

A Result in Best Approximation Theory for Family of Maps

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ABSTRACT

The aim of the present paper is to establish common fixed point and best approximation results for family of weakly compatible mappings which unify and generalize various known results.

Keywords

Common fixed point; Weakly Compatible maps; Best approximation.

AMS Subject Classification

41A50, 47H10, 54H25, 55M20

1. INTRODUCTION

The problem of best approximation began in 1853 with P. L. Chebyshev who considered the problem in the space of all real valued continuous functions defined on $[a, b]$, a closed real interval in \mathbb{R} . However, the result given by Meinardus [11] was the fundamental result in this direction. In 1963, Meinardus [11] employed the Schauder fixed point theorem to prove a result regarding invariant approximation in Banach spaces. In 1969, Brosowski [5] obtained celebrated results and generalized the Meinardus's result. Since then a number of results have been proved following in the direction of Brosowski [5]. In 1977, Subrahmanyam [18] extended the Meinardus result to a finite dimensional subspace. In 1979, Singh [17] further improved the results of Brosowski [5] in many ways. In 1982, Hicks and Humpheries [6] generalized the results regarding invariant approximation. In the year 1988, Shahab et al. [14] extended the results of Hicks and Humpheries [6] and Singh [17] by considering a pair of mappings wherein one is linear and the other one is nonexpansive.

In 1992, Beg et al. [4] proved results on the existence of the best approximant for relatively contractive commuting mappings on convex metric spaces which also generalized the core result of Sahab et al. [14]. In 1996, Al-Thagafi [1] established some theorems on invariant approximations for commuting maps.

By introducing the concept of noncommuting maps to this area Shahzad [15], Al-Thagafi and Shahzad [2, 3], Hussain and Jungck [8], Hussain [7], Hussain and Rhoades [9], Jungck and Hussain [10], O'Regan and Hussain [12] and Pathak and Hussain [13] extended and improved the above mentioned results to noncommuting maps such as pointwise \mathbb{R} -subweakly commuting maps, compatible maps, C_q -commuting maps and Banach operator pairs.

The aim of this paper is to extend the fixed point results of Singh and Tomar [16] for family of maps to best

approximation. We also extend the best approximation result of Sumitra et al. [19] to family of mappings.

Before going to main result we need some preliminaries and results as follows:

Definition 1.1: Let X be a non-empty set and $T: X \rightarrow X$ be a self map. Then $x \in X$ is a **fixed point** of T if $T(x) = x$. We denote by $\mathbf{Fix}(T)$, the set of all fixed points of T .

Definition 1.2: Let (X, d) be a metric space and $T, S: X \rightarrow X$. A point $x \in X$ is called:

(1) a **coincidence point** of the pair (T, S) if $Tx = Sx$,

(2) a **common fixed point** of the pair (T, S) if $x = Tx = Sx$.

Definition 1.3: Let (X, d) be a metric space and M be a nonempty subset of X . Let $T, S: X \rightarrow X$. The pair (T, S) is said to be **weakly compatible** if they commute at their coincidence points, i.e., $STx = TSx$ whenever $Sx = Tx$.

Definition 1.4: Let (X, d) be a metric space and M be any closed subset of X . If there exists a $y_0 \in M$ such that $d(x, y_0) = d(x, M) = \inf_{y \in M} d(x, y)$ then y_0 is called a **best approximation** to x out of M . We denote by $\mathbf{P}_M(x)$, the set of all best approximation to x out of M .

In 2003, Singh and Tomar [16] gave the following result:

Theorem A: Let $\{A_i\}$, $i=1,2,\dots$, S and T be selfmaps of a metric space (X, d) . If one of SX , TX or A_iX is a complete subspace of X such that

(i) $A_1X \subseteq TX$ and $A_iX \subseteq SX$,

(ii) $d(A_1x, A_2y) < \varphi(M_{12}(x, y))$ whenever $M_{12}(x, y) > 0$,

(iii) $d(A_1x, A_iy) < \varphi(M_{1i}(x, y))$ whenever $M_{1i}(x, y) \neq 0$.

where $M_{1i}(x, y) = \max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), \frac{d(A_1x, Ty) + d(A_iy, Sx)}{2}\}$ and $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an upper semicontinuous function such that $\varphi(t) < t$ for each $t > 0$.

Then:

(I) A_1 and S have a coincidence,

(II) A_i and T have a coincidence,

(III) A_1 and S have a common fixed point provided that they are weakly compatible,

(IV) A_i and T have a common fixed point provided that they are weakly compatible,

(V) A_i , S and T , for each i , have a unique common fixed point provided that (III) and (IV) both are true.

We will need the following general common fixed point result which is a consequence of Theorem A.

Theorem B: Let T and $\{A_i\}$, $i=1,2,\dots$, be selfmaps of a metric space (X, d) . If one of TX or A_iX is a complete subspace of X such that

(i) $\cup_i A_iX \subseteq TX$

(ii) $d(A_1x, A_iy) < \varphi(M_{1i}(x, y))$ whenever $M_{1i}(x, y) \neq 0$.

where $M_{1i}(x, y) = \max\{d(Tx, Ty), d(A_1x, Tx), d(A_iy, Ty), \frac{d(A_1x, Ty) + d(A_iy, Tx)}{2}\}$ and $\varphi: R_+ \rightarrow R_+$ be an upper semicontinuous function such that $\varphi(t) < t$ for each $t > 0$. If T is weakly compatible with each of A_i , then A_i and T , for each i , have a unique common fixed point.

2. MAIN RESULTS

Now, we give the main results on common fixed point and best approximation for family of weakly compatible maps satisfying generalized contractive condition in metric spaces.

Theorem 2.1: Let T and $\{A_i\}$, $i=1, 2,\dots$, be selfmaps of a complete metric space (X, d) . Suppose that $u \in X$ and A_i leaves T -invariant compact subset M of closed subspace $T(X)$ as invariant. For each $b \in P_M(u)$, let $d(x, A_i b) < d(x, Tb)$ and $Tb \in P_M(u)$. If T is weakly compatible with each of A_i and satisfy $d(A_1x, A_iy) < \varphi(M_{1i}(x, y))$, whenever $M_{1i}(x, y) \neq 0$, where $M_{1i}(x, y) = \max\{d(Tx, Ty), d(A_1x, Tx), d(A_iy, Ty), \frac{d(A_1x, Ty) + d(A_iy, Tx)}{2}\}$ for all $x, y \in P_M(u) \cup \{u\}$. Then $P_M(u) \cap \text{Fix}(T) \cap \text{Fix}(A_i) \neq \emptyset$.

Proof: Since M is a compact subset of $T(X)$ therefore $P_M(u) \neq \emptyset$. Now we show that $A_i(P_M(u)) \subseteq T(P_M(u))$ for each i .

Assume on the contrary that there exists $b \in P_M(u)$ with $A_i b \notin T(P_M(u))$. Consider

$$d(u, Tb) = d(u, M) \\ \leq d(u, A_i b)$$

$$< d(u, Tb), \text{ which is a contradiction.}$$

Hence $A_i(P_M(u)) \subseteq T(P_M(u))$. Now $T(P_M(u))$ being closed subset of complete metric space is complete. So by Theorem B, T and A_i , for each i , have a common fixed point in $P_M(u)$. Thus $P_M(u) \cap \text{Fix}(T) \cap \text{Fix}(A_i) \neq \emptyset$.

Theorem 2.2: Let T and $\{A_i\}$, $i=1, 2,\dots$, be selfmaps of a complete metric space (X, d) . Suppose that $u \in \text{Fix}(T) \cap \text{Fix}(A_i)$ for some u in X and A_i leaves T -invariant compact subset M of closed subspace $T(X)$ as invariant. If T is weakly compatible with each of A_i and satisfy $d(A_1x, A_iy) < \varphi(M_{1i}(x, y))$, whenever $M_{1i}(x, y) \neq 0$,

where $M_{1i}(x, y) = \max\{d(Tx, Ty), d(A_1x, Tx), d(A_iy, Ty), \frac{d(A_1x, Ty) + d(A_iy, Tx)}{2}\}$ for all $x, y \in P_M(u) \cup \{u\}$. Then $P_M(u) \cap \text{Fix}(T) \cap \text{Fix}(A_i) \neq \emptyset$.

Proof: Since M is a compact subset of $T(X)$, therefore $P_M(u) \neq \emptyset$. Now we show that $A_i(P_M(u)) \subseteq T(P_M(u))$ for each i .

Assume on the contrary that there exists $b \in P_M(u)$ with $A_i b \notin T(P_M(u))$. Consider

$$d(u, Tb) = d(u, M) \\ \leq d(u, A_i b) \\ = d(A_1 u, A_i b) \\ < \max\{d(Tu, Tb), d(A_1 u, Tu), d(A_i b, Tb), \frac{d(A_1 u, Tb) + d(A_i b, Tu)}{2}\} \\ < d(Tb, u), \text{ which is a contradiction.}$$

Hence $A_i(P_M(u)) \subseteq T(P_M(u))$. Now $T(P_M(u))$ being closed subset of complete metric space is complete. So by Theorem B, T and A_i , for each i , have a common fixed point in $P_M(u)$. Thus $P_M(u) \cap \text{Fix}(T) \cap \text{Fix}(A_i) \neq \emptyset$.

3. CONCLUSION

From the above discussion it is clear that our results generalize and extend the results on common fixed point and best approximation to family of mappings. The results on best approximation for two or more maps can be easily obtained from these results.

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