Weakly (πp, μ_y) – Continuous Functions on Generalized Topological Spaces

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ABSTRACT
This paper introduces and study a class of function namely weakly (πp, μ_y) – continuous functions. Some characterizations and properties concerning weakly (πp, μ_y) – continuous functions are obtained.

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1. INTRODUCTION AND PRELIMINARIES
In 2002, Á.Császár [5] introduced the notions of generalized topology and many authors [9, 10] have studied various types of continuity using weak forms of open sets in generalized topological spaces.

In this paper the weakly (πp, μ_y) – continuous function is introduced and studied. Moreover basic properties and preservation theorems of weakly (πp,μ_y)- continuous functions are investigated and the relationships between weakly (πp, μ_y) – continuous function and graphs are also investigated.

We recall some basic concepts and results.

Let X be a nonempty set and let exp(X) be the power set of X. μ⊂exp(X) is called a generalized topology [5](briefly, GT) on X, if ∅ ∈μ and unions of elements of μ belong to μ. The pair (X, μ) is called a generalized topological space (briefly, GTS). The elements of μ are called μ-open subsets of X and the complements are called μ-closed sets. If (X, μ) is a GTS and A⊆X, then the interior of (denoted by $i_\mu(A)$) is the union of all G⊆A, G∈μ and the closure of A (denoted by $c_\mu(A)$) is the intersection of all μ-closed sets containing A. Note that $c_\mu(A) = X - i_\mu(X-A)$ and $i_\mu(A) = X - c_\mu(X-A)$ [5].

Definition 1.1[6] Let (X, μ_x) be a generalized topological space and A⊆ X. Then A is said to be

(i) μ- semi open if $A \subseteq c_\mu(i_\mu(A))$.

(ii) μ- pre open if $A \subseteq i_\mu(c_\mu(A))$.

(iii) μ-α-open if $A \subseteq i_\mu(c_\mu(i_\mu(A)))$.

(iv) μ-β-open if $A \subseteq c_\mu(i_\mu(c_\mu(A)))$.

(v) μ-r-open[11] if $A = i_\mu(c_\mu(A))$

(vi) μ-rα-open[2] if there is a μ-r-open set U such that $U \subset A \subset c_\alpha(U)$.

Definition 1.2 [2] Let (X, μ_x) be a generalized topological space and A⊆ X. Then A is said to be μ-πrα closed set if $c_\pi(A) \subseteq U$ whenever A⊆U and U is μ-rα-open set. The complement of μ-πrα closed set is said to be μ-πrα open set.

The complement of μ-semi open (μ-pre open, μ-α-open, μ-β-open, μ-r-open, μ-rα-open) set is called μ- semi closed (μ- pre closed, μ-α- closed, μ-β- closed, μ-r- closed, μ-rα-closed) set.

Let us denote the class of all μ-semi open sets, μ-pre open sets, μ-α-open sets, μ-β-open sets, and μ-πrα open sets on X by σ(μ_x) (σ for short), π(μ_x) (π for short), α(μ_x) (α for short), β(μ_x) (β for short) and πp(μ_x) (πp for short) respectively. Let μ be a generalized topology on a non empty set X and S⊆X.

The μ-α-closure (resp. μ-semi closure, μ-pre closure, μ-β-closure, μ-πrα-closure) of a subset S of X denoted by $c_\alpha(S)$ (resp. $c_\sigma(S)$, $c_\pi(S)$, $c_\beta(S)$, $c_{\pi p}(S)$) is the intersection of μ-α-closed(resp. μ- semi closed, μ- pre closed, μ-β-closed, μ-πrα closed) sets including S.

The μ-α-interior (resp. μ-semi interior, μ-pre interior, μ-β-interior, μ-πrα-interior) of a subset S of X denoted by $i_\alpha(S)$ (resp. $i_\sigma(S)$, $i_\pi(S)$, $i_\beta(S)$, $i_{\pi p}(S)$) is the union of μ-α-open (resp. μ- semi open, μ- pre open, μ-β-open, μ-πrα open) sets contained in S.

Definition 1.3[2] A function f between the generalized topological spaces (X, μ_x) and (Y, μ_y) is called (μ_x, μ_y)-πrα continuous function if $f^{-1}(A) \in \pi p(\mu_x)$ for each A∈(Y, μ_y).

2. WEAKLY (ΠP, μ_Y) - CONTINUOUS FUNCTIONS
Definition 2.1 Let (X, μ_x) and (Y, μ_y) be GTS's. Then a function f: (X, μ_x)→(Y, μ_y) is said to be weakly (πp, μ_y)-continuous function if for each x ∈X and each μ_y-open set V of Y containing f(x), there exists U ∈πp(μ_x) such that $f(U) \subseteq c\mu_y(V)$.

Remark 2.2 Every (μ_x, μ_y)-πrα continuous function is weakly (πp, μ_y)- continuous function.

Example 2.3 Let X = {a, b, c, d}. Consider GTS's μ_x= {∅, {a}, {a, b}, {b, c}, {a, b, c}} and μ_y = {∅, {a, c}, {b, c}, {a, b, c}}. Define f: (X, μ_x)→(Y, μ_y) as follows f(a) = b, f(b) = f(c) = d and f(d) = c.

Then since $c\mu_y(\{a, c\}) = c\mu_y(\{b, c\}) = c\mu_y(\{a, b, c\}) = X$. It is obvious that f is weakly (πp, μ_y)- continuous. But f is not (μ_x, μ_y)-πrα continuous.

Theorem 2.4 Let (X, μ_x) and (Y, μ_y) be GTS's. Then for a function f: (X, μ_x)→(Y, μ_y) following statement are equivalent.

(i) f is weakly (πp, μ_y)- continuous function.

(ii) $f^{-1}(V) \subseteq i_{\pi p}(f^{-1}(c\mu_y(V)))$ for every μ_y-open subset V of Y.

(iii) $c_{\pi p}(f^{-1}(i\mu_y(F)))\subseteq f^{-1}(F)$ for every μ_y-closed set F of Y.

(iv) $c_{\pi p}(f^{-1}(i\mu_y(c\mu_y(B))))\subseteq f^{-1}(c\mu_y(B))$ for every set B of Y.

(v) $c_{\pi p}(f^{-1}(V))\subseteq f^{-1}(c\mu_y(V))$ for every μ_y-open subset V of Y.

Proof: (i) ⇒(ii). Let V be a μ_y-open subset of Y and $x\in f^{-1}(V)$. Then $f(x) \in V$. There exists a $U \in \pi p(\mu_x)$ such that $f(U)\subseteq c\mu_y(V)$.Thus obtain $x\in U\subseteq f^{-1}(c\mu_y(V))$.

This implies that $x\in i_{\pi p}(f^{-1}(c\mu_y(V))$ and consequently $f^{-1}(V)\subseteq i_{\pi p}(f^{-1}(c\mu_y(V))$.

(ii)⇒(iii). Let F be any μ_y-closed set of Y and $x\notin f^{-1}(F)$.Since Y\F is μ_y- open in Y and (ii),

$$x\in X\setminus f^{-1}(F) \subseteq f^{-1}(Y\setminus F)\subseteq i_{\pi p}(f^{-1}(c\mu_y(Y\setminus F)))$$

$$= i_{\pi p}(f^{-1}(Y\setminus i\mu_y(F)))$$

$$= i_{\pi p}(X\setminus f^{-1}(i\mu_y(F)))= X\setminus c_{\pi p}(f^{-1}(i\mu_y(F))).$$

Then $x\notin c_{\pi p}(f^{-1}(i\mu_y(F)))$.

Hence $c_{\pi p}(f^{-1}(i\mu_y(F)))\subseteq f^{-1}(F)$.

(iii)⇒(iv). Let B be any subset of Y. Then $c\mu_y(B)$ is closed in Y and by (iii), implies

$$c_{\pi p}(f^{-1}(i\mu_y(c\mu_y(B))))\subseteq f^{-1}(c\mu_y(B)).$$

(iv)⇒(v). Let V be any μ_y-open subset of Y.

By (iv) $c_{\pi p}(f^{-1}(V) = c_{\pi p}(f^{-1}(i\mu_y(V)))$

$$\subseteq c_{\pi p}(f^{-1}(i\mu_y(c\mu_y(V))))$$

$$\subseteq f^{-1}(c\mu_y(V)).$$

(v)⇒(i). Let $x\in X$ and V be any μ_y-open subset of Y containing f(x).

Then by (v), $x\in f^{-1}(V) \subseteq f^{-1}(i\mu_y(c\mu_y(V)))$

$$\subseteq f^{-1}(i_{\pi p}(c\mu_y(V)))$$

$$= X\setminus f^{-1}(c_{\pi p}(Y\setminus c\mu_y(V)))$$

$$\subseteq X \setminus c_{\pi p}(f^{-1}(Y\setminus c\mu_y(V)))$$

$$= i_{\pi p}(f^{-1}(c\mu_y(V))).$$

Therefore, there exists $U\in\pi p(\mu_x)$ such that $U \subseteq f^{-1}(c\mu_y(V))$.This shows that f is weakly(πp , μ_y) – continuous.

Theorem 2.5 Let (X, μ_x) and (Y, μ_y) be GTS's. Then for a function f: (X, μ_x)→(Y, μ_y) following statement are equivalent.

(i) f is weakly (πp, μ_y)- continuous function.

(ii) $c_{\pi p}(f^{-1}(i\mu_y\ (c\mu_y(V))))\subseteq f^{-1}((c\mu_y(V))$ for every μ_y-$\pi r\alpha$ closed set V.

(iii) $c_{\pi p}(f^{-1}(V)) \subseteq f^{-1}(c\mu_y(V))$ for every μ_y-pre open subset V of Y.

(iv) $f^{-1}(V)\subseteq i_{\pi p}(f^{-1}(c\mu_y(V)))$ for every μ_y-pre open subset set V of Y.

Proof: (i)⇒(ii). It follows from Theorem 2.4.

(ii)⇒(iii). Let V be μ_y- pre open set. Since every μ_y- pre open set is μ_y-$\pi r\alpha$ open set and by (ii)

$$c_{\pi p}(f^{-1}(V)) \subseteq c_{\pi p}(f^{-1}(i\mu_y\ (c\mu_y(V))))\subseteq f^{-1}(c\mu_y(V)).$$

(iii)⇒(iv).Let V be μ_y- pre open set.

Then by (iii) $f^{-1}(V)\subseteq f^{-1}(i\mu_y\ (c\mu_y(V)))$

$$\subseteq f^{-1}(Y\setminus (c\mu_y\ (Y\setminus c\mu_y(V))))$$

$$= X\setminus f^{-1}(c\mu_y\ (Y\setminus c\mu_y(V)))$$

$$\subseteq X \setminus c_{\pi p}(f^{-1}(Y\setminus c\mu_y(V)))$$

$$= X \setminus c_{\pi p}(X\setminus f^{-1}(c\mu_y(V)))$$

$$= i_{\pi p}(f^{-1}(c\mu_y(V))).$$

(iv)⇒(i).It follows from Theorem 2.4 since every μ_y- open set is μ_y- pre open set.

We recall that a GTS (X, μ) is said to be

(i) μ-$\pi r\alpha$ T_1[3] if for each pair of distinct points x and y in X, there exist two disjoint μ-$\pi r\alpha$ open sets U and V in X such that $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$.

(ii) μ-$\pi r\alpha$ T_2[3] if for each pair of distinct points x and y in X, there exist two disjoint μ-$\pi r\alpha$ open sets U and V containing x and y respectively.

(iii) Hausdorff space [7] if for each distinct points x and y of X, there exist disjoint μ-open sets U and V containing x and y respectively.

(iv) μ-Urysohn space [3] if for each pair of distinct points x and y of X, there exist μ-open sets U and V containing $x\in U$, $y\in V$ and $c_\mu(U)\cap c_\mu(V) = \emptyset$.

For a map f :(X, μ_x)→(Y, μ_y), the subset $\{(x, f(x)); x\in X\}\subset X\times Y$ is called the graph of f [3] and is denoted by $G_\mu(f)$.

Theorem 2.6 Let (X, μ_x) and (Y, μ_y) be two GTS's. If a function f :(X, μ_x)→(Y, μ_y) is weakly (πp, μ_y)- continuous injection function. Then the following hold:

(i) If Y is Urysohn then X is μ-$\pi r\alpha T_2$.

(ii) If Y is Hausdorff then X is μ-πrαT_1.

Proof: (i) Let x_1 and x_2 be any distinct points in X. Then $f(x_1) \neq f(x_2)$ and there exists μ_y- open sets U_1 and U_2 of Y containing $f(x_1)$ and $f(x_2)$ respectively such that $C\mu_y(U_1) \cap C\mu_y(U_2) = \emptyset$.

Since f is weakly (πp, μ_y)- continuous, there exist V_1, V_2 $\in\pi p(\mu_x)$ such that $f(V_1) \subseteq C\mu_y(U_1)$ and $f(V_2) \subseteq C\mu_y(U_2)$. Since $f^{-1}(C\mu_y(U_1))$ and $f^{-1}(C\mu_y(U_2))$ are disjoint, $V_1 \cap V_2 = \emptyset$. Hence X is μ-πrαT_2.

(ii)Let x_1 and x_2 be any distinct points in X. Since f is injective $f(x_1) \neq f(x_2)$.Since Y is Hausdorff there exists disjoint μ_y- open sets U_1 and U_2 of Y such that $f(x_1) \in U_1$ and $f(x_2) \in U_2$. Since $U_1 \cap U_2 = \emptyset$ then $C\mu_y(U_1) \cap U_2 = \emptyset$ and $U_1 \cap C\mu_y(U_2) = \emptyset$.Then obtain $f(x_2) \notin C\mu_y(U_1)$ and $f(x_1) \notin C\mu_y(U_2)$.Since f is weakly (πp, μ_y)- continuous, there exist $V_i \in \pi p(\mu_x)$ containing x_i such that $f(V_i) \subseteq C\mu_y(U_i)$, i = 1,2. Thus $x_1 \notin U_2$ and $x_2 \notin U_1$.

Hence μ-πrαT_1.

Definition 2.7 [3]

The graph $G_\mu(f)$ of a map f :(X, μ_x)→(Y, μ_y) between GTS's is said to be contra(πp, μ_y)- closed if for each $(x, y) \in (X \times Y) \setminus G_\mu(f)$, there exist an μ-πrα open set U in X containing x and a μ-closed set V in Y containing y such that $(U \times V) \cap G_\mu(f) = \emptyset$.

Proposition 2.8 [3] The following properties are equivalent for the graph $G_\mu(f)$ of a map f in GTS's.

(i) $G_\mu(f)$ is contra (πp, μ_y)- closed.

(ii) For each $(x, y) \in (X \times Y) \setminus G_\mu(f)$, there exist an μ-πrα open set U in X containing x and a μ-closed V in Y containing y such that $f(U) \cap V = \emptyset$.

Theorem 2.9 Let (X, μ_x) and (Y, μ_y) be two GTS's. If a function f :(X, μ_x)→(Y, μ_y) is weakly (πp, μ_y)- continuous function and (Y, μ_y) is a Hausdorff space, then the graph $G_\mu(f)$ is a contra (πp, μ_y)-closed set of X×Y.

Proof: Let $(x, y) \in (X \times Y) \setminus G_\mu(f)$.Then, we have $y \neq f(x)$. Since (Y, μ_y) is Hausdorff, there exist disjoint μ_y- open sets W and V such that $f(x) \in W$ and $y \in V$. Since f is weakly (πp, μ_y)- continuous function, there exist a μ_x-πrα open set U containing x such that $f(U) \subseteq C\mu_y(W)$. Since W and V are disjoint subsets of Y, then $V \cap C\mu_y(W) = \emptyset$. This shows that $(U \times V) \cap G_\mu(f) = \emptyset$ and $G_\mu(f)$ is contra (πp, μ_y)- closed set.

Definition 2.10 A generalized topological space (X, μ_x) is called μ_x- connected [1] if X is not the union of two disjoint non empty μ-open subsets of X.

Definition 2.11A generalized topological space (X, μ_x) is called πp- connected [2] if (X, πp(μ)) is connected.

Theorem 2.12 Let (X, μ_x) and (Y, μ_y) be two GTS's. If a function f : (X, μ_x)→(Y, μ_y) is a weakly (πp, μ_y)- continuous surjective function and (X, μ_x) is a πp-connected space, then Y is a connected space.

Proof: Assume that (Y, μ_y) is not connected. Then there exist non empty μ-open sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$.

Hence $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non empty subsets of X.

By Theorem 2.4 $f^{-1}(V_i) \subseteq i_{\pi p}(f^{-1}(C\mu_y(V_i)))$, i = 1, 2. Since V_i is μ- open and μ- closed and every μ-closed set is μ-πrα closed, $f^{-1}(V_i) \subseteq i_{\pi p}(f^{-1}(V_i))$ and hence $f^{-1}(V_i)$ is μ-πrα open for i = 1, 2. Therefore (X, μ_x) is not πp-connected. This is contradiction and hence (Y, μ_y) is connected.

Definition 2.13 A GTS (X, μ_x) is called

(i) πp-compact [4] if each cover of X composed of elements of μ-πrα open sets admits a finite subcover.

(ii) πp-closed space if every cover of X by μ-πrα open sets has a finite sub cover whose μ-πrα closure ($c_{\pi p}$)cover X.

Definition 2.14 A subset A of a GTS (X, μ_x) is said to be πp-closed relative to X if for every cover $\{V_\alpha: \alpha \in \Lambda\}$ of A by μ-πrα open sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{c_{\pi p}(V_\alpha) / \alpha \in \Lambda_0\}$.

Theorem 2.15 Let (X, μ_x) and (Y, μ_y) be two GTS's. If a function f : (X, μ_x)→(Y, μ_y) is a weakly (πp, μ_y)- continuous function and A is a πp-compact subset of (X, μ_x), then f(A) is πp- closed relative to (Y, μ_y).

Proof: Let $\{V_i: i \in \Lambda\}$ be any cover of f(K) by μ- open sets of Y.For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is weakly (πp, μ_y)- continuous, there exists $U_x \in \pi p(\mu_x)$ containing x such that $f(U) \subseteq C\mu_y(V_{\alpha(x)})$ The family $\{U_x / x \in \Lambda\}$is a cover of A by μ-πrα open sets of X. Since A is πp-compact, there exists a finite number of points, say, x_1, x_2, ………x_n in A such that $A \subseteq \cup\{Ux_k / x_k \in \Lambda, 1 \leq k \leq n\}$.

Therefore, $f(A) \subseteq \cup\{f(Ux_k) / x_k \in \Lambda, 1 \leq k \leq n\}$

$$\subseteq \cup\{c_{\pi p}(V_{\alpha(x_k)} / x_k \in \Lambda, 1 \leq k \leq n\}.$$

This shows that f(A) is πp-closed relative to (Y, μ_y).

Corollary 2.16 Let (X, μ_x) and (Y, μ_y) be two GTS's. If a function f : (X, μ_x)→(Y, μ_y) is a weakly (πp, μ_y)- continuous surjective function and the space (X, μ_x) is πp-compact, then (Y, μ_y) is a πp- closed space.

Definition 2.17 [8] If X is a generalized topological space and B⊆X, the frontier of B is denoted by $Fr_X(B)$ is defined as $Fr_X(B) = c_\mu(B) \cap c_\mu(X \setminus B)$.

Definition 2.18 Let A be a subset of a GTS (X, μ_x). Then the πp- frontier of A , denoted by πp-Fr(A) is defined as πp-Fr(A) = $c_{\pi p}(A) \cap c_{\pi p}(X \setminus A)$.

Theorem 2.19 Let (X, μ_x) and (Y, μ_y) be two GTS's and the set of all points x∈X at which a function f :(X, μ_x)→(Y, μ_y) is not weakly (πp, μ_y)- continuous if and only if the union of πp-frontier of the inverse images of the closure of μ-open sets containing f(x).

Proof: Suppose that f is not weakly (πp. μ_y)- continuous at x∈X. Then there exists an μ-open set V of Y containing f(x) such that f(U) is not contained in $c_\mu(V)$ for each $U_x \in \pi p(\mu_x)$. Hence $U \cap (X \setminus f^{-1}(c_\mu(V)) \neq \emptyset$,for each $U_x \in \pi p(\mu_x)$. So $x \in c_{\pi p}(X \setminus f^{-1}(c_\mu(V)))$.

On the other hand, $x \in f^{-1}(V) \subseteq c_{\pi p}(f^{-1}(c_\mu(V)))$.

Hence $x \in \pi p\text{-}Fr(f^{-1}(c_\mu(V)))$.

Conversely, suppose that f is weakly (πp, μ_y)- continuous at $x \in X$ and let V be any μ-open set of Y containing f(x), then there exists $U_x \in \pi p(\mu_x)$ such that $U \subseteq f^{-1}(c_\mu(V))$.

Hence by Theorem 2.4, $x \in f^{-1}(V) \subseteq i_{\pi p}(f^{-1}(c_\mu(V)))$ and hence $x \in \pi p\text{-}Fr(f^{-1}(c_\mu(V)))$, for each μ-open set V of Y containing f(x).

3. CONCLUSION

The study of this concept has led to certain findings and conclusions and constitutes a fundamental tool in the study of generalized topological spaces.

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