - \psi\left[\frac{1}{2}(G(x_n, x, x) + G(y_n, y, y))\right] \\ &\leq \phi\left[\frac{1}{2}(G(x_n, x, x) + G(y_n, y, y))\right]\end{aligned}$$

Letting $n \to \infty$, in the above inequality, We get

$$\begin{aligned}&\phi\left[\frac{1}{2}(G(x, F(x,y), F(x,y) + G(y,, F(y,x), F(y,x)))\right] \\ &\leq \phi(0) = 0.\end{aligned}$$

which shows, from (i_ϕ) and (iii_ψ), that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

$\square$

THEOREM 3.2. *In addition to the hypotheses of Theorem 3.1 suppose that for every (x, y), $(x', y') \in X \times X$ there exist a $(u, v) \in X \times X$ such that $(x, y, u, v) \in M$ and $(x', y', u, v) \in M$. Then F have a unique coupled fixed point.*

PROOF. From Theorem 3.1,the set of coupled fixed points of F is non empty. Suppose that (x, y) and $(x', y') \in X \times X$ are two coupled fixed points of F. We shall show that $x = x'$ and $y = y'$. Since (x, y), $(x', y') \in X \times X$, there exist $(u, v) \in X \times X$ such that (u, v) is comparable to (x, y) and (x', y').
Now, we construct the sequences $\{u_n\}$ and $\{v_n\}$ like as $u_0 = u$, $v_0 = v$, $u_{n+1} = F(u_n, v_n)$ and $v_{n+1} = F(v_n, u_n)$ for all $n \geq 0$. Now, set $x_0 = x$, $y_0 = y$, $x'_0 = x'$, $y'_0 = y'$ and in similar way, construct the sequences $\{x_n\}$, $\{y_n\}$, $\{x'_n\}$ and $\{y'_n\}$. That is as above for all $n \geq 0$,

$$\begin{aligned} x_{n+1} &= F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n), \\ x'_{n+1} &= F(x'_n, y'_n), \quad y'_{n+1} = F(y'_n, x'_n), \end{aligned}$$

Since M is F-invariant and $(x, y, u, v) = (x, y, u_0, v_0) \in M$, we get $(F(x, y), F(y, x), F(u_0, v_0), F(v_0, u_0)) = (x, y, u, v) \in M$. It is easy to show that $(x, y, u_n, v_n) \in M$.
Therefore, by Theorem 3.1,

$$\begin{aligned} &\phi\left[\frac{1}{2}(G(x, u_{n+1}, u_{n+1}) + G(y, v_{n+1}, v_{n+1}))\right] \\ &\quad = \phi\left[\frac{1}{2}(G(F(x, y), F(u_n, v_n), F(u_n, v_n)) \right. \\ &\qquad \left. + G(F(y, x), F(v_n, u_n), F(v_n, u_n)))\right] \\ &\leq \phi\left[\frac{1}{2}(G(x, u_n, u_n) + G(y, v_n, v_n))\right] \\ &\qquad - \psi\left[\frac{1}{2}(G(x, u_n, v_n) + G(y, v_n, v_n))\right] \end{aligned} \tag{14}$$

Using the property of ψ, we get

$$\begin{aligned} &\phi\left[\frac{1}{2}(G(x, u_{n+1}, u_{n+1}) + G(y, v_{n+1}, v_{n+1}))\right] \\ &\quad \leq \phi\left[\frac{1}{2}(G(x, u_n, u_n) + G(y, v_n, v_n))\right] \end{aligned}$$

As ϕ is non decreasing, so

$$\begin{aligned} &\left[\frac{1}{2}(G(x, u_{n+1}, u_{n+1}) + G(y, v_{n+1}, v_{n+1}))\right] \\ &\quad \leq \frac{1}{2}\left[G(x, u_n, u_n) + G(y, v_n, v_n)\right] \end{aligned}$$

Let

$$\delta_n = \frac{1}{2}\left[G(x, u_n, u_n) + G(y, v_n, v_n)\right], \quad n \geq 0.$$

So $\{\delta_n\}$ is non-decreasing. Hence, there exist $\alpha \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{1}{2}\left[G(x, u_n, u_n) + G(y, v_n, v_n)\right] = \alpha. \tag{15}$$

now, we shall show that $\alpha = 0$. Let if possible, $\alpha > 0$. Taking lim as $n \to \infty$ in (14). By (15), we hae

$$\begin{aligned} \phi(\alpha) &= \lim_{n\to\infty}\left[\phi\left(\frac{G(x, u_{n+1}, u_{n+1}) + G(y, v_{n+1}, v_{n+1})}{2}\right)\right] \\ &\leq \lim_{n\to\infty} \phi\left[\frac{G(x, u_n, u_n) + G(y, v_n, v_n)}{2}\right] \\ &\quad - \lim_{n\to\infty} \psi\left[\frac{G(x, u_n, u_n) + G(y, v_n, v_n)}{2}\right] \\ &= \phi(\alpha) - \lim_{n\to\infty} \psi\left[\frac{G(x, u_n, u_n) + G(y, v_n, v_n)}{2}\right] \end{aligned}$$

By property of ψ, we get $\phi(\alpha) < \phi(\alpha)$, which is a contradiction. Hence $\alpha = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[G(x, u_n, u_n) + G(y, v_n, v_n)\right] = 0$$

which shows that

$$\lim_{n\to\infty} G(x, u_n, u_n) = \lim_{n\to\infty} G(y, v_n, v_n) = 0$$

Similarly, we get

$$\lim_{n\to\infty} G(x', u_n, u_n) = \lim_{n\to\infty} G(y', v_n, v_n) = 0$$

and hence $x = x'$ and $y = y'$. $\square$

EXAMPLE 3.3. *Let $X = \mathbb{R}$, $G(x, y, z) = |x - y| + |y - z| + |z - x|$, and $F : X \times X \to X$ be such that*

$$F(x, y) = \frac{x + y}{24}, \quad (x, y) \in X^2.$$

The mapping F does not satisfy the mixed monotone property. It is easy to show that F satisfies (1) with $M = X^4$, $\phi(t) = \dfrac{t}{3}$, $\psi(t) = \dfrac{t}{6}$ and $(0, 0)$ is the unique coupled fixed point of F.

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