Comment on "Application of Improved (G'/G)- Expansion Method to Traveling Wave Solutions of Two Nonlinear Evolution Equations, Adv. Appl. Math. Mech. 4(2012) 122-130"

Elsayed M. E. Zayed
Mathematics Department,
Faculty of Sciences, Zagazig University, Zagazig, Egypt

Yasser A. Amer
Mathematics Department,
Faculty of Sciences, Zagazig University, Zagazig, Egypt

Reham M. A. Shohib
Mathematics Department,
Faculty of Sciences, Zagazig University, Zagazig, Egypt

ABSTRACT
The authors of the above article proposed the improved (G'/G)- expansion method and found some traveling wave solutions for each of two nonlinear evolution equations in mathematical physics, namely the Regularized Long Wave (RLW) equation and the Symmetric Regularized Long Wave (SRLW) equation. In the present article, we have noted that if we use a suitable transformation, the improved (G'/G)-expansion method can be reduced into the well -known generalized Riccati equation mapping method which provides us with much more traveling wave solutions, namely twenty seven solutions for each of these two nonlinear evaluation equations. Comparison between the results of these two methods is presented.

General Terms
02.30.Jr, 05.45.Yv, 02.30.Ik

Keywords
Improved (G'/G)- expansion method; Generalized Riccati equation mapping method; the nonlinear RLW equation; the nonlinear SRLW equation; traveling wave solutions.

1. INTRODUCTION
Traveling wave solutions for nonlinear evolution equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, optics, condensed matter physics, plasma physics and so on. In recent decades, many effective methods [1-25] have been established to obtain the exact traveling wave solutions of these equations. In Ref. [19] the authors proposed the improved (G'/G)- expansion method and found some solutions for each of two nonlinear evolution equations, namely the nonlinear RLW equation and the nonlinear SRLW equation. This method can be summarized as follows: Suppose that a nonlinear evaluation equation has the following from:

$$F(u,u_x,u_t,u_{xx},u_{tt},...)=0, \qquad (1.1)$$

where F is a polynomial in $u=u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. The wave transformation

$$u(x,t)=u(\xi),\ \xi=x-\omega t, \qquad (1.2)$$

where ω is a nonzero constant, reduces Eq. (1.1) to the following nonlinear ordinary differential equation (ODE) for $u(\xi)$:

$$H(u,u',u'',...)=0 \quad , \qquad (1.3)$$

where H is apolynomial in $u(\xi)$ and its total derivatives with respect to ξ.

The authors [19] assumed that Eq.(1.3) has the formal solution:

$$u(\xi)=\sum_{i=0}^{\ell}\alpha_i\left(\frac{G'(\xi)}{G(\xi)}\right)^i \ , \qquad (1.4)$$

where α_i are all real constants to be determined such that $\alpha_\ell\neq 0$, while $G(\xi)$ is the solution of the following nonlinear auxiliary ODE:

$$GG''=AG^2+BGG'+CG'^2, \qquad (1.5)$$

where A, B and C are real parameters such that $C\neq 1$, , while the positive integer ℓ in Eq. (1.4) is determined by balancing the nonlinear terms and the highest order derivatives. The authors [19] have obtained the following formulas:

Case 1. When $B\neq 0,\ \Delta=B^2+4A(1-C)\geq 0$, then

$$\frac{G'(\xi)}{G(\xi)}=-\frac{B}{2(1-C)}+\frac{B\sqrt{\Delta}}{2(1-C)}\left[\frac{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right)+c_2\exp\left(-\frac{\sqrt{\Delta}}{2}\xi\right)}{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right)-c_2\exp\left(-\frac{\sqrt{\Delta}}{2}\xi\right)}\right]. \qquad (1.6)$$

Case 2. When $B\neq 0,\ \Delta=B^2+4A(1-C)<0$, then

$$\frac{G'(\xi)}{G(\xi)}=\frac{B}{2(1-C)}+\frac{B\sqrt{-\Delta}}{2(1-C)}\left[\frac{ic_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)-c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right] \tag{1.7}$$

Case 3. When $B=0,\ \Delta=A(C-1)\geq 0$, then

$$\frac{G'(\xi)}{G(\xi)}=\frac{\sqrt{\Delta}}{(1-C)}\left[\frac{c_1\cos\left(\sqrt{\Delta}\xi\right)+c_2\sin\left(\sqrt{\Delta}\xi\right)}{c_1\sin\left(\sqrt{\Delta}\xi\right)-c_2\cos\left(\sqrt{\Delta}\xi\right)}\right]. \tag{1.8}$$

Case 4. When $B=0,\ \Delta=A(1-C)>0$, then

$$\frac{G'(\xi)}{G(\xi)}=\frac{\sqrt{-\Delta}}{(1-C)}\left[\frac{ic_1\cosh\left(\sqrt{-\Delta}\xi\right)-c_2\sinh\left(\sqrt{-\Delta}\xi\right)}{ic_1\sinh\left(\sqrt{-\Delta}\xi\right)-c_2\cosh\left(\sqrt{-\Delta}\xi\right)}\right], \tag{1.9}$$

where c1 and c2 are arbitrary constants, and $i=\sqrt{-1}$ while $C\neq 1$. After a careful revision of the cases (1.6) - (1.9), we have found that the constant" B" in the second terms of (1.6) and (1.7) should be omitted in order that both (1.6) and (1.7) satisfy the auxiliary ODE (1.5).

With reference to Sec. 3 of Ref. [19], the authors applied the improved (G'/G)- expansion method (1.4) and (1.5) and found the solutions(3.5)-(3.11) of [19] for the RLW equation (3.1) as well as the solutions (3.17)-(3.21) of [19] for the SLRW equation (3.12) which contain some minor errors due to the error in (1.6) and (1.7).

Let us now rewrite down the solutions (3.5)-(3.11) of Ref. [19] in the following corrected forms:

If we choose $B\neq 0$ and $\Delta_1=B^2+4A-4AC\geq 0$, then we have the solutions :

$$u(x,t)=\frac{\varepsilon b\omega\Delta_1}{2a}-\frac{3b\omega\Delta_1}{2a}\left[\frac{c_1e^{\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}+c_2e^{-\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}}{c_1e^{\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}-c_2e^{-\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}}\right]^2. \tag{1.10}$$

we choose $B\neq 0$ and $\Delta_1=B^2+4A-4AC<0$, we have the solutions:

$$u(x,t)=\frac{\varepsilon b\omega\Delta_1}{2a}+\frac{3b\omega\Delta_1}{2a}\left[\frac{ic_1\cos\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)-c_2\sin\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)}{ic_1\sin\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)+c_2\cos\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)}\right]^2. \tag{1.11}$$

If we choose $B=0$ and $\Delta_2=A(C-1)\geq 0$, we have the solutions

$$u(x,t)=\frac{-2\lambda b\omega\Delta_2}{a}-\frac{6b\omega\Delta_2}{a}\left[\frac{c_1\cos\left(\sqrt{\Delta_2}(x-\omega t)\right)+c_2\sin\left(\sqrt{\Delta_2}(x-\omega t)\right)}{c_1\sin\left(\sqrt{\Delta_2}(x-\omega t)\right)-c_2\cos\left(\sqrt{\Delta_2}(x-\omega t)\right)}\right]^2. \tag{1.12}$$

If we choose $B=0$ and $\Delta_2=A(C-1)<0$, we have the solutions

$$u(x,t)=\frac{-2\lambda b\omega\Delta_2}{a}+\frac{6b\omega\Delta_2}{a}\left[\frac{ic_1\cosh\left(\sqrt{-\Delta_2}(x-\omega t)\right)-c_2\sinh\left(\sqrt{-\Delta_2}(x-\omega t)\right)}{ic_1\sinh\left(\sqrt{-\Delta_2}(x-\omega t)\right)-c_2\cosh\left(\sqrt{-\Delta_2}(x-\omega t)\right)}\right]^2 \tag{1.13}$$

where A, B ,C, c_1, c_2 are real parameters and a, b are positive constants while ε equals to 1 or 3, so is λ. If ε equals to 1, we should choose $\omega=(b\Delta_1+1)^{-1}$, while if ε equals to 3, then $\omega=(1-b\Delta_1)^{-1}$. Similarly, if λ is equal to 1 or 3, ω is $(1-4b\Delta_2)^{-1}$ or $(1+4b\Delta_2)^{-1}$ respectively.

If we choose $c_1=-c_2$, then the solution (1.10) becomes in the form:

$$u(x,t)=\frac{\varepsilon b\omega\Delta_1}{2a}-\frac{3b\omega\Delta_1}{2a}\tanh^2\frac{\sqrt{\Delta_1}}{2}(x-\omega t), \tag{1.14}$$

while if we set $c_1=c_2$, we have the solution:

$$u(x,t)=\frac{\varepsilon b\omega\Delta_1}{2a}-\frac{3b\omega\Delta_1}{2a}\coth^2\left(\frac{\sqrt{\Delta_1}}{2}(x-\omega t)\right), \tag{1.15}$$

Also, we rewrite down the solutions (3.17)- (3.21) of the SLRW equation (3.12) of Ref. [19] in the corrected forms as follows:

If we choose $B\neq 0$ and $\Delta_1=B^2+4A-4AC\geq 0$, then

$$u(x,t)=\frac{\omega^2+1-2\omega^2\Delta}{\omega}+3\omega\Delta\left[\frac{c_1e^{\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}+c_2e^{-\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}}{c_1e^{\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}-c_2e^{-\frac{\sqrt{\Delta_1}}{2}(x-\omega t)}}\right]^2. \quad (1.16)$$

If we choose $B \neq 0$ and $\Delta_1 = B^2+4A-4AC<0$, then

$$u(x,t)=\frac{\omega^2+1-2\omega^2\Delta}{\omega}-3\omega\Delta\left[\frac{ic_1\cos\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)-c_2\sin\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)}{ic_1\sin\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)+c_2\cos\frac{\sqrt{-\Delta_1}}{2}(x-\omega t)}\right]^2. \quad (1.17)$$

If we choose $B=0$ and $\Delta = A(C-1)\geq 0$, then

$$u(x,t)=\frac{\omega^2+1+8\omega^2\Delta}{\omega}+12\omega\Delta\left[\frac{c_1\cos\sqrt{\Delta}(x-\omega t)+c_2\sin\sqrt{\Delta}(x-\omega t)}{c_1\sin\sqrt{\Delta}(x-\omega t)-c_2\cos\sqrt{\Delta}(x-\omega t)}\right]^2. \quad (1.18)$$

If we choose $B=0$ and $\Delta = A(C-1)<0$, then

$$u(x,t)=\frac{\omega^2+1+8\omega^2\Delta}{\omega}-12\omega\Delta\left[\frac{ic_1\cosh\left(\sqrt{-\Delta}(x-\omega t)\right)-c_2\sinh\left(\sqrt{-\Delta}(x-\omega t)\right)}{ic_1\sinh\left(\sqrt{-\Delta}(x-\omega t)\right)-c_2\cosh\left(\sqrt{-\Delta}(x-\omega t)\right)}\right]^2. \quad (1.19)$$

If we set $c_1=-c_2$ into (1.16), we get the solution

$$u(x,t)=\frac{\omega^2+1-2\omega^2\Delta}{\omega}+3\omega\Delta\tanh^2\frac{\sqrt{\Delta}}{2}(x-\omega t). \quad (1.20)$$

We have noted that the auxiliary nonlinear ODE (1.5) used in Ref. [19] can be rewritten in the form of the following generalized Riccati equation for (G'/G) as :

$$\left(\frac{G'}{G}\right)'=A+B\left(\frac{G'}{G}\right)+(C-1)\left(\frac{G'}{G}\right)^2. \quad (1.21)$$

If we use the simple transformation $\frac{G'}{G}=\phi(\xi)$, then Eqs. (1.4) and (1.21) can be rewritten in the form:

$$u(\xi)=\sum_{i=0}^{\ell}\alpha_i\phi^i(\xi), \quad (1.22)$$

where $\phi(\xi)$ satisfies the generalized Riccati equation

$$\phi'(\xi)=A+B\phi(\xi)+(C-1)\phi^2(\xi). \quad (1.23)$$

It is well -known [20-25] that the models (1.22) and (1.23) form the generalized Riccati equation mapping method where Eq. (1.23) has the following well- known twenty seven exact solutions:

Type 1: When $\Delta = B^2-4A(C-1)>0$ and $B(C-1)\neq 0$ or $A(C-1)\neq 0$ we have

$$\Phi_1(\xi)=\frac{1}{2(1-C)}[B+\sqrt{\Delta}\tanh(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$\Phi_2(\xi)=\frac{1}{2(1-C)}[B+\sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$\Phi_3(\xi)=\frac{1}{2(1-C)}[B+\sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi)\pm i\ \mathrm{sech}(\sqrt{\Delta}\xi))],$$

$$\Phi_4(\xi)=\frac{1}{2(1-C)}[B+\sqrt{\Delta}(\coth(\sqrt{\Delta}\xi)\pm \mathrm{csch}(\sqrt{\Delta}\xi))],$$

$$\Phi_5(\xi)=\frac{1}{4(1-C)}[2B+\sqrt{\Delta}(\tanh(\frac{\sqrt{\Delta}}{4}\xi)\pm\coth(\frac{\sqrt{\Delta}}{4}\xi))],$$

$$\Phi_6(\xi)=-\frac{1}{2(1-C)}[-B+\frac{\sqrt{\Delta(A_1^2+B_1^2)}-A_1\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi)+B_1}],$$

$$\Phi_7(\xi)=-\frac{1}{2(1-C)}[-B-\frac{\sqrt{\Delta(B_1^2-A_1^2)}+A_1\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi)+B_1}],$$

where A_1 and B_1 are two non-zero real constants satisfying $B_1^2-A_1^2>0$,

$$\Phi_8(\xi)=\frac{2A\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-B\cos h(\frac{\sqrt{\Delta}}{2}\xi)},$$

$$\Phi_9(\xi)=\frac{-2A\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)},$$

$$\Phi_{10}(\xi)=\frac{2A\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-B\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta}},i=\sqrt{-1}$$

$$\Phi_{11}(\xi)=\frac{2A\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{-B\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}},$$

$$\Phi_{12}(\xi)=\frac{4A\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)}{-2B\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}},$$

Type 2: When $\Delta=B^2+4A(1-C)<0$ and $-B(1-C)\neq 0$ or $-A(1-C)\neq 0$ we have

$$\Phi_{13}(\xi)=-\frac{1}{2(1-C)}[-B+\sqrt{-\Delta}\tan(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$\Phi_{14}(\xi)=-\frac{1}{2(1-C)}[B+\sqrt{-\Delta}\cot(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$\Phi_{15}(\xi)=-\frac{1}{2(1-C)}[-B+\sqrt{-\Delta}(\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi))],$$

$$\Phi_{16}(\xi)=\frac{1}{2(1-C)}[B+\sqrt{-\Delta}(\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi))],$$

$$\Phi_{17}(\xi)=-\frac{1}{4(1-C)}[-2B+\sqrt{-\Delta}(\tan(\frac{\sqrt{-\Delta}}{4}\xi)-\cot(\frac{\sqrt{-\Delta}}{4}\xi))],$$

$$\Phi_{18}(\xi)=-\frac{1}{2(1-C)}[-B+\frac{\pm\sqrt{-\Delta(A_1^2-B_1^2)}-A_1\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}],$$

$$\Phi_{19}(\xi)=-\frac{1}{2(1-C)}[-B-\frac{\pm\sqrt{-\Delta(A_1^2-B_1^2)}-A_1\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}],$$

where A_1 and B_1 are two non-zero real constants satisfying $A_1^2-B_1^2>0$,

$$\Phi_{20}(\xi)=-\frac{2A\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)},$$

$$\Phi_{21}(\xi)=\frac{2A\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)},$$

$$\Phi_{22}(\xi)=-\frac{2A\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}},$$

$$\Phi_{23}(\xi)=\frac{2A\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-B\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}},$$

$$\Phi_{24}(\xi)=\frac{4A\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)}{-2B\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)-\sqrt{-\Delta}}$$

Type 3: When $A=0 \, and \, B(1-C)\neq 0$ we have

$$\Phi_{25}(\xi)=\frac{Bd}{(1-C)[d+\cosh(B\xi)-\sinh(B\xi)]},$$

$$\Phi_{26}(\xi)=\frac{B[\cosh(B\xi)+\sinh(B\xi)]}{(1-C)[d+\cosh(B\xi)+\sinh(B\xi)]},$$

where d is an arbitrary constant.

Type 4: When $A=B=0 \, and \, (1-C)\neq 0$ we have

$$\Phi_{27}(\xi)=\frac{1}{(1-C)\xi-c_1},$$

where c_1 is an arbitrary constant.

The objective of this article is to apply the generalized Riccati equation mapping method (1.22) and (1.23) instead of the improved (G'/G)- expansion method (1.4) and (1.5) used in Ref. [19] for finding several solutions of the nonlinear RLW equation and the nonlinear SRLW equation. Comparison between our results in this article and the well known results obtained in Ref. [19] will be given in Sec. 3.

2. APPLICATIONS

In this section we will apply the generalized Riccati equation mapping method (1.22) and (1.23) to construct several solutions of the following nonlinear evolution equations:

2.1. Example 1. The nonlinear RLW equation

This equation is well- known [19] and has the form

$$u_t + u_x + a(u^2)_x - bu_{xxt} = 0 \qquad (2.1)$$

where a and b are positive constants. The wave transformation (1.2) reduces Eq. (2.1) into the nonlinear ODE:

$$(1-\omega)u' + 2auu' + b\omega u''' = 0. \qquad (2.2)$$

Integrating (2.2) once with respect to ξ with zero constant of integration, we get

$$(1-\omega)u + au^2 + bwu'' = 0.$$

(2.3)

Balancing u'' and u^2 we have $\ell = 2$. So Eq. (2.3) has the formal solution

$$u(\xi) = \alpha_0 + \alpha_1\phi(\xi) + \alpha_2\phi^2(\xi), \qquad (2.4)$$

where $\alpha_0, \alpha_1, \alpha_2$ are constants to be determined, such that $\alpha_2 \neq 0$.

Substituting (2.4) along with Eq. (1.23) into Eq. (2.3), collecting all the terms with the same order $\phi^k\,(k = 0,1,2,...)$ and setting all the coefficients to zero, we have the following algebraic equations:

$$Q^4 : 6\alpha_2 b\omega(1-C)^2 + a\alpha_2^2 = 0$$

$$Q^3 : b\omega(-10\alpha_2 B(1-C) + 2\alpha_1(1-C)^2) + 2a\alpha_1\alpha_2 = 0$$

$$Q^2 : b\omega[-8\alpha_2 A(1-C) + 4\alpha_2 B^2 - 3\alpha_1 B(1-C)] + (1-\omega)\alpha_2 + a(\alpha_1^2 + 2\alpha_0\alpha_2) = 0$$

$$Q : b\omega[6\alpha_2 AB - 2\alpha_1 A(1-C)) + \alpha_1 B^2] + 2a\alpha_0\alpha_1 + (1-\omega)\alpha_1 = 0$$

$$Q^0 : b\omega(\alpha_1 AB + 2\alpha_2 A^2) + a\alpha_0^2 + (1-\omega)\alpha_0 = 0$$

By solving these algebraic equations with the aid of Maple or Mathematica we can distinguish different cases as follows:

Case 1

$$\alpha_0 = -\frac{b\omega[B^2 - 2A(1-C)]}{a},\ \alpha_1 = \frac{6Bb\omega(1-C)}{a},$$

$$\alpha_2 = -\frac{6b\omega(1-C)^2}{a},\ \omega = \frac{1}{[1 + 4Ab(1-C) + bB^2]}$$

Case 2

$$\alpha_0 = \frac{6Ab\omega(1-C)}{a},\ \alpha_1 = \frac{6Bb\omega(1-C)}{a},$$

$$\alpha_2 = -\frac{6b\omega(1-C)^2}{a},\ \omega = \frac{1}{[1 - 4Ab(1-C) - bB^2]}$$

2.1.1. Exact solutions of the nonlinear RLW equation (2.1) for case 1.

By using case 1 and according to the values of solutions of type 1 when $\Delta = B^2 + 4A(1-C) > 0$, we obtain the following solutions :

$$u_1(x,t) = \frac{b\omega\Delta}{2a}\left[1 - 3\tanh^2(\frac{\sqrt{\Delta}}{2}\xi)\right];$$

$$u_2(x,t) = \frac{b\omega\Delta}{2a}\left[1 - 3\coth^2(\frac{\sqrt{\Delta}}{2}\xi)\right];$$

$$u_3(x,t) = \frac{b\omega\Delta}{2a}\left(1 - 3[\tanh(\sqrt{\Delta}\xi) \pm i\ \mathrm{sec}h(\sqrt{\Delta}\xi)]^2\right);$$

$$u_4(x,t) = \frac{b\omega\Delta}{2a}\left(1 - 3[\coth(\sqrt{\Delta}\xi) \pm \mathrm{csc}h(\sqrt{\Delta}\xi)]^2\right);$$

$$u_5(x,t) = \frac{b\omega\Delta}{2a}\left(1 - \frac{3}{4}\left[\tanh(\frac{\sqrt{\Delta}}{4}\xi) \pm \mathrm{csch}(\frac{\sqrt{\Delta}}{4}\xi)\right]^2\right);$$

$$u_6(x,t) = \frac{b\omega\Delta}{2a}\left(1 - 3\left(\frac{\sqrt{A_1^2 + B_1^2} - A_1\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi) + B_1}\right)^2\right);$$

$$u_7(x,t) = \frac{b\omega\Delta}{2a}\left(1 - 3\left(\frac{\sqrt{B_1^2 - A_1^2} + A_1\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi) + B_1}\right)^2\right);$$

where A_1 and B_1 are two non-zero real constant and satisfies $B_1^2 - A_1^2 > 0$.

$$u_8(x,t) = -\frac{b\omega}{a}\left\{B^2 + 2A(1-C) + \frac{12AB(C-1)\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi) - B\cosh(\frac{\sqrt{\Delta}}{2}\xi)]} + \frac{24A^2(1-C)^2\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi) - B\cosh(\frac{\sqrt{\Delta}}{2}\xi)]^2}\right\}$$

$$u_9(x,t) = -\frac{b\omega}{a}\left\{B^2 - 2A(1-C) - \frac{12AB(C-1)\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[B\sinh(\frac{\sqrt{\Delta}}{2}\xi) - \sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)]} + \frac{24A^2(1-C)^2\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[B\sinh(\frac{\sqrt{\Delta}}{2}\xi) - \sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)]^2}\right\}$$

$$u_{10}(x,t) = -\frac{b\omega}{a}\left\{B^2 - 2A(1-C) + \frac{12AB(C-1)\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - B\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}]} + \frac{24A^2(C-1)^2\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - B\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}]^2}\right\}$$

$$u_{11}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)+\frac{12AB(C-1)\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)-B\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}]}+\frac{4A^2(1-C)^2\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)-B\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}]^2}\right\}$$

$$u_{12}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)+\frac{12AB(C-1)\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[-B\sinh(\frac{\sqrt{\Delta}}{2}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{4}\xi)-\sqrt{\Delta}]}+\frac{24A^2(1-C)^2\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[-B\sinh(\frac{\sqrt{\Delta}}{2}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{4}\xi)-\sqrt{\Delta}]^2}\right\}$$

By using case 1 and according to the values of solutions of type 2 when $\Delta=B^2+4A(1-C)<0$, we obtain the following solutions :

$$u_{13}(x,t)=\frac{b\omega\Delta}{2a}\left[1+3\tan^2(\frac{\sqrt{-\Delta}}{2}\xi)\right];$$

$$u_{14}(x,t)=\frac{b\omega\Delta}{2a}\left[1+3\cot^2(\frac{\sqrt{-\Delta}}{2}\xi)\right];$$

$$u_{15}(x,t)=\frac{b\omega\Delta}{2a}\left[1+3[\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi)]^2\right];$$

$$u_{16}(x,t)=\frac{b\omega\Delta}{2a}\left[1+3[\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi)]^2\right];$$

$$u_{17}(x,t)=\frac{b\omega\Delta}{2a}\left[1+\frac{3}{4}\left(\tan(\frac{\sqrt{-\Delta}}{4}\xi)-\cot(\frac{\sqrt{-\Delta}}{4}\xi)\right)^2\right];$$

$$u_{18}(x,t)=\frac{b\omega\Delta}{2a}\left(1+3\left(\frac{\pm\sqrt{A_1^2-B_1^2}-A_1\cos(\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}\right)^2\right);$$

$$u_{19}(x,t)=\frac{b\omega\Delta}{2a}\left(1+3\left(\frac{\pm\sqrt{A_1^2-B_1^2}-A_1\sin(\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}\right)^2\right);$$

where A_1 and B_1 are two non-zero real constant and satisfies $A_1^2-B_1^2>0$.

$$u_{20}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)-\frac{12AB\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)]}+\frac{24A^2(1-C)^2\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)]^2}\right\}$$

$$u_{21}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)+\frac{12AB\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)]}+\frac{24A^2(1-C)^2\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)]^2}\right\}$$

$$u_{22}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)-\frac{12AB\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]}+\frac{24A^2(1-C)^2\cos^2(\sqrt{-\Delta}\xi)}{[\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]^2}\right\}$$

$$u_{23}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)-\frac{12AB\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)-B\sin(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]}+\frac{24A^2(1-C)^2\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)-B\sin(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]^2}\right\}$$

$$u_{24}(x,t)=-\frac{b\omega}{a}\left\{B^2-2A(1-C)-\frac{12AB\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)-\sqrt{\Delta}]}+\frac{24A^2(1-C)^2\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)-\sqrt{-\Delta}]^2}\right\}$$

where $\xi=x-\frac{1}{(1+4Ab(1-C)+bB^2)}t$.

By using case 1 and according to the values of solutions of type 3 when $A=0,\ -B(1-C)\neq 0$, we obtain the following solutions :

$$u_{25}(x,t)=-\frac{b}{a(1+bB^2)}\left\{B^2+\frac{6B^2d}{(1-C)[d+\cosh(B\xi)-\sinh(B\xi)]}+\frac{6B^2d^2}{[d+\cosh(B\xi)-\sinh(B\xi)]^2}\right\}$$

$$u_{26}(x,t)=-\frac{b}{a(1+bB^2)}\left\{B^2+\frac{B^2[\cosh(B\xi)+\sinh(B\xi)]}{(1-C)[d+\cosh(B\xi)+\sinh(B\xi)]}+\frac{B^2[\cosh(B\xi)+\sinh(B\xi)]^2}{[d+\cosh(B\xi)+\sinh(B\xi)]^2}\right\}$$

where d is an arbitrary constant and $\xi=x-\frac{1}{(1+bB^2)}t$.

By using case 1 and according to the values of solutions of type 4 when $A=B=0,\ (1-C)\neq 0$, we obtain the following solutions :

$$u_{27}(x,t)=\frac{-b}{a}\left\{\frac{6(1-C)^2}{[-(1-C)\xi+c_1]^2}\right\}$$

where c_1 is an arbitrary constant and $\xi = x - t$.

2.1.2. Exact solutions of the nonlinear RLW equation (2.1) for case 2.

By using case 2 and according to the values of solutions of type 1 when $\Delta = B^2 + 4A(1-C) > 0$, we obtain the following solutions:

$$u_1(x,t)=\frac{3b\omega\Delta}{2a}\operatorname{sech}^2(\frac{\sqrt{\Delta}}{2}\xi);$$

$$u_2(x,t)=\frac{-3b\omega\Delta}{2a}\operatorname{csc}h^2(\frac{\sqrt{\Delta}}{2}\xi);$$

$$u_3(x,t)=\frac{3b\omega\Delta}{2a}[1-(\tanh(\sqrt{\Delta}\xi)\pm i\ \operatorname{sec}h(\sqrt{\Delta}\xi))^2];$$

$$u_4(x,t)=\frac{3b\omega\Delta}{2a}[1+(\coth(\sqrt{\Delta}\xi)\pm \operatorname{csc}h(\sqrt{\Delta}\xi))^2];$$

$$u_5(x,t)=\frac{3b\omega\Delta}{8a}[\operatorname{sech}^2(\frac{\sqrt{\Delta}}{4}\xi)-\operatorname{csch}^2(\frac{\sqrt{\Delta}}{4}\xi)];$$

$$u_6(x,t)=\frac{3b\omega\Delta}{2a}\left\{1-\left(\frac{\sqrt{A_1^2+B_1^2}-A_1\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi)+B_1}\right)^2\right\};$$

$$u_7(x,t)=\frac{3b\omega\Delta}{2a}\left\{1-\left(\frac{\sqrt{B_1^2-A_1^2}+A_1\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi)+B_1}\right)^2\right\};$$

where A_1 and B_1 are two non-zero real constant and satisfies $B_1^2 - A_1^2 > 0$.

$$u_8(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-B\cosh(\frac{\sqrt{\Delta}}{2}\xi)]}-\frac{4A^2(1-C)\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-B\cosh(\frac{\sqrt{\Delta}}{2}\xi)]^2}\right\}$$

$$u_9(x,t)=\frac{6b\omega(1-C)}{a}\left\{A-\frac{2AB\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)]}-\frac{4A^2(1-C)\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)]^2}\right\}$$

$$u_{10}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-B\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta}]}-\frac{4A^2(1-C)\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-B\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta}]^2}\right\}$$

$$u_{11}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[-B\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}]}-\frac{4A^2(1-C)\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[-B\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)\pm\sqrt{\Delta}]^2}\right\}$$

$$u_{12}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[-B\sinh(\frac{\sqrt{\Delta}}{2}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{4}\xi)-\sqrt{\Delta}]}-\frac{4A^2(1-C)\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[-B\sinh(\frac{\sqrt{\Delta}}{2}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{4}\xi)-\sqrt{\Delta}]^2}\right\}$$

By using case 2 and according to the values of solutions of type 2 when $\Delta = B^2 + 4A(1-C) < 0$, we obtain the following solutions :

$$u_{13}(x,t)=\frac{3b\omega\Delta}{2a}\sec^2(\frac{\sqrt{-\Delta}}{2}\xi);$$

$$u_{14}(x,t)=\frac{3b\omega\Delta}{2a}\csc^2(\frac{\sqrt{-\Delta}}{2}\xi);$$

$$u_{15}(x,t)=\frac{3b\omega\Delta}{2a}[1+(\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi))^2];$$

$$u_{16}(x,t)=\frac{3b\omega\Delta}{2a}[1+(\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi))^2];$$

$$u_{17}(x,t)=\frac{3b\omega\Delta}{8a}[\sec^2(\frac{\sqrt{-\Delta}}{4}\xi)+\csc^2(\frac{\sqrt{-\Delta}}{4}\xi)];$$

$$u_{18}(x,t)=\frac{3b\omega\Delta}{2a}\left\{1+\left(\frac{\pm\sqrt{A_1^2+B_1^2}-A_1\cos(\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}\right)^2\right\};$$

$$u_{19}(x,t)=\frac{3b\omega\Delta}{2a}\left\{1+\left[\frac{\pm\sqrt{B_1^2-A_1^2}+A_1\cos(\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}\right]^2\right\};$$

where A_1 and B_1 are two non-zero real constant and satisfies $A_1^2 - B_1^2 > 0$.

$$u_{20}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A-\frac{2AB\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)]}-\frac{4A^2(1-C)\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)]^2}\right\}$$

$$u_{21}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)]}-\frac{4A^2(1-C)\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)]^2}\right\}$$

$$u_{22}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A-\frac{2AB\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]}-\frac{4A^2(1-C)\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]^2}\right\}$$

$$u_{23}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]}-\frac{4A^2(1-C)\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[-B\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]^2}\right\}$$

$$u_{24}(x,t)=\frac{6b\omega(1-C)}{a}\left\{A+\frac{2AB\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{\left(-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{4}\xi)-\sqrt{-\Delta}\right)}-\frac{4A^2(1-C)\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{\left(-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{4}\xi)-\sqrt{-\Delta}\right)^2}\right\}$$

$$\xi=x-\frac{t}{[1-4Ab(1-C)-bB^2]}.$$

By using case 2 and according to the values of solutions of type 3 when $A=0,\ -B(1-C)\neq 0$, we obtain the following solutions:

$$u_{25}(x,t)=\frac{6b\omega}{a}\left\{\frac{B^2d}{[d+\cosh(B\xi)-\sinh(B\xi)]}-\frac{B^2d^2}{[d+\cosh(B\xi)-\sinh(B\xi)]^2}\right\}$$

$$u_{26}(x,t)=\frac{6b\omega}{a}\left\{\frac{B^2[\cosh(B\xi)+\sinh(B\xi)]}{[d+\cosh(B\xi)+\sinh(B\xi)]}-\frac{B^2[\cosh(B\xi)+\sinh(B\xi)]^2}{[d+\cosh(B\xi)+\sinh(B\xi)]^2}\right\}$$

where d is an arbitrary constant and $\xi=x-\frac{t}{(1-bB^2)}$.

By using case 2 and according to the values of solutions of type 4 when $A=B=0,-(1-C)\neq 0$, we obtain the following solutions

$$u_{27}(x,t)=-\frac{6b\omega(1-C)^2}{a[-(1-C)\xi+c_1]^2}$$

where c_1 is an arbitrary constant and $\xi=x-t$.

2.2. Example 2. The nonlinear SRLW equation

This equation is well- known [19] and has the form

$$u_{tt}+u_{xx}+uu_{xt}+u_xu_t+u_{xxtt}=0, \tag{2.5}$$

where a and b are positive constants. The wave transformation (1.2) reduces Eq. (2.5) into the ODE:

$$(\omega^2+1)u''-\omega(uu')'+\omega^2u''''=0. \tag{2.6}$$

Integrating (2.6) once with respect to ξ with zero constant of integration, we get

$$(\omega^2+1)u'-\omega uu'+\omega^2u'''=0. \tag{2.7}$$

Balancing u''' and uu' we have $\ell=2$. So Eq. (2.7) has the formal solution

$$u(\xi)=\alpha_0+\alpha_1\phi(\xi)+\alpha_2\phi^2(\xi), \tag{2.8}$$

where $\alpha_0,\alpha_1,\alpha_2$ are constants to be determined, such that $\alpha_2\neq 0$.

Substituting (2.8) along with Eq. (1.23) into Eq. (2.7), collecting all the terms with the same order $\phi^k\ (k=0,1,2,\ldots)$ and setting all the coefficients to zero, we have the following algebraic equations:

$$Q^5:-2\alpha_2^2\omega(C-1)+24\alpha_2(C-1)^3=0$$

$$Q^4:-\omega[3\alpha_1\alpha_2(C-1)+2\alpha_2^2B]+\omega^2[54\alpha_2B(C-1)^2+6\alpha_1(C-1)^3]=0$$

$$Q^3: 2\alpha_2(1+\omega^2)(C-1)-\omega[2\alpha_0\alpha_2(C-1)+\alpha_1^2B(C-1)+3\alpha_1\alpha_2B+2\alpha_2^2A]+\omega^2[40\alpha_2A(C-1)^2+38\alpha_2B^2(C-1)+12\alpha_1B(C-1)^2]=0$$

$$Q^2: (1+\omega^2)[\alpha_1(C-1)+2\alpha_2B]-\omega[\alpha_0\alpha_1(C-1)+2\alpha_0\alpha_2B+\alpha_1^2B+3\alpha_1\alpha_2A]+\omega^2[52\alpha_2AB(C-1)^2+8\alpha_1A(C-1)^2+7\alpha_1B^2(C-1)+8\alpha_2B^3]=0$$

$$Q: (1+\omega^2)[\alpha_1B+2\alpha_2A]-\omega[\alpha_0\alpha_1B+2\alpha_0\alpha_2A+\alpha_1^2A]+\omega^2[16\alpha_2A^2(C-1)+8\alpha_1AB(C-1)+14\alpha_2AB^2+\alpha_1B^3]=0$$

$$Q^0: (1+\omega^2)\alpha_1A-\omega\alpha_0\alpha_1A+\omega^2[2\alpha_1A^2(C-1)+6\alpha_2A^2B+\alpha_1AB^2]=0$$

By solving these algebraic equations with the aid of Maple or Mathematic we have the result:

$$\alpha_0=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega},\ \alpha_1=12B\omega(C-1),\ \alpha_2=12\omega(C-1)^2,\ \omega=\omega$$

(2.9)

2.2.1. Exact solutions of the nonlinear SRLW (2.5)

By using (2.9) and according to the values of solutions of type 1 when $\Delta=B^2+4A(1-C)>0$, we obtain the following solutions:

$$u_1(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta[2-3\tanh^2(\frac{\sqrt{\Delta}}{2}\xi)];$$

$$u_2(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta[2-3\coth^2(\frac{\sqrt{\Delta}}{2}\xi)];$$

$$u_3(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left(2-3[\tanh(\sqrt{\Delta}\xi)\pm i\,\mathrm{sec}h(\sqrt{\Delta}\xi)]^2\right);$$

$$u_4(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left(2-3[\coth(\sqrt{\Delta}\xi)\pm \mathrm{csc}h(\sqrt{\Delta}\xi)]^2\right);$$

$$u_5(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left(2-\frac{3}{4}[\tanh(\frac{\sqrt{\Delta}}{4}\xi)\pm\coth(\frac{\sqrt{\Delta}}{4}\xi)]^2\right);$$

$$u_6(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left\{2-3\left(\frac{\sqrt{A_1^2+B_1^2}-A_1\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi)+B_1}\right)^2\right\};$$

$$u_7(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left\{2-3\left(\frac{\sqrt{B_1^2-A_1^2}-A_1\cosh(\sqrt{\Delta}\xi)}{A_1\sinh(\sqrt{\Delta}\xi)+B_1}\right)^2\right\};$$

where A_1 and B_1 are two non-zero real constant and satisfies $B_1^2-A_1^2>0$.

$$u_8(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}+\frac{24AB\omega(C-1)\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-B\cosh(\frac{\sqrt{\Delta}}{2}\xi)]}+\frac{48A^2\omega(C-1)^2\cosh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-B\cosh(\frac{\sqrt{\Delta}}{2}\xi)]^2}$$

$$u_9(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}-\frac{24AB\omega(C-1)\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)]}+\frac{48A^2\omega(C-1)^2\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)]^2}$$

$$u_{10}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}-\frac{24AB\omega(C-1)\cosh(\sqrt{\Delta}\xi)}{[\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-B\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta}]}+\frac{48A^2\omega(C-1)^2\cosh^2(\sqrt{\Delta}\xi)}{[\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-B\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta}]^2}$$

$$u_{11}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}+\frac{24AB\omega(C-1)\sinh(\sqrt{\Delta}\xi)}{[\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)-B\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}]}+\frac{48A^2\omega(C-1)^2\sinh^2(\sqrt{\Delta}\xi)}{[\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)-B\sinh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta}]^2}$$

$$u_{12}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}+\frac{24AB\omega(C-1)\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)-B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}]}+\frac{48A^2\omega(C-1)^2\sinh^2(\frac{\sqrt{\Delta}}{2}\xi)}{[\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)-B\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}]^2}$$

By using (2.9) and according to the values of solutions of type 2 when $\Delta=B^2+4A(1-C)<0$, we obtain the following solutions:

$$u_{13}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta[2+3\tan^2(\frac{\sqrt{-\Delta}}{2}\xi)];$$

$$u_{14}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta[2+3\cot^2(\frac{\sqrt{-\Delta}}{2}\xi)];$$

$$u_{15}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left(2+3[\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi)]^2\right);$$

$$u_{16}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left(2+3[\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi)]^2\right);$$

$$u_{17}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left(2+\frac{3}{4}[\tan(\frac{\sqrt{-\Delta}}{4}\xi)-\cot(\frac{\sqrt{-\Delta}}{4}\xi)]^2\right);$$

$$u_{18}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left\{2+3\left(\frac{\pm\sqrt{A_1^2-B_1^2}-A_1\cos\sqrt{-\Delta}\xi)}{A_1\sin(\sqrt{-\Delta}\xi)+B_1}\right)^2\right\};$$

$$u_{19}(x,t)=\frac{\omega^2+1}{\omega}-\omega\Delta\left\{2+3\left(\frac{\pm\sqrt{A_1^2-B_1^2}-A_1\sin(\sqrt{-\Delta}\xi)}{A_1\sinh(\sqrt{-\Delta}\xi)+B_1}\right)^2\right\};$$

where A_1 and B_1 are two non-zero real constant and satisfies $A_1^2-B_1^2>0$.

$$u_{20}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}+\frac{24AB\omega(C-1)\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)]}+\frac{48A^2\omega(C-1)^2\cos^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+B\cos(\frac{\sqrt{-\Delta}}{2}\xi)]^2}$$

$$u_{21}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}-\frac{24AB\omega(C-1)\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)]}+\frac{48A^2\omega(C-1)^2\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)]^2}$$

$$u_{22}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}-\frac{24AB\omega(C-1)\cos(\sqrt{-\Delta}\xi)}{[\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]}+\frac{48A^2\omega(C-1)^2\cos^2(\sqrt{-\Delta}\xi)}{[\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)+B\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]^2}$$

$$u_{23}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+]}{\omega}+\frac{24AB\omega(C-1)\sin(\sqrt{-\Delta}\xi)}{[\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)-B\sin(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]}+\frac{48A^2\omega(C-1)^2\sin^2(\sqrt{-\Delta}\xi)}{[\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)-B\sin(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}]^2}$$

$$u_{24}(x,t)=\frac{1+\omega^2[8A(C-1)+B^2+1]}{\omega}+\frac{24AB\omega(C-1)\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{[2\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)-\sqrt{-\Delta}]}+\frac{48A^2\omega(C-1)^2\sin^2(\frac{\sqrt{-\Delta}}{2}\xi)}{[2\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)-B\sin(\frac{\sqrt{-\Delta}}{2}\xi)-\sqrt{-\Delta}]^2}$$

By using (2.9) and according to the values of solutions of type 3 when $A=0,\ B(C-1)\neq 0$, we obtain the following solutions :

$$u_{25}(x,t)=\frac{1+(1+B^2)\omega^2}{\omega}-\frac{12\omega B^2 d}{[d+\cosh(B\xi)-\sinh(B\xi)]}+\frac{12\omega B^2 d^2}{[d+\cosh(B\xi)-\sinh(B\xi)]^2}$$

$$u_{26}(x,t)=\frac{1+(1+B^2)\omega^2}{\omega}-\frac{12B^2\omega[\cosh(B\xi)+\sinh(B\xi)]}{[d+\cosh(B\xi)+\sinh(B\xi)]}+\frac{12B^2\omega[\cosh(B\xi)+\sinh(B\xi)]^2}{[d+\cosh(B\xi)+\sinh(B\xi)]^2},$$

where d is an arbitrary constant .

By using (2.9) and according to the values of solutions of type 4 when $A=B=0,\ (C-1)\neq 0$, we obtain the following solutions

$$u_{27}(x,t)=\frac{1+\omega^2}{\omega}-\frac{12\omega(C-1)^2}{[(C-1)\xi+c_1)]^2},$$

where c_1 is an arbitrary constant .

3. CONCLUSIONS AND DISCUSSIONS

Liu et al [19] have used the improved (G'/G) – expansion method (1.4) and (1.5) to find the exact solutions of the two nonlinear evolution equations (2.1) and (2.5) with the aid of formulas (1.6)-(1.9). We have shown that there is a minor error in the formulas (1.6) and (1.7) which have been corrected. This leads to some errors in the solutions of these equations which have been corrected too.

In the present article we have shown that the improved (G'/G) – expansion method (1.4) and (1.5) can be reduced to the well- known generalized Riccati equation mapping method (1.22) and (1.23). We have noted that the second method gives much more solutions of the two nonlinear equations (2.1) and (2.5) than the first one, where most of them are new and the others are well-known. Furthermore, we have shown that some solutions obtained using the second method are equivalent to some solutions obtained using the first one as follows:

(i)If we choose $\varepsilon=1$ or $\varepsilon=3$ in (1.14) and (1.15) obtained in Ref. [19] we deduce that the resultant solutions are equivalent to our results $u_1(x,t)$ and $u_2(x,t)$ obtained in case 1 or case 2, respectively.

(ii)If we choose $\varepsilon=1$ or $\varepsilon=3$ and $c_1=0, c_2\neq 0$ in (1.11) obtained in Ref. [19] we deduce that the resultant solutions are equivalent to our result $u_{13}(x,t)$ obtained in case 1, or our result $u_{13}(x,t)$ obtained in case 2, respectively.

(iii)If we choose $\varepsilon=1$ or $\varepsilon=3$ and $c_1\neq 0$ $c_2=0,$ in (1.11) obtained in Ref. [19] we deduce that the resultant solutions are equivalent to our result $u_{14}(x,t)$ obtained in case 1, or our result $u_{14}(x,t)$ obtained in case 2, respectively .

(iv)If we choose $c_1=-c_2$ and $c_1=c_2$ in (1.16) obtained in Ref. [19] we deduce that the resultant solutions are equivalent to our results $u_1(x,t)$ and $u_2(x,t)$ of Eq. (2.5) respectively.

(v)If we choose $c_1=0,\ c_2\neq 0$ and $c_2=0, c_1\neq 0$ in (1.17) obtained in Ref. [19] we deduce that the resultant solutions are equivalent to our results $u_{13}(x,t)$ and $u_{14}(x,t)$ of Eq. (2.5) respectively.

Finally, with the aid of the Maple, we have assured the correctness of the obtained solutions in this article by putting them back into the original equations.

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