Nonsplit Geodetic Number of a Lict Graph

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ABSTRACT
A set $S \subseteq V(\eta(G))$ is a non split geodetic set of $\eta(G)$, if S is a geodetic set and $\langle V-S \rangle$ is connected. The nonsplit geodetic number of a lict graph $\eta(G)$, denoted by $g_{ns}(\eta(G))$, is the minimum cardinality of a nonsplit geodetic set of $\eta(G)$. The bounds on non split geodetic number in terms of elements of G and covering number of G. Further the relationship between nonsplit geodetic number and geodetic number of a graph is established.

General Terms
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Keywords
Cartesian product, Distance, Edge covering number, geodetic number, Vertex covering number.

1. INTRODUCTION
In this paper we follow the notations and undefined terms of [3]. As usual n = |V | and m = |E| denote the number of vertices and edges of a graph G respectively. The graphs considered here are finite, undirected and simple and have at least one component which is not complete or at least two non trivial components. We refer [3] for unexplained terminology and notation. For any graph G=(V,E), the Lict graph $\eta(G)$ [4],whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices of $\eta(G)$ adjacent if and only if the corresponding edges of G are adjacent or the corresponding members of G are incident. The distance d (u, v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. It is well known that this distance is a metric on the vertex set V (G). For a vertex v of G, the eccentricity e(V) is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G and the maximum eccentricity is the diameter, diam G.

A u-v path of length d(u, v) is called a u-v geodesic. We define I [u, v] to the set (interval) of all vertices lying on some u-v geodesic of G and for a nonempty subset S of V (G), $I[S]=U_{u,v\in S}\, I(u,v)$.

A set S of vertices of G is called a geodetic set in G if I[S] = V (G) and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number g(G).

Nonsplit geodetic number of a graph was studied in [6]. A geodetic set S of a graph G = (V, E) is a nonsplit geodetic set if the induced subgraph < V − S > is connected. The nonsplit geodetic number $g_{ns}(G)$ is the minimum cardinality of a nonsplit geodetic set. Geodetic number of a lict graph was studied in [6]. Geodetic number of a lict graph $\eta(G)$ is a set S' of vertices $\eta(G)$=H is called a geodetic set in H if $I[S']$ =V (H), and a geodetic set of minimum cardinality is a Lict geodetic number of G and is denoted by $g(\eta(G))$. Now we define nonsplit geodetic number of a lict graph. A set S' of vertices of $\eta(G)$ = H is called the nonsplit geodetic set in H if the induced subgraph $V(H)-S'$ is connected and a nonsplit geodetic set of minimum cardinality is the nonsplit geodetic number of $\eta(G)$ and is denoted by $g_{ns}(\eta(G))$.

A vertex v is an extreme vertex in a graph G, if the subgraph induced by its neighbors is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G. An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G. The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G.

2. PRELIMINARY RESULTS
Theorem 2.1[1] Every geodetic set of a graph contains its extreme vertices.

Theorem 2.2 [1] For any tree T with k end edges, g(T) = k.

Theorem 2.3[3] For any graph G of order n, $\alpha_1(G)+\beta_1(G)=n.$

Theorem 2.4[1] If G is a nontrivial connected graph, then $g(G)\le g(G\times K_2)$

Theorem 2.5[3] For any path P_n, the edge covering number is $\alpha_1(P_n)=\begin{cases}\frac{n}{2} & if\ n\, is\, even\\ \frac{n+1}{2} & if\ n\, is\, odd\end{cases}$

Theorem 2.6[3] For any path P_n, the vertex covering number is $\alpha_0(P_n)=\begin{cases}\frac{n}{2} & if\ n\ is\ even\\ \frac{n-1}{2} & if\ n\ is\ odd\end{cases}$

Theorem 2.7[5] For any tree T, $g(\eta(T))=n$.

Theorem 2.8 For any tree T, $\eta(T)$ is a Block graph.

Theorem 2.9 For any graph $g(\eta(G))\le g_{ns}(\eta(G))$

3. MAIN RESULTS

Theorem 3.1 For any tree of order n, $g_{ns}[\eta(T)]=n$, where n be the number of vertices of T.

Proof. Let S be the set of all extreme vertices of a lict graph of a tree. By the Theorem 2.1, we obtain $g[\eta(T)]\ge S.$ This implies $g[\eta(T)]\ge S.$ On the other hand for an internal vertex v of T there exists end vertices x, y of T, such that $v\in I(S)$ and $I(S)=V(T).$ Thus, $g_{ns}[\eta(T)]\le S.$ Therefore $g_{ns}[\eta(T)]=S.$ Also, every geodetic set of S' of T must contain S, is the minimum geodetic set and $V-S$ is connected. Thus S itself a minimum nonsplit geodetic set of G= $\eta(T)$ i.e $g_{ns}[\eta(T)]=S.$ Now by the definition, the pendant edges and cut vertices of a tree are the extreme vertices of a lict graph. Since the number of vertices in T is the sum of cut vertices and the number of pendant edges. Hence the minimum nonsplit geodetic set is the number of vertices in tree T. Thus $g_{ns}[\eta(T)]=n.$ Hence the proof.

Theorem 3.2 For any path P_n of order n, $g_{ns}[\eta(P_n)]=n$ where n is the number of vertices.

Proof . Proof follows from the above Theorem.

Theorem 3.3 For any path P_n of order n, $g_{ns}[\eta(P_n)]=\alpha_1(P_n)+\beta_1(P_n)$.

Proof. Since $\alpha_1(P_n)+\beta_1(P_n)=n$ and also $g_{ns}[\eta(P_n)]=n$,we have $g_{ns}[\eta(P_n)]=\alpha_1(P_n)+\beta_1(P_n)$. Hence the proof.

Theorem 3.4 For any path P_n of order n, $g_{ns}[\eta(P_n)]=\omega(P_n)+c_i$ where $\omega(P_n)$ and c_i are the clique and cut vertices of P_n respectively.

Proof. Let $S=\{u_1,u_2,u_3,\dots,u_k\}$ be the geodetic set of $\eta(P_n)$. Now, consider any set $S'=S-\{u_i\}$, with extreme vertex $u_i\in S$. Let us consider any vertices u_j,u_k of S'. Since each block in $\eta(P_n)$ is K_3, u_i does not lies in u_j-u_k geodesic. So S' is not a geodetic set of $\eta(P_n)$. Hence S is a minimum nonsplit geodetic set, further $V-S$ is connected. So $|S|=n=\omega(P_n)+c_i$. Hence the proof.

Theorem 3.5 For any path P_n of order n, $g_{ns}[\eta(P_n)]=\begin{cases}2\alpha_0(P_n), & n\ is\ even\\ 2\alpha_0(P_n)+1, & n\ is\ odd\end{cases}$

Proof. Let P_n be the path with $n\ge 4$ vertices. Consider $V=\{v_1,v_2,v_3,\dots,v_n\}$ be the vertices and $E=\{(v_i,v_{i+1})\},i=1,2,3,\dots$ be the edge set of path P_n. By the Theorem 2.6, the vertex covering number is a minimum cardinality of vertex cover of P_n. We have the following cases,

Case(i). Suppose n is even, by the Theorem 2.6 we have

$$\alpha_0(P_n)=\frac{n}{2}$$
$$\Rightarrow\quad n=2\alpha_0(P_n)$$

Since $g_{ns}[\eta(P_n)]=n$, we have $g_{ns}[\eta(P_n)]=2\alpha_0(P_n)$.

Case(ii). Suppose n is odd , by the Theorem 2.6 we have

$$\alpha_0(P_n)=\frac{n-1}{2}$$
$$\Rightarrow\quad n=2\alpha_0(P_n)+1$$

Since $g_{ns}[\eta(P_n)]=n$, we have $g_{ns}[\eta(P_n)]=2\alpha_0(P_n)+1$.

Hence the proof.

Theorem 3.6 For any path P_n of order n, $g_{ns}[\eta(P_n)]=\begin{cases}2\alpha_1(P_n), & n\ is\ even\\ 2\alpha_1(P_n)-1, & n\ is\ odd\end{cases}$

Proof. Let P_n be the path with $n\ge 4$ vertices. Consider $V=\{v_1,v_2,v_3,\dots,v_n\}$ be the vertices and $E=\{(v_i,v_{i+1})\},i=1,2,3,\dots$ be the edge set of path P_n. By the Theorem 2.5 the edge covering number is a minimum cardinality of edge cover of P_n. We have the following cases

Case(i). Suppose n is even, by the Theorem 2.5, we have

$$\alpha_1(P_n)=\frac{n}{2}$$
$$\Rightarrow\quad n=2\alpha_1(P_n)$$

Since $g_{ns}[\eta(P_n)]=n$, we have $g_{ns}[\eta(P_n)]=2\alpha_1(P_n)$.

Case(ii). Suppose n is odd, by the Theorem 2.5, we have

$$\alpha_1(P_n)=\frac{n+1}{2}$$
$$\Rightarrow\quad n+1=2\alpha_1(P_n)$$
$$\Rightarrow\quad n=2\alpha_1(P_n)-1$$

Since $g_{ns}[\eta(P_n)]=n$, we have $g_{ns}[\eta(P_n)]=2\alpha_1(P_n)-1$.

Hence the proof

4. ADDING AN END EDGE

For an edge e = (u, v) of a graph G with deg(u) = 1 and deg(v) > 1, we call e an end-edge and u an end-vertex.

Theorem 4.1 If G' is a graph obtained by adding a pendant vertex to $G=C_n$ then $g_{ns}[\eta(G')]=k+3$, k be the number of pendant vertices.

Proof: Let $\{e_1,e_2,\ldots,e_n,e_1\}$ be a cycle with n vertices which is even and G' be the graph obtained from G= C_n by adding end edges {u v_i}, i=1, 2, …,k such that $u\in G$ and $v_i\notin G$. By the definition of lict graph $\eta(G')$ has $\langle K_{k+3}\rangle$ as an induced subgraph. Also the edges {u v_i}, i=1, 2, …,k and the cutvertex u becomes the vertices of $\eta(G')$. These belongs to some geodetic set of $\eta(G')$. Hence $\{u,e_1,e_2,\ldots,e_k,e_l,e_m\}$ are the vertices of $\eta(G')$ where e_l,e_m are the edges incident on the antipodal vertex of u in G' and these vertices belongs to some geodetic set of $\eta(G')$. Thus $\eta(G')=C_n\,U\,K_{k+3}$. Let $S=\{u,e_1,e_2,\ldots e_k,e_l,e_m\}$ be the geodetic set. Suppose $S'=\{u,e_1,e_2,\ldots e_k\}$ be the set vertices $\eta(G')$ such that $|S'|<|S|$ then S' is not a geodetic set of $\eta(G')$. Thus, S is the minimum geodetic set and V-S is connected. Hence S is the minimum nonsplit geodetic set, therefore $g_{ns}[\eta(G')]=k+3$. Hence the proof.

Theorem 4.2 If G' is a graph obtained by adding a pendant vertex to $G=C_n$ then $g_{ns}[\eta(G')]=k+2$, k be the number of pendant vertices.

Proof: Let $\{e_1,e_2,\ldots,e_n,e_1\}$ be a cycle with n vertices which is odd and G' be the graph obtained from G=Cn by adding pendant edges {u vi}, i=1, 2, …,k such that $u\in G$, $v_i\notin G$. By the definition of lict graph $\eta(G')$ has $\langle K_{k+2}\rangle$ as an induced sub graph, also the edges {u v_i}, i=1, 2,…, k becomes vertices of $\eta(G')$. Further the cutvertex u is also a vertex of $\eta(G')$ and it belongs to some geodetic set of $\eta(G')$. Hence $S=\{u,e_1,e_2,\ldots,e_k,e_l,e_m\}$ are the geodetic set of $\eta(G')$ where e_l,e_m are the edges on the vertex of u in G'. Let $e_i=\{a,b\}\in G$ such that $d(u,a)=d(u,b)$, in the $\eta(G')$, two element subsets of $\eta(G')$ has the property that $I(S)=V[\eta(G')]$. Since V-S is connected, thus S itself a minimum nonsplit geodetic set of $\eta(G')$. Therefore $g_{ns}[\eta(G')]=k+2$.

Hence the proof.

5. CARTESIAN PRODUCT OF GRAPHS

The Cartesian product of two graphs G, H by $G\times H$, and it is the graph with the vertex set $V(G)\times V(H)$ specified by putting (u, v) adjacent to (u',v') if and only if (1) u= u' and $vv'\in E(H)$ or (2) v= v' and $uu'\in E(G)$. A vertex v is an extreme vertex in a graph G, and the subgraph induced by its neighbors is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G.

Theorem 5.1 For any graph G, $g_{ns}[K_3\times G]=n$.

Proof. Consider $G=\eta(P_n)$. Let $K_3\times G$ be the graph formed from two copies G_1 and G_2 of G and S be a minimum nonsplit geodetic set of $K_3\times G$. Now, we define S' to be the union of those vertices of S in G_1 and the vertices of G_1 corresponding to vertices of G_2 belonging to S. Let $v\in V(G_1)$ lies on some $x-y$ geodesic where $x,y\in S$. Since S is a nonsplit geodetic set, by the theorem we have $g_{ns}[G]=n$ at least one of x and y belong to V_1. If both $x,y\in V_1$ then $x,y\in S'$. Hence we may assume that $x\in V_1$ and $y\in V_2$. If y corresponds to x then $v=x\in S'$. Hence, we may assume that y corresponds to $y'\in S'$ where $y\neq x$. Since $d(x,y)=d(x,y')+1$ and the vertex v lies on an $x-y$ geodesic in $K_3\times G$. Hence v lies on an $x-y$ geodesic in G that is $g(G)\leq g_{ns}(K_3\times G)$.

Further, let S contains a vertex x with the property that every vertex of G_1 lies on an $x-w$ geodesic in G_1 for some $w\in S$. Let S' consists of x together with those vertices of G_2 corresponding to those vertices in $S-\{x\}$. Thus $|S'|=|S|$. We show that S' is a nonsplit geodetic set of $K_3\times G$. Hence $g_{ns}(K_3\times G)\leq g(G)$. Therefore $g_{ns}(K_3\times G)=g(G)=n$.

Hence the proof.

Theorem 5.2 For any path P_n,

$$g_{ns}[K_3\times\eta(P_n)]=\begin{cases}2\alpha_0(P_n), & n \text{ is even}\\ 2\alpha_0(P_n)+1, & n \text{ is odd}\end{cases}$$

Proof. Let $\alpha_0(P_n)$ be a vertex covering of a graph is a minimum cardinality of an vertex cover of a path P_n. We have the following cases,

Case(i). Suppose n is even, by the Theorem 2.6, we have

$$\alpha_0(P_n)=\frac{n}{2}$$

$$\Rightarrow n=2\alpha_0(P_n)$$

Since $g_{ns}[K_3\times\eta(P_n)]=n$, we have $g_{ns}[K_3\times\eta(P_n)]=2\alpha_0(P_n)$.

Case(ii). Suppose n is odd, by the Theorem 2.6, we have

$$\alpha_0(P_n)=\frac{n-1}{2}$$

$$\Rightarrow \quad n = 2\alpha_0(P_n) + 1$$

Since $g_{ns}[K_3 \times \eta(P_n)] = n$, we have
$g_{ns}[K_3 \times \eta(P_n)] = 2\alpha_0(P_n) + 1$.

Hence the proof.

6. CONCLUSION

In this paper we studied the nonsplit geodetic number of lict graph of a graph and obtain some results on the Cartesian product of a graph. Further, the results on adding an end edge to a cycle C_n and its lict graph characterised.

7. REFERENCES

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