

$\mathcal{M}_X G\zeta^*$ -Interior and $\mathcal{M}_X G\zeta^*$ -Closure in \mathcal{M} -Structures

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ABSTRACT

In this paper, we introduce $\mathcal{M}_X G\zeta^*$ -Interior, $\mathcal{M}_X G\zeta^*$ -Closure and some of its basic properties.

Keywords

$\mathcal{M}_X G\zeta^*$ -open, $\mathcal{M}_X G\zeta^*$ -closed, $\mathcal{M}_X Int_{G\zeta^*}(A)$, $\mathcal{M}_X Cl_{G\zeta^*}(A)$.

1. INTRODUCTION

N.Levine [14] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Balachandran, Sundaram and Maki [5], Bhattacharyya and Lahiri[6], Arockiarani[3], Dunham[9], Gnanambal[10], Malghan [18], Palaniappan and Rao[21], Park[22], Arya and Gupta[4] and Devi[8] have worked on generalized closed sets, their generalizations and related concepts in general topology.

In 2000, V.Popa and T.Noiri [23] introduced the notion of minimal structure. Also they introduce the notion of m_X -open sets and m_X -closed sets and characterized those sets using m_X -closure and m_X -interior respectively. Further they introduced m -continuous functions and studied some of its basic properties. V.Popa and T.Noiri [24] obtained the definitions and characterizations of separation axioms by using the concept of minimal structures. They achieved many important results compatible by the general topology case. Some other results about minimal spaces can be found in [1], [2], [7], [15], [16], [19] and [20].

Already Kokilavani V [12] et al. introduce $\mathcal{M}_X G\zeta^*$ - closed set. In this paper, the notion of $\mathcal{M}_X G\zeta^*$ -interior is defined and some of its basic properties are studied. Also we introduce the concept of $\mathcal{M}_X G\zeta^*$ -closure in topological spaces using the notions of $\mathcal{M}_X G\zeta^*$ -closed sets, and we obtain some related results.

2. PRELIMINARIES

In this paper, we introduce the notion of $\mathcal{M}_X G\zeta^*$ -interior is defined and some of its basic properties are studied. Also we introduce the concept of $\mathcal{M}_X G\zeta^*$ -closure in topological spaces using the notions of $\mathcal{M}_X G\zeta^*$ -closed sets, and we obtain some related results.

DEFINITION 2.1. [17] Let X be a nonempty set and let $m_X \subseteq P(X)$, where $P(X)$ denote the power set of X where m_X is an \mathcal{M} -structure (or a minimal structure) on X , if φ and X belong to m_X .

The members of the minimal structure m_X are called m_X -open sets, and the pair (X, m_X) is called an m -space. The complement of m_X -open set is said to be m_X -closed.

DEFINITION 2.2. [17] Let X be a nonempty set and m_X an \mathcal{M} -structure on X . For a subset A of X , m_X -closure of A and m_X -interior of A are defined as follows:

$$m_X-cl(A) = \bigcap \{F : A \subseteq F, X-F \in m_X\}$$

$$m_X-int(A) = \bigcup \{F : F \subseteq A, F \in m_X\}$$

LEMMA 2.3. [17] Let X be a nonempty set and m_X an \mathcal{M} -structure on X . For subsets A and B of X , the following properties hold:

- $m_X-cl(X - A) = X - m_X-int(A)$ and $m_X-int(X - A) = X - m_X-cl(A)$.
- If $X - A \in m_X$, then $m_X-cl(A) = A$ and if $A \in m_X$ then $m_X-int(A) = A$.
- $m_X-cl(\varphi) = \varphi$, $m_X-cl(X) = X$ and $m_X-int(\varphi) = \varphi$, $m_X-int(X) = X$.
- If $A \subseteq B$ then $m_X-cl(A) \subseteq m_X-cl(B)$ and $m_X-int(A) \subseteq m_X-int(B)$.
- $A \subseteq m_X-cl(A)$ and $m_X-int(A) \subseteq A$.
- $m_X-cl(m_X-cl(A)) = m_X-cl(A)$ and $m_X-int(m_X-int(A)) = m_X-int(A)$.
- $m_X-int(A \cap B) = (m_X-int(A)) \cap (m_X-int(B))$ and $(m_X-int(A)) \cup (m_X-int(B)) \subseteq m_X-int(A \cup B)$.
- $m_X-cl(A \cup B) \subseteq (m_X-cl(A)) \cup (m_X-cl(B))$ and $m_X-cl(A \cap B) \subseteq (m_X-cl(A)) \cap (m_X-cl(B))$.

LEMMA 2.4. [16] Let (X, m_X) be an m -space and A be a subset of X . Then $x \in m_X-cl(A)$ if and only if $U \cap A \neq \varphi$ for every $U \in m_X$ containing x .

DEFINITION 2.5. [25] Let X be a nonempty set and m_X an \mathcal{M} -structure on X . For a subset A of X , am -closure of A and am -interior of A are defined as follows:

$$amcl(A) = \bigcap \{F : A \subseteq F, F \text{ is } am\text{-closed in } X\}$$

$$amint(A) = \bigcup \{F : F \subseteq A, F \text{ is } am\text{-open in } X\}$$

DEFINITION 2.6. A subset A of an m -space (X, m_X) is called as

(i) $m_X g^\# \alpha$ -closed set [12] if $amcl(A) \subseteq U$ whenever $A \subseteq U$ and U is mg -open in X .

(ii) $m_X^\# g \alpha$ -closed set [12] if $amcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $m_X g^\# \alpha$ -open in X .

(iii) $\mathcal{M}_X G\zeta^*$ -closed set [12] if $amcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $m_X^\# g \alpha$ -open in X .

The complement of $\mathcal{M}_X G\zeta^*$ -closed set is called as $\mathcal{M}_X G\zeta^*$ -open set.

DEFINITION 2.7. [13] A subset S is said to be an $\mathcal{M}_X G\zeta^*$ -neighbourhood of a point x of X if there exists a $\mathcal{M}_X G\zeta^*$ -open set U such that $x \in U \subset S$.

DEFINITION 2.8. [23] A minimal structure m_X on a nonempty set X is said to have the property \mathcal{B} if the union of any family of subsets belonging to m_X belongs to m_X .

REMARK 2.9. A minimal structure m_X with the property \mathcal{B} coincides with a generalized topology on the sense of Lugojan.

LEMMA 2.10. [2] Let X be a nonempty set and m_X an \mathcal{M} -structure on X satisfying the property \mathcal{B} . For a subset A of X , the following property hold:

- (a) $A \in m_X$ iff $m_X - int(A) = A$.
- (b) $A \in m_X$ iff $m_X - cl(A) = A$.
- (c) $m_X - int(A) \in m_X$ and $m_X - cl(A) \in m_X$.

3. $\mathcal{M}_X G\zeta^*$ -INTERIOR AND $\mathcal{M}_X G\zeta^*$ -CLOSURE

We introduce the following definition:

DEFINITION 3.1. Let X be a nonempty set and m_X an \mathcal{M} -structure on X . For a subset A of X , $m_X ag$ -closure of A and $m_X ag$ -interior of A are defined as follows:

$$m_X ag-cl(A) = \bigcap \{F : A \subseteq F, F \text{ is } m_X ag\text{-closed in } X\}$$

$$m_X ag-int(A) = \bigcup \{F : F \subseteq A, F \text{ is } m_X ag\text{-open in } X\}$$

DEFINITION 3.2. Let A be a subset of X . A point $x \in A$ is said to be $\mathcal{M}_X G\zeta^*$ -interior point of A if A is a $\mathcal{M}_X G\zeta^*$ -nbhd of x . The set of all $\mathcal{M}_X G\zeta^*$ -interior points of A is called the $\mathcal{M}_X G\zeta^*$ -interior of A and is denoted by $\mathcal{M}_X Int_{G\zeta^*}(A)$.

$$\mathcal{M}_X Int_{G\zeta^*}(A) = \bigcup \{F : F \subseteq A, F \text{ is } \mathcal{M}_X G\zeta^*\text{-open in } X\}$$

THEOREM 3.3. If A be a subset of X . Then $\mathcal{M}_X Int_{G\zeta^*}(A) = \bigcup \{G : G \text{ is } \mathcal{M}_X G\zeta^*\text{-open}, G \subset A\}$.

PROOF. Let A be a subset of X .

$x \in \mathcal{M}_X Int_{G\zeta^*}(A) \Leftrightarrow x$ is a $\mathcal{M}_X G\zeta^*$ -interior point of A .

$\Leftrightarrow A$ is a $\mathcal{M}_X G\zeta^*$ -nbhd of point x .

\Leftrightarrow there exists $\mathcal{M}_X G\zeta^*$ -open set G such that $x \in G \subset A$.

$\Leftrightarrow x \in \bigcup \{G : G \text{ is } \mathcal{M}_X G\zeta^*\text{-open}, G \subset A\}$.

Hence $\mathcal{M}_X Int_{G\zeta^*}(A) = \bigcup \{G : G \text{ is } \mathcal{M}_X G\zeta^*\text{-open}, G \subset A\}$.

THEOREM 3.4. Let A and B be subsets of X . Then

(i) $\mathcal{M}_X Int_{G\zeta^*}(X) = X$ and $\mathcal{M}_X Int_{G\zeta^*}(\emptyset) = \emptyset$.

(ii) $\mathcal{M}_X Int_{G\zeta^*}(A) \subset A$.

(iii) If B is any $\mathcal{M}_X G\zeta^*$ -open set contained in A , then $B \subset \mathcal{M}_X Int_{G\zeta^*}(A)$.

(iv) If $A \subset B$, then $\mathcal{M}_X Int_{G\zeta^*}(A) \subset \mathcal{M}_X Int_{G\zeta^*}(B)$.

(v) $\mathcal{M}_X Int_{G\zeta^*}(\mathcal{M}_X Int_{G\zeta^*}(A)) = \mathcal{M}_X Int_{G\zeta^*}(A)$.

PROOF.

(i) Since X and \emptyset are $\mathcal{M}_X G\zeta^*$ -open sets, by Theorem 3.3. $\mathcal{M}_X Int_{G\zeta^*}(X) = \bigcup \{G : G \text{ is } \mathcal{M}_X G\zeta^*\text{-open}, G \subset X\} = X \cup \{\text{all } \mathcal{M}_X G\zeta^*\text{-open sets}\} = X$. That is $\mathcal{M}_X Int_{G\zeta^*}(X) = X$. Since \emptyset is the only $\mathcal{M}_X G\zeta^*$ -open set contained in \emptyset , $\mathcal{M}_X Int_{G\zeta^*}(\emptyset) = \emptyset$.

(ii) Let $x \in \mathcal{M}_X Int_{G\zeta^*}(A) \Rightarrow x$ is a $\mathcal{M}_X G\zeta^*$ -interior point of A .

$\Rightarrow A$ is a $\mathcal{M}_X G\zeta^*$ -nbhd of x .

$\Rightarrow x \in A$.

Thus $x \in \mathcal{M}_X Int_{G\zeta^*}(A) \Rightarrow x \in A$. Hence $\mathcal{M}_X Int_{G\zeta^*}(A) \subset A$.

(iii) Let B be any $\mathcal{M}_X G\zeta^*$ -open sets such that $B \subset A$. Let $x \in B$, then since B is a $\mathcal{M}_X G\zeta^*$ -open set contained in A , x is a $\mathcal{M}_X G\zeta^*$ -interior point of A . That is $x \in \mathcal{M}_X Int_{G\zeta^*}(A)$. Hence $B \subset \mathcal{M}_X Int_{G\zeta^*}(A)$.

(iv) Let A and B be subsets of X such that $A \subset B$. Let $x \in \mathcal{M}_X Int_{G\zeta^*}(A)$. Then x is a $\mathcal{M}_X G\zeta^*$ -interior point of A and so A is $\mathcal{M}_X G\zeta^*$ -nbhd of x . Since $B \supset A$, B is also a $\mathcal{M}_X G\zeta^*$ -

nbhd of x . This implies that $x \in \mathcal{M}_X Int_{G\zeta^*}(B)$. Thus we have shown that $x \in \mathcal{M}_X Int_{G\zeta^*}(A) \Rightarrow x \in \mathcal{M}_X Int_{G\zeta^*}(B)$. Hence

$$\mathcal{M}_X Int_{G\zeta^*}(A) \subset \mathcal{M}_X Int_{G\zeta^*}(B)$$

(v) Let A be any subset of X . By the definition of $\mathcal{M}_X G\zeta^*$ -interior, $\mathcal{M}_X Int_{G\zeta^*}(A) = \bigcup \{F : F \subseteq A, F \text{ is } \mathcal{M}_X G\zeta^*\text{-open in } X\}$, if $A \subset F \in \mathcal{M}_X G\zeta^*O(X)$, then $\mathcal{M}_X Int_{G\zeta^*}(A) \subset F$.

Since F is $\mathcal{M}_X G\zeta^*$ -open set containing $\mathcal{M}_X Int_{G\zeta^*}(A)$, by

(iii) $\mathcal{M}_X Int_{G\zeta^*}(\mathcal{M}_X Int_{G\zeta^*}(A)) \subset F$. Hence

$$\mathcal{M}_X Int_{G\zeta^*}(\mathcal{M}_X Int_{G\zeta^*}(A)) \subset \bigcup \{F : F \subseteq A, F \text{ is } \mathcal{M}_X G\zeta^*\text{-open in } X\} = \mathcal{M}_X Int_{G\zeta^*}(A)$$

That is

$$\mathcal{M}_X Int_{G\zeta^*}(\mathcal{M}_X Int_{G\zeta^*}(A)) = \mathcal{M}_X Int_{G\zeta^*}(A)$$

THEOREM 3.5. If a subset A of space X is $\mathcal{M}_X G\zeta^*$ -open, then $\mathcal{M}_X Int_{G\zeta^*}(A) = A$.

PROOF. Let A be $\mathcal{M}_X G\zeta^*$ -open subset of X . We know that $\mathcal{M}_X Int_{G\zeta^*}(A) \subset A$. Also, A is $\mathcal{M}_X G\zeta^*$ -open set contained in A . From Theorem 3.4.

(iii) $A \subset \mathcal{M}_X Int_{G\zeta^*}(A)$. Hence $\mathcal{M}_X Int_{G\zeta^*}(A) = A$.

The converse of the above Theorem need not be true, as seen from the following example.

EXAMPLE 3.6. Let $X = \{a, b, c, d\}$ with m_X -open set =

$$\{\emptyset, X, \{a\}, \{b\}, \{a, b, d\}, \{a, b, c\}\}$$

Then $\mathcal{M}_X G\zeta^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Note that $\mathcal{M}_X Int_{G\zeta^*}(\{c, d\}) = \{c\} \cup \{d\} \cup \emptyset = \{c, d\}$, But $\{c, d\}$ is not a $\mathcal{M}_X G\zeta^*$ -open set in X .

THEOREM 3.7. If A and B are subsets of X , then $\mathcal{M}_X Int_{G\zeta^*}(A) \cup \mathcal{M}_X Int_{G\zeta^*}(B) \subset \mathcal{M}_X Int_{G\zeta^*}(A \cup B)$.

PROOF. We know that $A \subset A \cup B$ and $B \subset A \cup B$. We have, Theorem 3.4. (iv), $\mathcal{M}_X Int_{G\zeta^*}(A) \subset \mathcal{M}_X Int_{G\zeta^*}(A \cup B)$ and $\mathcal{M}_X Int_{G\zeta^*}(B) \subset \mathcal{M}_X Int_{G\zeta^*}(A \cup B)$. This implies that $\mathcal{M}_X Int_{G\zeta^*}(A) \cup \mathcal{M}_X Int_{G\zeta^*}(B) \subset \mathcal{M}_X Int_{G\zeta^*}(A \cup B)$.

THEOREM 3.8. If A and B are subsets of X , then $\mathcal{M}_X Int_{G\zeta^*}(A \cap B) = \mathcal{M}_X Int_{G\zeta^*}(A) \cap \mathcal{M}_X Int_{G\zeta^*}(B)$.

PROOF. We know that $A \cap B \subset A$ and $A \cap B \subset B$. We have, Theorem 3.4. (iv), $\mathcal{M}_X Int_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Int_{G\zeta^*}(A)$ and $\mathcal{M}_X Int_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Int_{G\zeta^*}(B)$. This implies that $\mathcal{M}_X Int_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Int_{G\zeta^*}(A) \cap \mathcal{M}_X Int_{G\zeta^*}(B) \rightarrow (1)$. Again, let $x \in \mathcal{M}_X Int_{G\zeta^*}(A) \cap \mathcal{M}_X Int_{G\zeta^*}(B)$. Then $x \in \mathcal{M}_X Int_{G\zeta^*}(A)$ and $x \in \mathcal{M}_X Int_{G\zeta^*}(B)$. Hence x is a $\mathcal{M}_X G\zeta^*$ -interior point of each of sets A and B . It follows that A and B are $\mathcal{M}_X G\zeta^*$ -nbhds of x , so that their intersection $A \cap B$ is also a $\mathcal{M}_X G\zeta^*$ -nbhd of x . Hence

$$x \in \mathcal{M}_X Int_{G\zeta^*}(A \cap B) \text{. Thus } x \in \mathcal{M}_X Int_{G\zeta^*}(A) \cap \mathcal{M}_X Int_{G\zeta^*}(B) \text{ implies that } x \in \mathcal{M}_X Int_{G\zeta^*}(A \cap B) \text{. Therefore } \mathcal{M}_X Int_{G\zeta^*}(A) \cap \mathcal{M}_X Int_{G\zeta^*}(B) \subset \mathcal{M}_X Int_{G\zeta^*}(A \cap B) \rightarrow (2) \text{. From (1) and (2), we get } \mathcal{M}_X Int_{G\zeta^*}(A \cap B) = \mathcal{M}_X Int_{G\zeta^*}(A) \cap \mathcal{M}_X Int_{G\zeta^*}(B) \text{.}$$

THEOREM 3.9. If A is a subset of X , then

$$m_X - int(A) \subset \mathcal{M}_X Int_{G\zeta^*}(A)$$

PROOF. Let A be a subset of a space X .

Let $x \in m_X - int(A) \Rightarrow x \in \bigcup \{G : G \text{ is } m_X - \text{open}, G \subset A\}$.

\Rightarrow there exists a m_X -open set G such that $x \in G \subset A$.

\Rightarrow there exist a $\mathcal{M}_X G\zeta^*$ -open set G such that

$$x \in G \subset A, \text{ as}$$

Every m_X -open set is a $\mathcal{M}_X G\zeta^*$ -open set in X

$$\Rightarrow x \in \bigcup \{G : G \text{ is } \mathcal{M}_X G\zeta^*\text{-open}, G \subset A\}$$

$$\Rightarrow x \in \mathcal{M}_X Int_{G\zeta^*}(A)$$

Thus $x \in m_X\text{-int}(A) \Rightarrow x \in \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$. Hence $m_X\text{-int}(A) \subset \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$.

REMARK 3.10. Containment relation in the above Theorem 3.9 may be proper as seen from the following example.

EXAMPLE 3.11. Let $X = \{a, b, c\}$ with m_X -open set $= \{\varphi, X, \{a\}, \{a, b\}\}$. Then $\mathcal{M}_X G\zeta^*O(X) = \{\varphi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Let $A = \{a, c\}$. Now $\mathcal{M}_X\text{Int}_{G\zeta^*}(A) = \{a, c\}$ and $m_X\text{-int}(A) = \{a\}$. It follows that $m_X\text{-int}(A) \subset \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$ and $m_X\text{-int}(A) \neq \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$.

THEOREM 3.12. If A is a subset of X , then $amint(A) \subset \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$ where $amint(A)$ is given by $amint(A) = \cup\{G: G \text{ is a } am\text{-open}, G \subset A\}$.

PROOF. Let A be a subset of a space X .
 Let $x \in amint(A) \Rightarrow x \in \cup\{G: G \text{ is } am\text{-open}, G \subset A\}$.
 \Rightarrow there exists a am -open set G such that $x \in G \subset A$.
 \Rightarrow there exist a $\mathcal{M}_X G\zeta^*$ -open set G such that $x \in G \subset A$, as

Every am -open set is a $\mathcal{M}_X G\zeta^*$ -open set in X .
 $\Rightarrow x \in \cup\{G \subset X: G \text{ is } \mathcal{M}_X G\zeta^* \text{-open}, G \subset A\}$.
 $\Rightarrow x \in \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$.

Thus $x \in amint(A) \Rightarrow x \in \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$. Hence $amint(A) \subset \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$.

REMARK 3.13. Containment relation in the above Theorem 3.12 may be proper as seen from the following example.

EXAMPLE 3.14. Let $X = \{a, b, c\}$ with m_X -open set $= \{\varphi, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Then $\mathcal{M}_X G\zeta^*O(X) = \{\varphi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $amO(X) = \{\varphi, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Let $A = \{a, b\}$. Now $\mathcal{M}_X\text{Int}_{G\zeta^*}(A) = X$ and $am\text{-int}(A) = \varphi$. It follows that $am\text{-int}(A) \subset \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$ and $am\text{-int}(A) \neq \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$.

THEOREM 3.15. If A is a subset of X , then $\mathcal{M}_X\text{Int}_{G\zeta^*}(A) \subset m_X ag\text{-int}(A)$ where $m_X ag\text{-int}(A)$ is given by $m_X ag\text{-int}(A) = \cup\{G \subset X: G \text{ is a } m_X ag\text{-open}, G \subset A\}$.

PROOF. Let A be a subset of a space X .
 Let $x \in \mathcal{M}_X\text{Int}_{G\zeta^*}(A) \Rightarrow x \in \cup\{G: G \text{ is } \mathcal{M}_X G\zeta^* \text{-open}, G \subset A\}$.
 \Rightarrow there exists a $\mathcal{M}_X G\zeta^*$ -open set G such that $x \in G \subset A$.
 \Rightarrow there exist a $m_X ag$ -open set G such that $x \in G \subset A$, as

Every $\mathcal{M}_X G\zeta^*$ -open set is $m_X ag$ -open set in X .
 $\Rightarrow x \in \cup\{G \subset X: G \text{ is } m_X ag\text{-open}, G \subset A\}$.
 $\Rightarrow x \in m_X ag\text{-int}(A)$.

Thus $x \in \mathcal{M}_X\text{Int}_{G\zeta^*}(A) \Rightarrow x \in m_X ag\text{-int}(A)$. Hence $\mathcal{M}_X\text{Int}_{G\zeta^*}(A) \subset m_X ag\text{-int}(A)$.

EXAMPLE 3.16. Let $X = \{a, b, c\}$ with m_X -open set $= \{\varphi, X, \{a\}, \{b\}\}$. Then $\mathcal{M}_X G\zeta^*O(X) = \{\varphi, X, \{a\}, \{b\}\}$ and $m_X agO(X) = \{\varphi, X, \{a\}, \{b\}, \{c\}, \{a, b\}\}$. Let $A = \{a, b\}$. Now $\mathcal{M}_X\text{Int}_{G\zeta^*}(A) = \varphi$ and $m_X ag\text{-int}(A) = \{a, b\}$. It follows that $\mathcal{M}_X\text{Int}_{G\zeta^*}(A) \subset m_X ag\text{-int}(A)$ and $m_X ag\text{-int}(A) \neq \mathcal{M}_X\text{Int}_{G\zeta^*}(A)$.

Analogous to closure in a space X , we define $\mathcal{M}_X G\zeta^*$ -closure in a space X as follows.

DEFINITION 3.17. Let A be a subset of a space X . We define the $\mathcal{M}_X G\zeta^*$ -closure of A to be the intersection of all $\mathcal{M}_X G\zeta^*$ -closed sets containing A . In symbols, $\mathcal{M}_X Cl_{G\zeta^*}(A) = \cap\{F: A \subseteq F, F \text{ is } \mathcal{M}_X G\zeta^* \text{-closed in } X\}$

THEOREM 3.18. Let A and B are subsets of a space X . Then

- (i) $\mathcal{M}_X Cl_{G\zeta^*}(X) = X$ and $\mathcal{M}_X Cl_{G\zeta^*}(\varphi) = \varphi$.
- (ii) $A \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$.
- (iii) If B is any $\mathcal{M}_X G\zeta^*$ -closed set containing A , then $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset B$.
- (iv) If $A \subset B$ then $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset \mathcal{M}_X Cl_{G\zeta^*}(B)$.
- (v) $\mathcal{M}_X Cl_{G\zeta^*}(A) = \mathcal{M}_X Cl_{G\zeta^*}(\mathcal{M}_X Cl_{G\zeta^*}(A))$.

PROOF.

(i) By the definition of $\mathcal{M}_X G\zeta^*$ -closure, X is the only $\mathcal{M}_X G\zeta^*$ -closed set containing X . Therefore $\mathcal{M}_X Cl_{G\zeta^*}(X) = \text{Intersection of all the } \mathcal{M}_X G\zeta^* \text{-closed sets containing } X = \cap\{X\} = X$. That is $\mathcal{M}_X Cl_{G\zeta^*}(X) = X$. By the definition of $\mathcal{M}_X G\zeta^*$ -closure, $\mathcal{M}_X Cl_{G\zeta^*}(\varphi) = \text{Intersection of all the } \mathcal{M}_X G\zeta^* \text{-closed sets containing } \varphi = \varphi \cap \text{any } \mathcal{M}_X G\zeta^* \text{-closed sets containing } \varphi = \varphi$. That is $\mathcal{M}_X Cl_{G\zeta^*}(\varphi) = \varphi$.
 (ii) By the definition of $\mathcal{M}_X G\zeta^*$ -closure of A , it is obvious that $A \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$.

(iii) Let B be any $\mathcal{M}_X G\zeta^*$ -closed set containing A . Since $\mathcal{M}_X Cl_{G\zeta^*}(A)$ is contained in every $\mathcal{M}_X G\zeta^*$ -closed set containing A , $\mathcal{M}_X Cl_{G\zeta^*}(A)$ is contained in every $\mathcal{M}_X G\zeta^*$ -closed set containing A . Hence in particular $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset B$.

(iv) Let A and B be subsets of X such that $A \subset B$. By the definition of $\mathcal{M}_X G\zeta^*$ -closure, $\mathcal{M}_X Cl_{G\zeta^*}(B) = \cap\{F: B \subseteq F, F \text{ is } \mathcal{M}_X G\zeta^* \text{-closed in } X\}$. If $B \subset F \in \mathcal{M}_X G\zeta^* C(X)$, then $\mathcal{M}_X Cl_{G\zeta^*}(B) \subset F$. Since $A \subset B, A \subset B \subset F \in \mathcal{M}_X G\zeta^* C(X)$, we have $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset F$. Therefore $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset \cap\{F: B \subset F \in \mathcal{M}_X G\zeta^* C(X)\} = \mathcal{M}_X Cl_{G\zeta^*}(B)$. That is $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset \mathcal{M}_X Cl_{G\zeta^*}(B)$.

(v) Let A be any subset of X . By the definition of $\mathcal{M}_X G\zeta^*$ -closure, $\mathcal{M}_X Cl_{G\zeta^*}(A) = \cap\{F: A \subset F \in \mathcal{M}_X G\zeta^* C(X)\}$, if $A \subset F \in \mathcal{M}_X G\zeta^* C(X)$, then $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset F$. Since F is $\mathcal{M}_X G\zeta^*$ -closed set containing $\mathcal{M}_X Cl_{G\zeta^*}(A)$, by (iii) $\mathcal{M}_X Cl_{G\zeta^*}(\mathcal{M}_X Cl_{G\zeta^*}(A)) \subset F$. Hence

$$\mathcal{M}_X Cl_{G\zeta^*}(\mathcal{M}_X Cl_{G\zeta^*}(A)) \subset \cap\{F: A \subset F \in \mathcal{M}_X G\zeta^* C(X)\} = \mathcal{M}_X Cl_{G\zeta^*}(A). \quad \text{That is } \mathcal{M}_X Cl_{G\zeta^*}(\mathcal{M}_X Cl_{G\zeta^*}(A)) = \mathcal{M}_X Cl_{G\zeta^*}(A).$$

THEOREM 3.19. If a subset A of space X is $\mathcal{M}_X G\zeta^*$ -closed, then $\mathcal{M}_X Cl_{G\zeta^*}(A) = A$.

PROOF. Let A be $\mathcal{M}_X G\zeta^*$ -closed subset of X . We know that $A \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$. Also, A is $\mathcal{M}_X G\zeta^*$ -closed set contained in A . From Theorem 3.18. (iii) $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset A$. Hence $\mathcal{M}_X Cl_{G\zeta^*}(A) = A$.

The converse of the above theorem need not be true as seen from the following example.

EXAMPLE 3.20. Let $X = \{a, b, c, d\}$ with m_X -open set $= \{\varphi, X, \{a\}, \{b\}, \{a, b, d\}, \{a, b, c\}\}$. Then $\mathcal{M}_X G\zeta^* C(X) = \{\varphi, X, \{d\}, \{c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Note that $\mathcal{M}_X Cl_{G\zeta^*}(\{a, b\}) = \{a, b\}$, but $\{a, b\}$ is not a $\mathcal{M}_X G\zeta^*$ -closed set in X .

THEOREM 3.21. If A and B are subsets of X , then $\mathcal{M}_X Cl_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Cl_{G\zeta^*}(A) \cap \mathcal{M}_X Cl_{G\zeta^*}(B)$.

PROOF. Let A and B be subsets of X . Clearly $A \cap B \subset A$ and $A \cap B \subset B$. We have, Theorem 3.18.

(iv), $\mathcal{M}_X Cl_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$ and $\mathcal{M}_X Cl_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Cl_{G\zeta^*}(B)$. This implies that $\mathcal{M}_X Cl_{G\zeta^*}(A \cap B) \subset \mathcal{M}_X Cl_{G\zeta^*}(A) \cap \mathcal{M}_X Cl_{G\zeta^*}(B)$.

THEOREM 3.22. If A and B are subsets of a space X , then $\mathcal{M}_X Cl_{G\zeta^*}(A \cup B) = \mathcal{M}_X Cl_{G\zeta^*}(A) \cup \mathcal{M}_X Cl_{G\zeta^*}(B)$.

PROOF. Let A and B be subsets of X . We know that $A \subset A \cup B$ and $B \subset A \cup B$. Hence $\mathcal{M}_X Cl_{G\zeta^*}(A) \cup \mathcal{M}_X Cl_{G\zeta^*}(B) \subset \mathcal{M}_X Cl_{G\zeta^*}(A \cup B) \rightarrow (1)$. Now to prove $\mathcal{M}_X Cl_{G\zeta^*}(A \cup B) \subset \mathcal{M}_X Cl_{G\zeta^*}(A) \cup \mathcal{M}_X Cl_{G\zeta^*}(B)$. Again, let $x \in \mathcal{M}_X Cl_{G\zeta^*}(A \cup B)$ and suppose $x \notin \mathcal{M}_X Cl_{G\zeta^*}(A) \cup \mathcal{M}_X Cl_{G\zeta^*}(B)$. Then there exists $\mathcal{M}_X G\zeta^*$ -closed sets A_1 and B_1 with $A \subset A_1, B \subset B_1$ and $x \notin A_1 \cup B_1$. We have $A \cup B \subset A_1 \cup B_1$ and $A_1 \cup B_1$ is $\mathcal{M}_X G\zeta^*$ -closed set by the theorem 4.1 in [11] such that $x \notin A_1 \cup B_1$. Thus $x \notin \mathcal{M}_X Cl_{G\zeta^*}(A \cup B)$ which is a contradiction to $x \in \mathcal{M}_X Cl_{G\zeta^*}(A \cup B)$. Hence $\mathcal{M}_X Cl_{G\zeta^*}(A \cup B) \subset \mathcal{M}_X Cl_{G\zeta^*}(A) \cup \mathcal{M}_X Cl_{G\zeta^*}(B) \rightarrow (2)$. From (1) and (2), we have $\mathcal{M}_X Cl_{G\zeta^*}(A \cup B) = \mathcal{M}_X Cl_{G\zeta^*}(A) \cup \mathcal{M}_X Cl_{G\zeta^*}(B)$.

THEOREM 3.23. For an $x \in X, x \in \mathcal{M}_X Cl_{G\zeta^*}(A)$ if and only if $V \cap A \neq \emptyset$ for every $\mathcal{M}_X G\zeta^*$ -open sets V containing x .

PROOF. Let $x \in X$ and $x \in \mathcal{M}_X Cl_{G\zeta^*}(A)$. To prove $V \cap A \neq \emptyset$ for every $\mathcal{M}_X G\zeta^*$ -open sets V containing x . Prove the result by contradiction. Suppose there exists a $\mathcal{M}_X G\zeta^*$ -open set V containing x such that $V \cap A = \emptyset$. Then $A \subset X - V$ and $X - V$ is $\mathcal{M}_X G\zeta^*$ -closed. We have $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset X - V$. This shows that $x \notin \mathcal{M}_X Cl_{G\zeta^*}(A)$, which is contradiction. Hence $V \cap A \neq \emptyset$ for every $\mathcal{M}_X G\zeta^*$ -open set V containing x .

Conversely, let $V \cap A \neq \emptyset$ for every $\mathcal{M}_X G\zeta^*$ -open set V containing x . To prove $x \in \mathcal{M}_X Cl_{G\zeta^*}(A)$. We prove that result by contradiction. Suppose $x \notin \mathcal{M}_X Cl_{G\zeta^*}(A)$. Then there exists a $\mathcal{M}_X G\zeta^*$ -closed subset F containing A such that $x \notin F$. Then $x \in X - F$ and $X - F$ is $\mathcal{M}_X G\zeta^*$ -open. Also $(X - F) \cap A = \emptyset$, which is a contradiction. Hence $x \in \mathcal{M}_X Cl_{G\zeta^*}(A)$.

THEOREM 3.24. If A is subset of a space X , then $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset m_X-cl(A)$.

PROOF. Let A be a subset of a space X . By the definition of closure, $cl(A) = \cap \{F \subset X: A \subset F \in C(X)\}$. If $A \subset F \in C(X)$, then $A \subset F \in \mathcal{M}_X G\zeta^*C(X)$, because every m_X -closed set is $\mathcal{M}_X G\zeta^*$ -closed. That is $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset F$. Therefore $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset \cap \{F \subset X: A \subset F \in C(X)\} = cl(A)$. Hence $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset cl(A)$.

REMARK 3.25. Containment relation in the above Theorem 3.24. may be proper as seen from following example.

EXAMPLE 3.26. Let $X = \{a, b, c, d\}$ with m_X -open set $= \{\emptyset, X, \{c, d\}, \{a, b, c\}\}$. Then $\mathcal{M}_X G\zeta^*C(X) = \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, b, d\}\}$. Let $A =$

$\{a, b, d\}$. Note that $\mathcal{M}_X Cl_{G\zeta^*}(\{a, b, d\}) = \{a, b, d\}$ and $m_X-cl(\{a, b, d\}) = X$. It follows that $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset m_X-cl(A)$ and $\mathcal{M}_X Cl_{G\zeta^*}(A) \neq m_X-cl(A)$.

THEOREM 3.27. If A is subset of a space X , then $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset amcl(A)$, where $amcl(A)$ is given by $amcl(A) = \cap \{F \subset X: A \subset F \text{ and } F \text{ is } am\text{-closed set in } X\}$.

PROOF. Let A be a subset of X . By definition of am -closure $amcl(A) = \cap \{F \subset X: A \subset F \text{ and } F \text{ is } am\text{-closed subset of } X\}$. If $A \subset F$ and F is am -closed subset of X , then $A \subset F \in \mathcal{M}_X G\zeta^*C(X)$, because every am -closed is $\mathcal{M}_X G\zeta^*$ -closed subset in X . That is $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset F$. Therefore $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset \cap \{F \subset X: A \subset F \text{ and } F \text{ is } am\text{-closed}\} = amcl(A)$. Hence $\mathcal{M}_X Cl_{G\zeta^*}(A) \subset amcl(A)$.

REMARK 3.28. Containment relation in the above Theorem 3.27. may be proper as seen from following example.

EXAMPLE 3.29. Let $X = \{a, b, c\}$ with m_X -open set $= \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Then $\mathcal{M}_X Cl_{G\zeta^*}(A) =$

$\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $amcl(A) = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Let $A = \{a, b\}$. Then $\mathcal{M}_X Cl_{G\zeta^*}(\{a, b\}) = \{a, b\}$ and $amcl(\{a, b\}) = X$. That is

$\mathcal{M}_X Cl_{G\zeta^*}(A) \subset amcl(A)$ and $\mathcal{M}_X Cl_{G\zeta^*}(A) \neq amcl(A)$.

THEOREM 3.30. If A is a subset of a space X , then $m_X \alpha g-cl(A) \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$ where $m_X \alpha g-cl(A)$ is given by $m_X \alpha g-cl(A) = \cap \{F \subset X: A \subset F \in m_X \alpha GC(X)\}$.

PROOF. Let A be a subset of X . By the definition of $\mathcal{M}_X G\zeta^*$ -closure, $\mathcal{M}_X Cl_{G\zeta^*}(A) = \cap \{F \subset X: A \subset F \in \mathcal{M}_X G\zeta^*C(X)\}$. If $A \subset F \in \mathcal{M}_X G\zeta^*C(X)$, then $A \subset F \in m_X \alpha GC(X)$, because every $\mathcal{M}_X G\zeta^*$ -closed set is $m_X \alpha g$ -closed set. That is $m_X \alpha g-cl(A) \subset F$. Therefore $m_X \alpha g-cl(A) \subset \cap \{F \subset X: A \subset F \in \mathcal{M}_X G\zeta^*C(X)\} = \mathcal{M}_X Cl_{G\zeta^*}(A)$. Hence $m_X \alpha g-cl(A) \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$.

REMARK 3.31. Containment relation in the above Theorem 3.30. may be proper as seen from following example.

EXAMPLE 3.32. Let $X = \{a, b, c\}$ with m_X -open set $= \{\emptyset, X, \{a\}, \{b, c\}\}$. Then $\mathcal{M}_X Cl_{G\zeta^*}(A) = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $m_X \alpha g-cl(A) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $A = \{c\}$. Then $\mathcal{M}_X Cl_{G\zeta^*}(\{c\}) = \{b, c\}$ and $m_X \alpha g-cl(\{c\}) = \{c\}$. That is $m_X \alpha g-cl(A) \subset \mathcal{M}_X Cl_{G\zeta^*}(A)$ and $m_X \alpha g-cl(A) \neq \mathcal{M}_X Cl_{G\zeta^*}(A)$.

THEOREM 3.31. Let A be any subset of X . Then
 (i) $(\mathcal{M}_X Int_{G\zeta^*}(A))^c = \mathcal{M}_X Cl_{G\zeta^*}(A^c)$
 (ii) $\mathcal{M}_X Int_{G\zeta^*}(A) = (\mathcal{M}_X Cl_{G\zeta^*}(A^c))^c$
 (iii) $\mathcal{M}_X Cl_{G\zeta^*}(A) = (\mathcal{M}_X Int_{G\zeta^*}(A^c))^c$

PROOF. (i) Let $x \in (\mathcal{M}_X Int_{G\zeta^*}(A))^c$. Then $x \notin \mathcal{M}_X Int_{G\zeta^*}(A)$. That is every $\mathcal{M}_X G\zeta^*$ -open set U containing x such that $U \not\subset A$. That is every $\mathcal{M}_X G\zeta^*$ -open set U containing x such that $U \cap A^c \neq \emptyset$. By theorem 3.23., $x \in \mathcal{M}_X Cl_{G\zeta^*}(A^c)$ and therefore $(\mathcal{M}_X Int_{G\zeta^*}(A))^c \subset \mathcal{M}_X Cl_{G\zeta^*}(A^c)$. Conversely,

let $x \in \mathcal{M}_X Cl_{G\zeta^*}(A^c)$. Then by Theorem 3.23., every $\mathcal{M}_X G\zeta^*$ -open set U containing x such that $U \cap A^c \neq \emptyset$. That is every $\mathcal{M}_X G\zeta^*$ -open set U containing x such that $U \not\subset A$. This implies by Definition of $\mathcal{M}_X G\zeta^*$ -interior of A , $x \notin \mathcal{M}_X Int_{G\zeta^*}(A)$. That is $x \in (\mathcal{M}_X Int_{G\zeta^*}(A))^c$ and $\mathcal{M}_X Cl_{G\zeta^*}(A^c) \subset (\mathcal{M}_X Int_{G\zeta^*}(A))^c$. Thus $(\mathcal{M}_X Int_{G\zeta^*}(A))^c = \mathcal{M}_X Cl_{G\zeta^*}(A^c)$.

(ii) Follows by taking complements in (i).

(iii) Follows by replacing A by A^c in (i).

4. REFERENCES

- [1] Alimohammady M, Roohi M, Linear minimal spaces, *Chaos, Solitons and Fractals*, 33(4) (2007), 1348-1354.
- [2] Alimonammady M, Roohi M, Fixed Point in Minimal Spaces, *Nonlinear Analysis: Modelling and Control*, 2005, Vol. 10, No.4, 305 – 314.
- [3] I.Arockiarani, Studies on Generalizations of Generalized Closed Sets and Maps in Topological Spaces, *Ph.D Thesis, Bharathiar University, Coimbatore*, (1997).
- [4] S.P.Arya and R.Gupta, On Strongly Continuous Mappings, *Kyungpook Math. J.* 14(1974), 131 – 143.
- [5] K.Balachandran, P.Sundaram and H.Maki, On Generalized Continuous Maps in Topological Spaces, *Mem. I ac Sci. Kochi Univ. Math.*, 12(1991), 5 – 13.
- [6] P.Bhattacharyya and B.K.Lahiri, Semi-generalized Closed sets in Topology, *Indian J. Math.*, 29 (1987), 376 – 382.
- [7] Cszasz A, Generalized topology: generalized continuity, *Acta. Math. Hungar.*, 96, pp. 351 – 357, 2002.6.S.Lugojan. Generalized Topology, *Stud. Cerc. Math.*, 34, pp.348 – 360, 1982.
- [8] R.Devi, K.Balachandran and H.Maki, On Generalized α -continuous Maps, *Far. East J. Math. Sci. Special Volume, Part I* (1997), 1 – 15.
- [9] W.Dunham, A New Closure Operator for non- T_1 Topologies, *Kyungpook Math. J.*, 22 (1982), 55 – 60.
- [10] Y.Gnanambal, On Generalized Pre-regular Closed Sets in Topological Spaces, *Indian J.Pure Appl. Math.*, 28 (1997), 351 – 360.
- [11] Kokilavani V, Myvizhi M and VivekPrabhu M, Generalized ζ^* -Closed sets in Topological Spaces, *International Journal of Mathematical Archive*, 4950, 2013, 274 – 279.
- [12] Kokilavani V, Myvizhi M, On $\mathcal{M}_X g\zeta^*$ -closed sets in \mathcal{M} -structures (Submitted).
- [13] Kokilavani V, Myvizhi M, On $(M, Q, G\zeta^*)$ -Open functions in \mathcal{M} -structures, *International Journal of Mathematics Trends and Technology*, 4(2013), 386 – 393.
- [14] N.Levine, Generalized Closed Sets in Topology, *Rend. Circ. Mat. Palermo*, 19 (1970), 89 – 96.
- [15] Maki H, On generalizing semi-open sets and pre-open sets, in: *Meeting on Topological Spaces Theory and its Application, August 1996*, pp. 13 – 18.
- [16] Maki H, Rao K C and NagoorGani A, On generalizing semi-open and preopen sets, *Pure Appl. Math. Sci.*, 49(1999), 17–29.
- [17] Maki H, Umehara J, Noiri T, Every topological space is pre $T_{1/2}$, *Mem. Fac. Sci. Kochi Univ. Ser.Math.*, pp. 33 – 42.
- [18] S.R.Malghan, Generalized Closed Maps, *J. Karnatak Univ. Sci.*, 27(1982), 82 – 88.
- [19] Noiri T, On Λ_m -sets and related spaces, in: *Proceedings of the 8th Meetings on Topological Spaces Theory and its Applications, August 2003*, pp. 31 – 41.
- [20] Noiri T, 11th meetings on topological spaces theory and its applications, *Fukuoka University Seminar House*, 2006, 1 – 9.
- [21] N.Palaniappan and K.C.Rao, Regular Generalized Closed Sets, *Kyungpook, Math. J.*, 33(1993), 211 – 219.
- [22] J.K.Park and J.H.Park, Mildly Generalized Closed Sets, Almost Normal and Mildly Normal Spaces, *Chaos, Solutions and Fractals*, 20(2004), 1103 – 1111.
- [23] Popa V, Noiri T, On M-continuous functions, *Anal. Univ. "Dunarea Jos" – Galati, Ser. Mat. Fiz, Mec. Teor. Fasc.II*, 18(23), pp. 31 – 41, 2000.
- [24] Popa V, Noiri T, On m - D -continuous axioms, *J. Math. Univ. Istanbul Fac.Sci.*, 61 / 62 (2002 / 2003), 15 – 28.
- [25] Won Keun Min, am -open sets and am -continuous functions, *Commun. Korean Math. Soc.* 25 (2010), N0.2, pp. 251 – 256.