Convexity Preserving Interpolation by GC^2-Rational Cubic Spline

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ABSTRACT

A weighted rational cubic spline interpolation has been constructed using rational spline with quadratic denominator. GC^1-piecewise rational cubic spline function involving parameters has been constructed which produces a monotonic interpolant to given monotonic data . The degree of smoothness of this spline is GC^2 in the interpolating interval when the parameters satisfy a continuous system. It is observed that under certain conditions the interpolant preserve the convexity property of the data set. We have discussed the constrains for GC^2-rational spline interpolant in section. Also the error estimate formula of this interpolation are obtained.

Keywords:

Interpolation, shape parameters, monotonicity, convexity, approximation

1. INTRODUCTION

Spline interpolation is a useful and powerful tool in CAGD. In many cases the rational spline curves better approximating functions than the usual spline functions. It has been observed that many simple shapes including conic section and quadratic surfaces can not be represented exactly by usual spline, whereas rational splines can exactly represent all these conic sections and quadratic surfaces in an easy manner. Shape preserving interpolation is a powerful tool to visualize the data in the form of curves and surfaces. The problem of curve interpolation to the given data has been studied with various requirements. One may be concerned with the smoothness of the interpolating curves, the preservation of the underlying shape features of the data, the computational complexity, or the fulfillment of certain constraints. Shape preserving signifies preserving the three basic crucial features such as positivity, monotonicity and convexity of the data.

Convexity preserving C^2 rational quadratic trigonometric spline were presented in [4]. Duan et.al.[3] represented the construction and shape preserving analysis of a new weighted rational cubic interpolation and its approximation. Sarfraz et.al.[7] uses the piecewise C^1 rationdal cubic function developed by Delbourgo and gregory[1] to preserve the shape of positive data. Since their rational function has only a single parameters there is no freedom for curve modification and hence the method is not suitable for interactive curve design. Sarfraz[6] developed a rational cubic function with quadratic denominator that involves two free parameters. The rational function in [6] attained C^2 continuity by imposing constraints on first derivatives at the knots and is unable to interpolate the data with specified derivatives. In this paper we have discussed the monotonicity, convexity and approximation properties of rational spline with cubic numerator and quadratic denominator. The shape parameters play a crucial role in preserving the convexity and monotonicit of GC^2 rational spline

The paper is organized as follows: The piecewise rational (cubic /quadratic) spline interpolant is developed in section 2. the approximation properties of the rational interpolation are studied in section 3. Monotonicity and convexity are studied in section 4. We have discussed the constrains for GC^2-rational spline interpolant in section 5.

2. THE RATIONAL (CUBIC/QUADRATIC) SPLINE INTERPOLATION

A rational (cubic/quadratic) spline with based on function values and derivatives was given in [5]. Given a data set $\{(t_i, f_i, d_i), i = 1, 2, ...n\}$ where f_i and d_i are the function values and the derivative values defined at the knots, respectively, and $a = t_1 < t_2 < ..... < t_n = b$ are the knots. let $h_i = t_{i+1} - t_i, \theta = \frac{t-t_i}{h_i}$, and let α_i and β_i and λ_i be positive parameters .

Denote

$$P(t) = \frac{p_i(t)}{q_i(t)} \quad (2.1)$$

Where

$$p_i(t) = (1-\theta)^2\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^2\beta_i f_{i+1}$$
$$q_i(t) = (1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i$$

and

$$V_i = f_i + \alpha_i h_i \lambda_i d_i$$
$$W_i = f_{i+1} - \beta_i h_i d_{i+1}$$

This rational cubic spline P(t) satisfies

$$P(t_i) = f_i, P^{'}(t_i) = \lambda_i d_i$$
$$P(t_{i+1}) = f_{i+1}, P^{'}(t_{i+1}) = d_{i+1} \quad (2.2)$$

Where $d_i^{'}$ s denote the derivative values at the knots t_i. These derivative parameters are usually note given and can be determined by using the method as discussed in [2]. .

3. ERROR ESTIMATION OF INTERPOLATION

This section investigates the estimation of the approximation error incurred when the rational cubic function (2.1) is used to interpolate data from an arbitrary function that is $GC^2[a,b]$. Locality of the interpolation allows error estimation in each subinterval $[t_i, t_{i+1}]$ without loss of generality. Consider the case that the knots are equally spaced, namely $h_i = h = \frac{t_n - t_0}{n}$ for all i = 1,2...n. The Peano Kernel theorem [8] is used to estimate the error in each subinterval $[t_i, t_{i+1}]$ as

R[f] = f(t) - P(t) = $\int_{t_i}^{t_{i+2}} f^2(\tau) R_t[(t-\tau)_+] d\tau, t\epsilon[t_i, t_{i+1}]$ (3.1)

Where

$R_t[(t-\tau)_+] =$

$\{ p(\tau), \quad t_i < \tau < t$

$q(\tau), \quad t < \tau < t_{i+1}$

Where

$$p(\tau) = (t-\tau) - \frac{\theta^2((1-\theta)+\beta_i)(t_{i+1}-\tau) - \theta^2(1-\theta)\beta_i h}{(1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i},$$

and

$$q(\tau) = -\frac{\theta^2((1-\theta)+\beta_i)(t_{i+1}-\tau) - \theta^2(1-\theta)\beta_i h}{(1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i},$$

Then

$$\|R(f)\| = \|f(t) - P(t)\|$$

$$\leq \|f^2(t)\| [\int_{t_i}^{t} |p(\tau)| d\tau + \int_{t}^{t_{i+1}} |q(\tau)| d\tau] \quad (3.2)$$

For $q(\tau)$, since

$$q(t) = -\frac{\theta^2(1-\theta)^2 h}{(1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i} \leq 0$$

$$q(t_{i+1}) = \frac{\theta^2(1-\theta)\beta_i h}{(1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i} \geq 0$$

It is easy to see that the root $\tau^\star$ of $q(\tau)$is

$\tau^\star = t_{i+1} - \frac{(1-\theta)\beta_i h}{(1-\theta)+\beta_i}$ Thus

$$\int_{t_i}^{t_{i+1}} |q(\tau)| d\tau = \int_{t}^{\tau^\star} -q(\tau) d(\tau) + \int_{\tau^\star}^{t_{i+1}} q(\tau) d\tau$$

$$= \frac{\theta^2(1-\theta)^2(1+\beta_i)^2 h^2}{2((1-\theta)+\beta_i)((1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i)}$$

similarly since,

p(t)= q(t) ≤ 0, $p(t_i) = \frac{\theta(1-\theta)^2\alpha_i h}{(1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i} \geq 0$

and the root $\tau_\star$ of $p(\tau)$ in $[t_i, t]$ is

$$\tau_\star = t_{i+1} - \frac{(\theta + (1-\theta)\alpha_i)}{\theta + \alpha_i}$$

So that

$$\int_{t_i}^{t} |p(\tau)| d\tau = \int_{t_i}^{\tau_\star} p(\tau) d\tau + \int_{\tau_\star}^{t} -p(\tau) d\tau$$

$$= \frac{\theta^2(1-\theta)^2(\theta^2+\alpha_i^2)h^2}{2(\theta+\alpha_i)((1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i)}$$

From the calculation above, it can be shown that

$$\|R[f]\| = \|f(t) - P(t)\| \ \leq \|f^2(t)\| h^2 W(\theta, \alpha_i, \beta_i)$$

Where

$$W(\theta, \alpha_i, \beta_i) = W_1(\theta, \alpha_i, \beta_i) + W_2(\theta, \alpha_i, \beta_i) \quad (3.3)$$

$$W_1(\theta, \alpha_i, \beta_i) = \frac{\theta^2(1-\theta)^2(\theta^2+\alpha_i^2)h^2}{2(\theta+\alpha_i)((1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i)}$$

$$W_2(\theta, \alpha_i, \beta_i = \frac{\theta^2(1-\theta)^2(1+\beta_i)^2 h^2}{2((1-\theta)+\beta_i)((1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i)}$$

from the calculation above, it can be shown that

$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^2(t)\| h^2 W(\theta, \alpha_i, \beta_i)$

The above can be summarized as :

Theorem 3.1. The error of rational (cubic/quadratic) function defined in (2.1) for $f(t)\epsilon GC^2[a,b]$ in each subinterval $[t_i, t_{i+1}]$ is

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^2(t)\| h^2 m_i$$

$$m_i = max_{0 \leq \theta \leq 1} W(\theta, \alpha_i, \beta_i)$$

4. MONOTONICITY AND CONVEXITY PRESERVING SPLINE INTERPOLANT

4.1. Monotonicity

We assume a monotonic increasing data, so that

$$\lambda_i > 0, f_1 \leq f_2 \leq ...... \leq f_n \quad (4.1)$$

or equivalently

$$\Delta_i \geq 0, (i = 1, 2, ...n-1)$$

To have a monotonic interpolant P(t), it is necessary that the derivative parameters d_i should satisfy:

$$d_i \geq 0 \ i = 1, 2........n \quad (4.2)$$

P(t) is monotonic if and only if

$$P^{'}(t) \geq 0, t\epsilon[a,b]$$

After simplification, it can be shown that for $t\epsilon[t_i, t_{i+1}]$

$$P^{'}(t) = \frac{(1-\theta)^4 A_{1i} + \theta(1-\theta)^3 B_{1i} + \theta^2(1-\theta)^2 C_{1i} + \theta^2(1-\theta^2) D_{1i} + \theta(1-\theta)^2 E_{1i} + \theta^4 F_{1i}}{((1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i)^2} \quad (4$$

Where

$$A_{1i} = \lambda_i d_i \alpha_i^2$$

$$B_{1i} = (\Delta_i - 2\beta_i d_{i+1})\alpha_i$$

$$C_{1i} = 2\alpha_i\beta_i\Delta_i + \Delta_i - \lambda_i d_i \alpha_i(1+\beta_i - \beta_i(1+\alpha_i)d_{i+1})$$

$$D_{1i} = \beta_i \Delta_i$$

$$E_{1i} = \alpha_i(1+2\beta_i)\Delta_i$$

$$F_{1i} = \beta_i^2 d_{i+1}$$

We observe that $P^{'}(t)$ is positive if each of $A_{1i}, B_{1i}, C_{1i}, D_{1i}, E_{1i}$ and F_{1i} are positive. Since A_{1i}, D_{1i}, E_{1i} and F_{1i} are automatically positive.

Thus the sufficient condition for the interpolant P(t) be monotone is that $B_{1i} \geq 0$ and $C_{1i} \geq 0$

Now

$\beta_i \geq 0$ if $(\Delta_i - 2\beta_i d_{i+1})\alpha_i \geq 0$

ie $\beta_i \leq \frac{\Delta_i}{2d_{i+1}}$ (A)

$C_{1i} \geq 0$

if $2\alpha_i\beta_i\Delta_i + \Delta_i - \lambda_i d_i \alpha_i(1+\beta_i) - \beta_i(1+\alpha_i)d_{i+1}) \geq 0$

ie $\beta_i \geq \frac{\alpha_i \lambda_i d_i}{(1-\alpha_i)d_{i+1}}$ (B)

Therefore the spline interpolant is monotone if

$$\frac{\alpha_i \lambda_i d_i}{(1-\alpha_i)d_{i+1}} \leq \beta_i \leq \frac{\Delta_i}{2d_{i+1}} \quad (4.4)$$

Therefore $P^{'}(t)$ is monotone if (4.4) holds.

We have thus proved the following theorem.

Theorem 4.1. Given a monotonic increasing set of data satisfying (4.1) and the derivative values satisfying (4.2), there exist a monotone rational (cubic /quadratic) spline interpolant $P\epsilon GC^{'}[a,b]$ involving the parameters $\alpha_i \geq 0$ and $\beta_i \geq 0$ which satisfies the interpolatary conditions (2.2) provided (4.4) holds.

4.2 Convexity

Engineering practice usually requires the interpolating function retains the shape of the given data. In order to get the condition for the interpolation to keep convexity or concavity in the interpolating interval, consider the condition for the second order derivative to remain positive or negative in the interpolating interval. With assumption a strictly convex data then

$$\Delta_1 \leq \Delta_2 \leq .... \leq \Delta_{n-1} \quad (4.5)$$

For a convex interpolant P(t), it is necessary that the derivative parameters to be:

$d_1 < \Delta_1 < .... < \Delta_{i-1} < d_i < \Delta_i < .... < \Delta_{n-1} < d_n$ (4.6)

P(t) is convex if

$$P^{''}(t) \geq 0 \quad (4.7)$$

for all $t\epsilon[a,b]$. For $t\epsilon[t_i, t_{i+1}]$, the second derivative is

$$P^{''}(t) = \frac{(1-\theta)^3 A_{2i} + \theta(1-\theta)^2 B_{2i} + \theta^2(1-\theta) C_{2i} + \theta^3 D_{2i}}{h((1-\theta)^2\alpha_i + \theta(1-\theta) + \theta^2\beta_i)^3} \quad (4.8)$$

Where

$$A_{2i} = 2\alpha_i^2(\beta_i(\Delta_i - d_{i+1}) + (\Delta_i - \lambda_i d_i))$$

$$B_{2i} = 6\alpha_i^2\beta_i(\Delta_i - \lambda_i d_i)$$

$$C_{2i} = 6\alpha_i\beta_i^2(d_{i+1} - \Delta_i)$$

$$D_{2i} = 2\beta_i^2(\alpha_i(\lambda_i d_i - \Delta_i) + (d_{i+1} - \Delta_i)) \quad (4.9)$$

we observe that $P^{''}(t)$ is non negative if each A_{2i}, B_{2i}, C_{2i} and D_{2i} is non negative since we are assuming that (4.6) holds, B_{2i} is positive when $\lambda_i d_i < \Delta_i$ and C_{2i} is automatically positive. Thus the sufficient condition for the interpolant P(t) to be convex is that

$A_{2i} \geq 0$, $B_{2i} \geq 0$ and $D_{2i} \geq 0$

$A_{2i} \geq 0$ if

$2\alpha_i^2(\beta_i(\Delta_i - d_{i+1}) + (\Delta_i - \lambda_i d_i)) \geq 0$
ie $\beta_i \geq \frac{\Delta_i - \lambda_i d_i}{d_{i+1} - \Delta_i}$ (C)
$B_{2i} \geq 0$ if
$6\alpha_i^2\beta_i(\Delta_i - \lambda_i d_i) \geq 0$
ie $\lambda_i \leq \frac{\Delta_i}{d_i}$ (D)
and $D_{2i} \geq 0$ if
$2\beta_i^2(\alpha_i(\lambda_i d_i - \Delta_i) + (d_{i+1} - \Delta_i)) \geq 0$
ie $\frac{\Delta_i - \lambda_i d_i}{d_{i+1} - \Delta_i} \geq \frac{1}{\alpha_i}$ (E)
Therefor the spline interpolant is convex if
$\frac{1}{\alpha_i} \leq \frac{\Delta_i - \lambda_i d_i}{d_{i+1} - \Delta_i} \leq \beta_i$ (4.10).
Thus the spline interpolant is convex if (4.10) together with (4.6) holds.
we have thus proved the following theorem.

Theorem 4.2. For a given set of strict convex data a convex rational spline interpolant $P \epsilon GC^1[a, b]$ involving the parameters α_i and β_i exist, which satisfy the interpolating condition (2.2), the derivative parameters $d_i^{'}s$ satisfying (4.6) and (4.10) holds.

GC^2 **rational spline interpolant**

Rational spline could even be C^2 in the interpolating interval [a, b] infect let
$P^{''}(t_i+) = \lambda_i^2 P^{''}(t_i-)$, i = 2,3,......n-1
the condition lead to the following continuous system of linear equations:
$h_i\alpha_i\alpha_{i-1}\lambda_i^2 d_{i-1} + (h_i\alpha_i\lambda_i^2 + h_{i-1}\beta_{i-1}\lambda_i)d_i + h_{i-1}\beta_i\beta_{i-1}d_{i+1}$
$= h_i(1 + \alpha_{i-1})\alpha_i\Delta_{i-1}\lambda_i^2 + h_{i-1}(1 + \beta_i)\beta_{i-1}\Delta_i$
(4.11)
Therefor, if the successive parameters $(\alpha_{i-1}, \beta_{i-1})$and (α_i, β_i) satisfy (4.11) at i= 2,3.......n-1, namely for the positive parameters $\alpha_{i-1}, \beta_{i-1}$ and the selected β_i, if
$\alpha_i = \frac{h_{i-1}\beta_{i-1}((\Delta_i - \lambda_i d_i) + \beta_i(\Delta_i - d_{i+1}))}{h_i\lambda_i^2((d_i - \Delta_{i-1}) + \alpha_{i-1}(\lambda_{i-1}d_i - \Delta_{i-1}))}$
then
$P(t)\epsilon GC^2[a, b]$..

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