A Combined Study of New Escape Time Fractal for Sine and Inverse Tangent Functions

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ABSTRACT
Complex graphics of dynamical system have been a subject of intense research nowadays. The fractal geometry is the base of these beautiful graphical images. Many researchers and authors have worked to study the complex nature of the two most popular sets in fractal geometry, the Julia set and the Mandelbrot set, and proposed their work in various forms using existing tools and techniques. The generation of fractals and study of the dynamics of transcendental function is one of the emerging and interesting fields of research nowadays. Recently, Ashish Negi, Rajeshri Rana and Yashwant S. Chauhan are among those researchers who have contributed a lot in the area of Fractal Geometry applications. In this paper we review the recently done work on sine and inverse tangent functions.

Keywords
Fractals, Relative Superior Mandelbrot Set, Relative Superior Julia Set, Ishikawa Iteration.

1. INTRODUCTION
In 1918, French mathematician Gaston Julia [12] investigated the iteration process of complex function and attained a Julia set, which is a landmark in the field of fractal theory. The object Mandelbrot set on the other hand was given by Benoit B. Mandelbrot [14] in 1979. The visual complexity, beauty and self-similarity of these structures have made these subjects of a wide area of intense research right from its advent. The various extensions and variants of both of these sets have been extensively studied using Picard's iterations. We have applied in this research article a new iteration process called Ishikawa iteration.

Fractals are mathematical fireworks thriving on the new horizons of research in modern analysis and computers. Fractal theory is an exciting branch of mathematical sciences, whose mere existence have worried the founders of modern analysis and so in recent more sympathetic light has been shed on these entities. They can be a geometrical representation of ubiquitous natural objects like clouds, rivers and forests. These all are fractals in nature and can be modeled on a computer using a recursive algorithm of computer graphics.

In this paper we studied and reviled the dynamical behavior of inverse tangent function also termed as arc tangent function, along with the sine function, which falls under the category of transcendental functions. Fixed points are determined using relative superior Ishikawa iterates to develop an entirely new class of fractal images for these functions. Escape criteria of polynomials are used to generate relative superior Mandelbrot sets and relative superior Julia sets.

The study of dynamical behavior of the transcendental function were initiated by Fatou[10]. For transcendental function, points with unbounded orbits are not in Fatou sets but they must lie in Julia sets. Attractive points of a function have a basin of attraction, which may be disconnected. A point z in Julia for cosine function has an orbit that satisfies |Im z|>=50.

A Julia set thus, satisfies the following properties:

(i) Closed

(ii) Nonempty

(iii) Forward invariant t (If z J(F)), then F(z) J(F), where F is the function

(iv) Backward invariant

(v) Equal to the closure of the set of repelling cycles of F.

2. ELABORATION OF CONCEPTS INVOLVED

2.1 Mandelbrot Set:
Definition 1. The Mandelbrot set M for the quadratic $Q_C(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of point 0 is bounded, that is, $M = \{c \in C : \{Q_c^n(0)\}; n = 0,1,2,3\ldots \text{ is bounded}\}$

An equivalent formulation is

$$M = \{c \in C : \{Q_c^n(0) \text{ does not tends to } \infty \text{ as } n \to \infty\}\}$$

We choose the initial point 0, as 0 is the only critical point of Q_c .

2.2 Julia Set:
Definition 2. The set of points K whose orbits are bounded under the iteration function of $Q_c(z)$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of $Q_c(z)$.

2.3 Ishikawa Iteration:
Definition 3. Ishikawa Iterates [11, 13, 16]: Let X be a subset of real or complex number and $f : X \to X$ for

all $x_0 \in X$, we have the sequence {xn} and {yn} in X in the following manner:

$$y_n = S'_n f(x_n) + (1 - S'_n) x_n$$

$$x_{n+1} = S_n f(y_n) + (1 - S_n) x_n$$

Where $0 \le S'_n \le 1,\ 0 \le S_n \le 1$ and S'_n & S_n are both convergent to non-zero number.

2.4 Relative Superior Orbit:

Definition 4. [13] The sequence x_n and y_n constructed above is called Ishikawa sequence of iteration or relative superior sequence of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$.

Notice that $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1$ is $SO(x_0, s_n, t)$ i.e. Mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduce to $O(x_0, t)$.

We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit.

2.5 Relative Superior Mandelbrot Set :

Now we define Mandelbrot set for the function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 5. [13] Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where n = 1,2,3,… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded i.e. $RSM = \{c \in C : Q_c^k(0) : k = 0, 1, 2, 3...\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number.

2.6 Relative Superior Julia Set :

Definition 6. [2] The set of points RSK whose orbits are bounded under relative superior iteration of function Q(z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK.

3. ESCAPE CRITERION FOR RELATIVE SUPERIOR JULIA AND MANDELBROT SET

3.1 Escape Criterion for Quadratics:

[13] Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z| \to \infty$ as $n \to \infty$. So, $|z| \ge |c|$ and $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

3.2 Escape Criterion for Cubic:

[13] Suppose,

$|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives the escape criterion for cubic polynomials.

3.3 General Escape Criterion:

[13] Consider $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$ is the escape criterion.

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to (z^n + c)^{-1}$. However, instead of starting with z_0 = infinity, it is simpler to start with $z_1 = c$, which yields the same result. A critical point of $z \to F(z) + c$ is a point where $F'(z) = 0$.

4. SIMULATIONS AND RESULTS

4.1 Generation of relative superior Mandelbrot Set (Sine Quad. function)

Fig.1 : Relative Superior Mandelbrot Set for s= s'=1

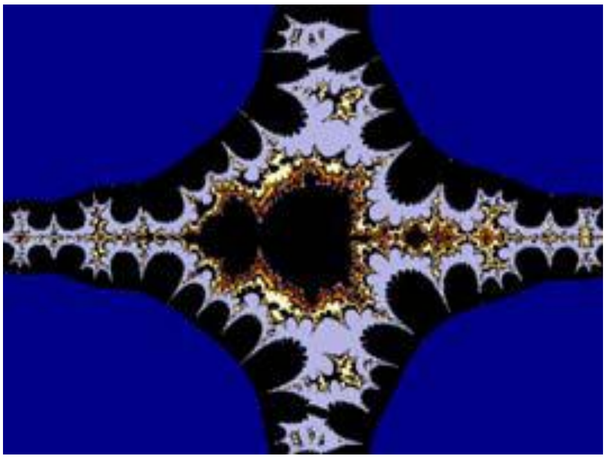
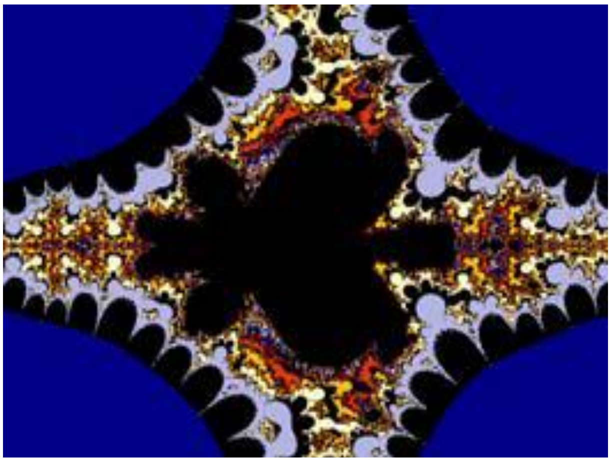


Fig.2 : Relative Superior Mandelbrot Set for s=0.3, s'=0.7



4.2 Generation of relative superior Mandelbrot Set (Sine Cubic function)

Fig.3 : Relative Superior Mandelbrot Set for s= s'=1

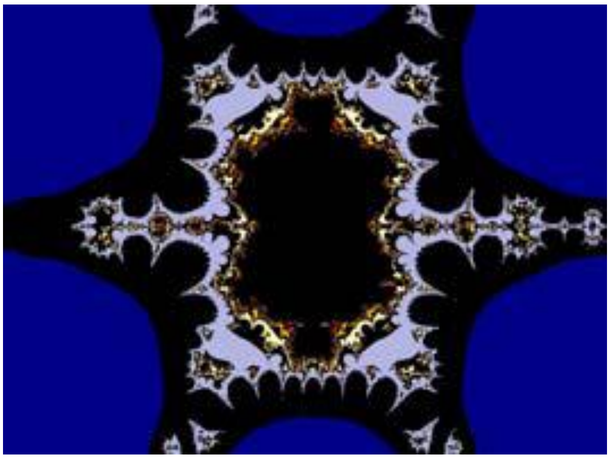
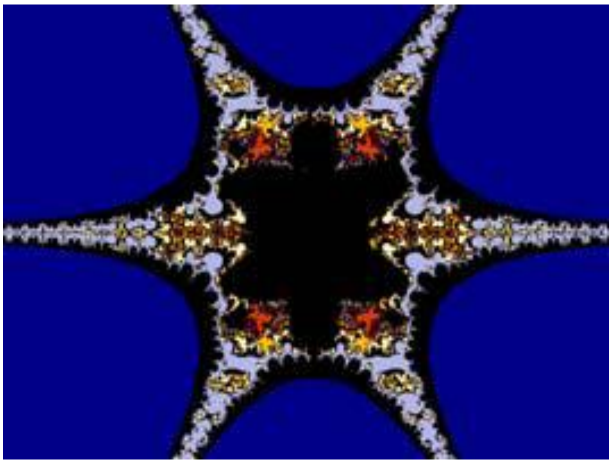


Fig.4 : Relative Superior Mandelbrot Set for s=0.3, s'=0.7



4.3 Generation of relative superior Mandelbrot Set (Tangent Quad. function)

Fig.5 : Relative Superior Mandelbrot Set for s= s'=1

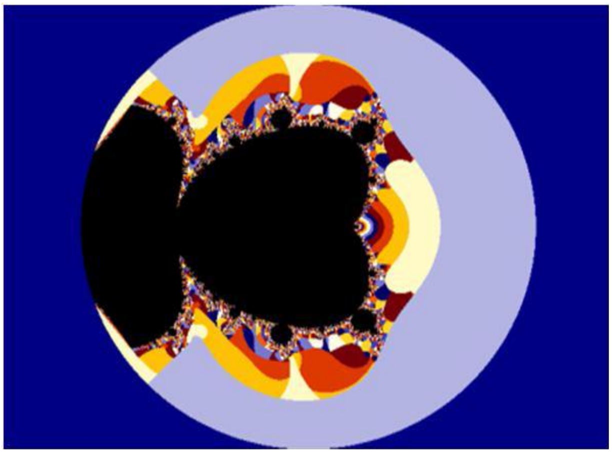
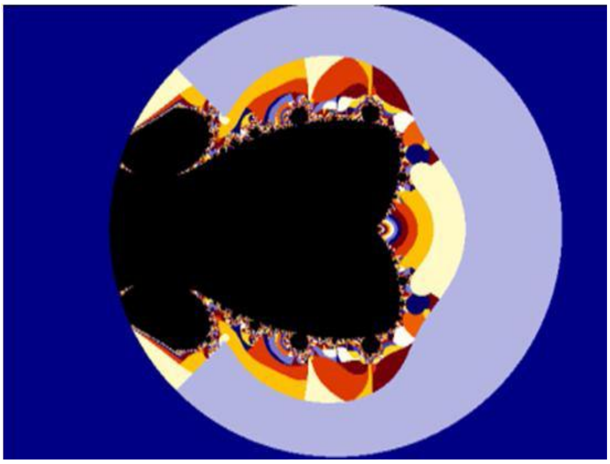


Fig.6 : Relative Superior Mandelbrot Set for s=0.9, s'=0.1



4.4 Generation of relative superior Mandelbrot Set (Tangent Cubic function)

Fig.7 : Relative Superior Mandelbrot Set for s= s'=1

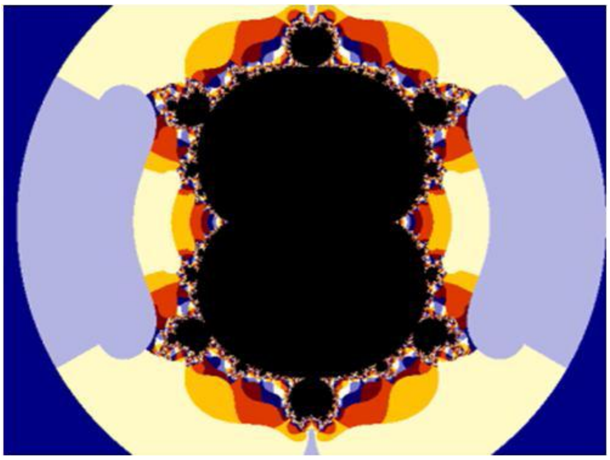
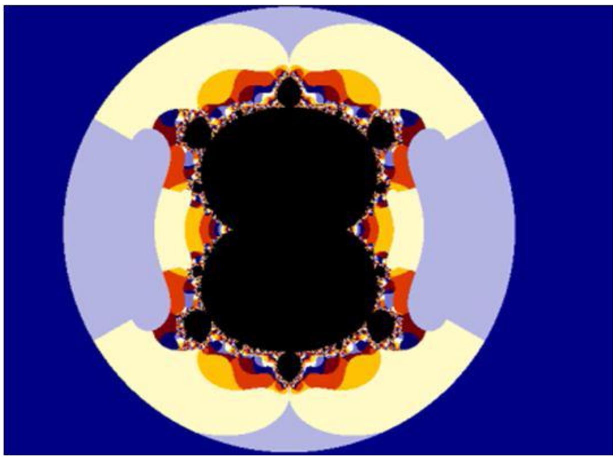
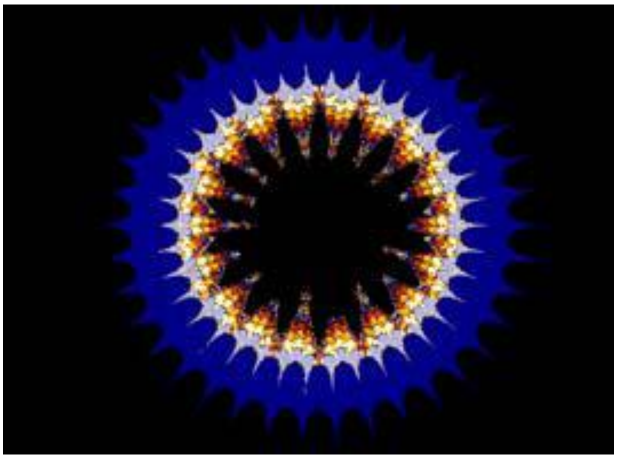


Fig.8 : Relative Superior Mandelbrot Set for s=0.9, s'=0.1



4.5 Generalization of relative superior Mandelbrot Set

Fig.9 : Relative Superior Mandelbrot Set for s=0.1, s'=0.5 n=19



4.6 Generation of relative superior Julia sets

Fig.10 : Relative Superior Julia Set(Sine Quad. function) for s=0.3, s'=0.7 c=-0.1848425651+0.2453514273i

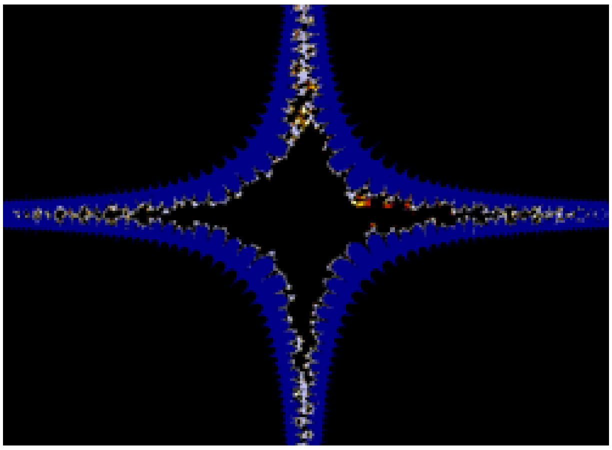


Fig.10 : Relative Superior Julia Set (Sine Cubic Function) for s=0.3, s'=0.7 c= 0.06553079165+1.052110021i

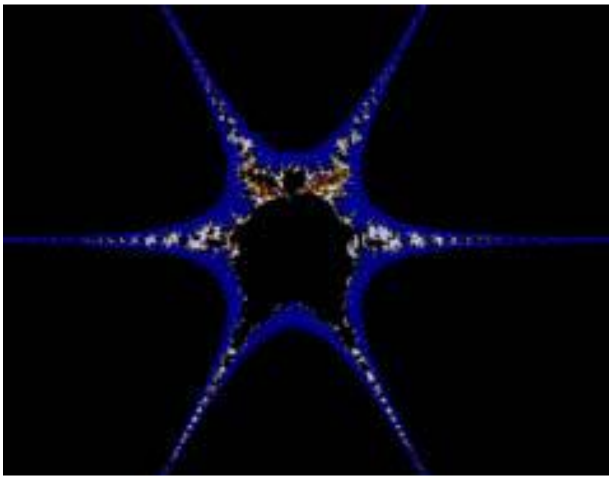


Fig.11 : Relative Superior Julia Set (Tangent Quad. Function) for s=0.6, s'=0.3, c=-0.61455061980+0.00900541716i

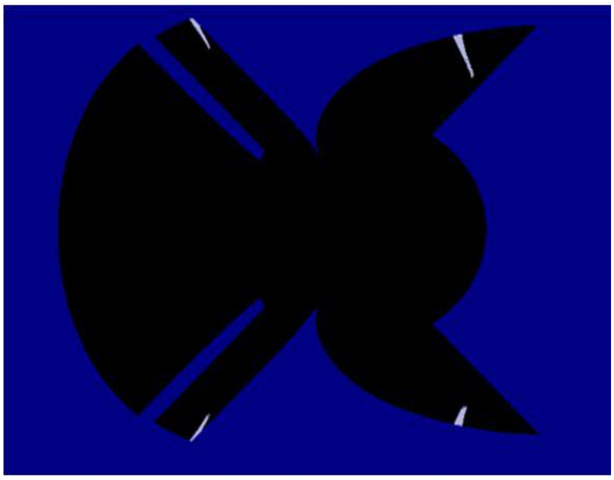
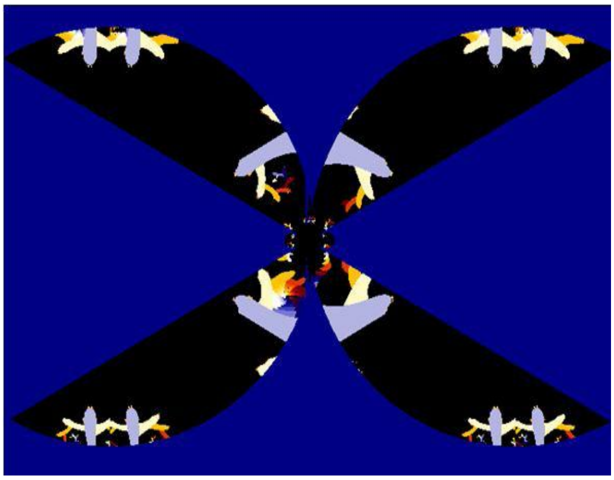


Fig.12 : Relative Superior Julia Set (Tangent Cubic Function) for s=0.9, s'=0.1, c=-0.03352184239-0.03135431148i



4.7 Fixed Point Calculations

Table 1. Fixed points of Sine quadratic polynomial, Orbit of F(z) at s=0.1 and s'=0.5 for (z0=3934870291+0i)

Number of iteration i	|F(Z)|	Number of iteration i	|F(Z)|
61	0.4235	69	0.4237
62	0.4236	70	0.4237
63	0.4236	71	0.4237
64	0.4236	72	0.4237
65	0.4233	73	0.4237
66	0.4234	74	0.4238
67	0.4237	75	0.4238
68	0.4237	76	0.4238

Here we skipped 60 iterations and observed that the value converges to a fixed points after 73 iterations.

Fig.13 : Orbit of F(z) at s=0.1 and s'=0.5 for (z0= -0.3934870291+0i)

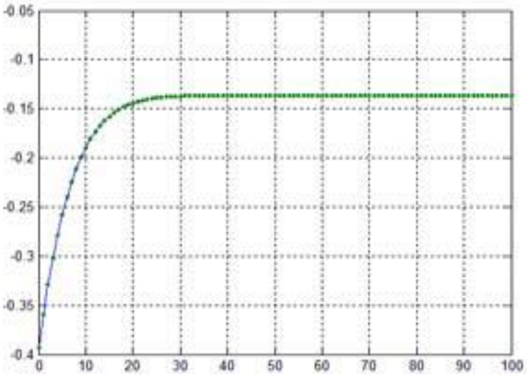


Table 2. Fixed points of Sine Cubic polynomial, Orbit of F(z) at s=0.1 and s'=0.5 for (z0= 0.07944042258 + 0.03670696339i)

Number of iteration i	|F(Z)|	Number of iteration i	|F(Z)|
51	0.4161	58	0.4163
52	0.4162	59	0.4163
53	0.4162	60	0.4163
54	0.4162	61	0.4163
55	0.4163	62	0.4164
56	0.4163	63	0.4164
57	0.4163	64	0.4164

Here we skipped 50 iterations and observed that the value converges to a fixed points after 61 iterations.

Fig.14 : Orbit of F(z) at s=0.1 and s'=0.5 for (z0= 0.07944042258+0.03670696339i)

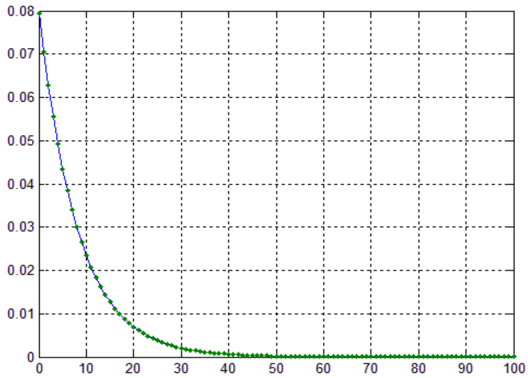


Table 3. Fixed points of Tangent quadratic polynomial, Orbit of F(z) at s=0.6 and s'=0.3 for (z0 =-0.61455061980+0.00900541716i)

Number of iteration i	|F(Z)|	Number of iteration i	|F(Z)|
1	0.6146	9	0.4168
2	0.3143	10	0.4169
3	0.3667	11	0.4169
4	0.4072	12	0.4169
5	0.4185	13	0.4169
6	0.4186	14	0.4169
7	0.4174	15	0.4169
8	0.4169	16	0.4169

Here we observed that the value converges to a fixed point after 9 iterations.

Fig.15 : Orbit of F(z) at at s=0.6 and s'=0.3 for (z0 =-0.61455061980+0.00900541716i)

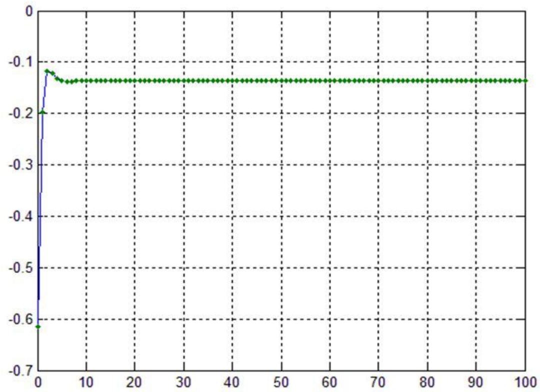
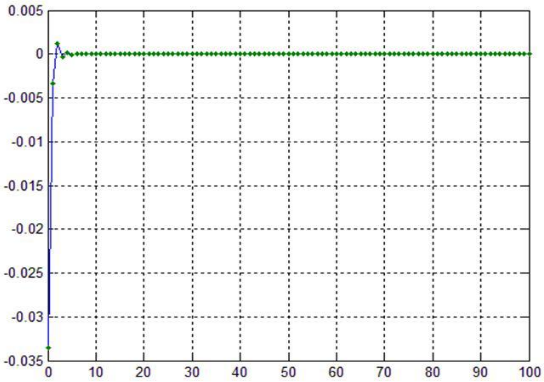


Table 4. Fixed points of Tangent Cubic polynomial, Orbit Orbit of F(z) at s=0.9 and s'=0.1 for (z0=-0.03352184239-0.03135431148 i)

Number of iteration i	|F(Z)|	Number of iteration i	|F(Z)|
1	0.0459	7	0.4237
2	0.4469	8	0.4238
3	0.4161	9	0.4238
4	0.4261	10	0.4238
5	0.4230	11	0.4238
6	0.4230	12	0.4238

Here we observed that the value converges to a fixed point after 7 iterations.

Fig.16 : Orbit of F(z) at s=0.9 and s'=0.1 for (z0=-0.03352184239-0.03135431148 i)



5. CONCLUSION

In this paper we studied the generation of fractal geometries using the two functions: the sine function and the inverse tangent function. Both the functions belong to same class of function families known as transcendental family. For Sine function, the fixed point 0 for S(z) = Sin z also satisfies S'(0)=1. The orbit on the real axis tends to zero and the orbit on the imaginary axis tends to infinity, in case of sine function. For inverse tangent function, the relative superior Julia set possesses n+ 1 wing, whereas, in case of sine function this count is 2n. We also observed in our results that for tangent function, in case of even powers, relative superior Mandelbrot sets are symmetrical only along the real axis, while on the other hand, for odd terms, body maintains its symmetry along both the axis. Also, in the past literature the Julia sets were shown to be disconnected, which in our research, were found to be connected for relative superior Ishikawa iterates.

6. ACKNOWLEDGMENTS

We express our thanks to **Dr. A.K. Swami**, Principal, G.B.Pant Engineering College, Ghurdauri, for providing necessary infrastructure for the research work. We would also like to thanks **Dr. Priti Dimri**, Head, Department of Computer Science and Applications, G.B.Pant Engineering College, Ghurdauri, for her unconditional and valuable support in writing this paper.

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