Fine Weakly Clopen and Fine Strongly $\theta-b-$Continuous Functions

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ABSTRACT
Powar P. L. and Rajak K., have introduced fine-topological space which is a special case of generalized topological sspace. In this paper, the author has introduced a new class of functions called fine weakly BR-continuous functions on fine topological space (cf. [12]). Weakly BR-continuous functions has been introduced by Ekici [3] which includes the class of strongly $\theta-b-$continuous functions due to Park [11]. Characterizations and properties of the class of fine-weakly BR-continuous functions are investigated.

KEYWORDS
Fine open sets, $\theta-$open set, $\theta-$closed set, $b-$open set and $b-$closed set.

1. INTRODUCTION
The notion of continuity is one of the most important tools in Mathematics and many different forms of generalizations of continuity have been introduced and investigated. The notion of weakly continuous functions was introduced by Levin [6]. In 2007, Son et. Al. [14] have introduced a new class of functions called weakly clopen function including the class of almost clopen functions. The class of almost clopen functions is a generalization of perfectly continuous functions regular set-connected functions and clopen functions.

The notion of fine-topological has been introduced and investigated by Powar P. L. and Rajak K in 2010 (cf. [12]). Fine –Topological space is a special case of generalized topological space. In this space fine –open sets have introduced which includes all semi-open sets, pre-open sets, $\alpha-$open sets , $\beta-$open sets, regular open sets etc.

In this paper, the author has introduced new generalizations of strongly $\theta-b-f-$contnuous functions and fine-weakly clopen functions. We obtain some characterizations of fine weakly BR-continuous functions and the relationships among fine weakly BR-continuous functions, fine-weakly-clopen functions.

2.PRELIMINARIES
Throughout this paper, space X and Y always mean topological spaces. For a subset A of a space X, the closure and interior of A is denoted by cl(A) and int(A). We use the following basic definitions:

2.1 Definition A subset S of a topological space (X, τ) is said to be

1) $\alpha-$ open [7] if S $\subseteq$ Int(cl(Int(S))).
2) Semi-open [7] if S $\subseteq$ cl(Int(S)).
3) Pre-open [7] if S $\subseteq$Int(cl(S))).
4) $\beta-$-open [7] if $S\subseteq$ cl(Int(cl(S))).
5) Regular-open [7] if $S\subseteq$ Int(cl(S)).

2.2 Definition The largest b-open set contained in a subset A is called b-interier and is denotad by b- int (A) (cf. [1]).

2.3 Definition The smallest b-open set containing a subset A is called b-closure of A and is denoted by b-cl(A) (cf. [3]).

2.4 Definition A point x of X is called a b-$\theta-$cluster point of a subset A if b-cl(U) $\cap$A $\neq\phi$ for every open set U containing x. The set of all b-$\theta-$ cluster point is called b- θ-closure of A and is denoted by b- θ-cl (A) (cf. [3]).

2.5 Definition A subset A is said to be $b-\theta-$closed if A = $b-\theta-$cl(A). The complement of a $b-\theta-$closed set is said to be $b-\theta-$open (cf. [3]).

2.6 Definition A point x of X is called a $b-\theta-$ interior point of a subset A if there exist a b-regular set U containing x such that $U\subset A$ and is denoted by $x\in b-\theta-int(A)$ (cf. [3]).

2.7 Definition A function f: X → Y is said to be

1) Strongly $\theta-$continuous if for each $x\in X$ and each open set V of Y containing f(x), there exist an open set U of X containing x such that $f(cl(U))\subset V$ (cf. [8]).
2) Strongly $\theta-$semi-continuous if for each $x\in X$ and each open set V of Y containing f(x), there exists a semi-open set U of X containing x such that $f(s-cl(U))\subset V$ (cf. [5]).
3) Strongly $\theta-$pre-continuous if for each $x\in X$ and each open set V of Y containing f(x), there exists a pre-open set U of X containing x such that $f(p-clU\subset V$ (cf. [8]).
4) Strongly $\theta-b-$continuous if for each $x\in X$ and each open set V of Y containing f(x), there exists a b-open set U of X containing x such that $f(b-clU\subset V$ (cf. [10]).

2.8 Definition A function f: X → Y is said to be

1) Weakly BR-continuous at $x\in X$ if for each open set U of Y containing f(x), there exist a b-regular set V containing x such that $f(V)\subset cl(U)$ (cf. [3]).
2) Weakly BR-continuous if for each$x\in X$, f is weakly BR-continuous at $\in X$ (cf. [3]).

2.9 Definition A function f: X → Y is said to be BR-continuous if $f^{-1}(U) \in BR(X)$ for each open set U of Y (cf. [3]).

2.10 Definition A function f: X → Y is said to be

1) Perfectly continuous if $f^{-1}(U)$ is clopen in X for each open set U of Y (cf. [9]).
2) Clopen if for each $x \in X$ and each open set V of Y containing f(x), there exists a clopen set U of X containing x such that f(U) ⊂ V (cf. [15]).
3) Almost clopen if for each $x \in X$ and each open set V of Y containing f(x), there exists a clopen set U of X containing x such that f(U) ⊂ $int(cl(V))$ (cf. [4]).
4) Regular set connected if $f^{-1}(U)$ is clopen in X for each regular open set U of Y (cf. [2]).
5) Weakly clopen if for each $x \in X$ and each open set V of Y containing f(x), there exists a clopen set U of X containing x such that f(U) ⊂ $cl(V)$ (cf. [14]).

2.11 Definition A function f: X → Y is said to be contra b-θ −continuous if $f^{-1}(V)$ is $b - \theta$ −closed in X for each open set V of Y.

2.12 Definition Let (X, τ) be a topological space we define

$\tau(A_\alpha) = \tau_\alpha$ (*say*) = $\{G_\alpha(\neq X) : G_\alpha \cap A_\alpha = \phi$, for $A_\alpha \in \tau$ and $A_\alpha \neq \phi, X$, for some $\alpha \in J$, where *J* is the index set.$\}$

Now, we define

$$\tau_f = \{\phi, X, \cup_{\{\alpha \in J\}} \{\tau_\alpha\}\}$$

The above collection τ_f of subsets of *X* is called the fine collection of subsets of *X* and (X, τ, τ_f) is said to be the fine space *X* generated by the topology τ on *X* (cf [12]).

2.13 Definition A subset *U* of a fine space *X* is said to be a fine-open set of *X*, if *U* belongs to the collection τ_f and the complement of every fine-open sets of *X* is called the fine-closed sets of *X* and we denote the collection by F_f (cf [12]).

Definition Let A be a subset of a fine space *X*, we say that a point $x \in X$ is a fine limit point of *A* if every fine-open set of *X* containing *x* must contains at least one point of A other than *x* (cf [12]).

2.14 Definition Let *A* be the subset of a fine space *X*, the fine interior of *A* is defined as the union of all fine-open sets contained in the set *A* i.e. the largest fine-open set contained in the set *A* and is denoted by f_{Int} (cf [12]).

2.15 Definition Let *A* be the subset of a fine space *X*, the fine closure of *A* is defined as the intersection of all fine-closed sets containing the set *A* i.e the smallest fine-closed set containing the set *A* and is denoted by f_{cl} (cf [12]).

2.17 Definition A function $f : (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is called fine-irresolute (or f-irresolute) if $f^{-1}(V)$ is fine-open in *X* for every fine-open set *V* of *Y* (cf. [12]) .

3 FINE-b-OPEN SETS AND FINE $b - \theta$ −CONTINUOUS FUNCTIONS

In this section, author has define a new class of open sets called f-b-open sets and also introduced a new class of functions called fine $b - \theta$ −continuous functions.

3.1 Definition A subset A of a fine-topological space (X, τ, τ_f) is said to be f-b-open if $A \subseteq f_{int}(f_{cl}(A)) \cup f_{cl}(f_{int}(A))$ and the complement of f-b-open sets is called f-b-closed.

3.2 Example Let X = $\{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$,

$$\tau_f = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\},$$

$$F_f = \phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\},$$

The fine-b-open sets of fine-topological space (X, τ, τ_f) are $\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$ and the fine-closed sets are $\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}$.

3.3 Definition A point $x \in X$ is said to be in the $f - \theta$ −closure of a subset A of X, denoted by $f - \theta - cl(A)$, if $f_{cl}(G) \cap A \neq \phi$ for each fine open set G of X containing x.

3.4 Example In the above example 3.1, let $A = \{a\} \subset X$ for a point $a \in X \exists$ fine-open sets $\{a\}, \{a, b\}, \{a, c\}$ and $f_{cl}\{a\} = \{a\}, f_{cl}\{a, b\} = X, f_{cl}\{a, c\} = \{a, c\}$ such that $A \cap \{a\} \neq \phi, A \cap X \neq \phi, A \cap \{a, c\} \neq \phi$ it implies that a be in the $f - \theta - cl(A)$. Similarly, we can check that $b \in X$ not in the $b - \theta - cl(A)$.

3.5 Definition A subset A of a fine space X is called $f - \theta$ −closed if $A = f - \theta - cl(A)$. The complement of $f - \theta$ −closed sets are called $f - \theta$ −open.

3.6 Example Consider the example 3.1, the only $f - \theta$ −closed sets are $\phi, X, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}$ and $f - \theta$ −open sets are $, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$.

3.7 Definition Let A be a subset of X. The smallest $f - b$ −closed set containing the set A is called $f - b$ −closure of A and is denoted by $f - b - cl(A)$.

3.8 Definition Let A be a subset of X. The largest fine-b-open set contained in the set A is called $f - b$ −interor of A and is denoted by $f - b - int(A)$.

3.9 Example Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_f = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b\}, \{a, c\}, \{b, c\}, F_f = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}$.

3.10 Definition A subset A of a topological space X is said to be $f-b-$regular if A is both $f-b-$open and $f-b-$closed.

3.11Example In the above example 3.4, the $f-b-$regular sets are $\phi, X\ \{a\}, \{b\}, \{b,c\}, \{a,c\}$.

3.12 Definition A point $x \in X$ is called a $f-b-\theta-$cluster point of a subset A if $f-b-cl(U) \cap A \neq \phi$ for every $f-b-$open set U containing x. The set of all $f-b-\theta-$cluster point of A is called $f-b-\theta$-closure of A and is denoted by $f-b-\theta-cl(A)$.

3.13 Definition A subset A is said to be $f-b-\theta-$closed if $A = f-b-\theta-cl(A)$. The complement of $f-b-\theta-$closed set is called $f-b-\theta-$open.

3.14 Remark

- Every b-open set is fine –open.
- Every $\theta-$open set is fine-open.
- Every $b-\theta-$open set is fine-open.

3.10 Remark From definition 2.12, definieion2.7 and remark 3.14 it can be observe that

Fine- irresolute mapping
$\Rightarrow$ Strongly$\theta-$continuous
$\Rightarrow$ Strongly θ-semi continuous
$\Rightarrow$ Strongly $\theta-$pre continuous
$\Rightarrow$ Strongly $\theta-b-$continuous

3.15 Definition A function $f: X \to Y$ is said to be strongly $f-\theta-$continuous if for each $x \in X$ and each fine open set V of Y containing $f(x)$, there exists a fine-open set U of X containing x such that $f(f_{cl}(U)) \subset V$.

3.16 Example Let $X = \{a,b,c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{a,b\}\}$, $\tau_f = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}, \{b\}, \{b,c\}\}$, $F_f = \{\phi, X, \{b,c\}, \{c\}, \{b\}, \{a,c\}, \{a\}\}$ and let $Y = \{1,2,3\}$ with the topology $\tau' = \{\phi, X, \{1\}\}$, $\tau'_f = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$, $F'_f = \{\phi, X, \{2,3\}, \{3\}, \{2\}\}$. We define a map $f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f')$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It can be easily check that, the function f is strongly $f-\theta-$continuous.

3. 17 Definition A function $f: X \to Y$ is said to be strongly $f-\theta-$ semi-continuous if for each $x \in X$ and each fine open set V of Y containing $f(x)$, there exists a fine-semi-open set U of X containing x such that $f(f_{scl}(U)) \subset V$.

3.18 Example Let $X = \{a,b,c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a,b\}\}$, $\tau_f = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}, \{b\}, \{b,c\}\}$, $F_f = \{\phi, X, \{b,c\}, \{c\}, \{b\}, \{a,c\}, \{a\}\}$ and let $Y = \{1,2,3\}$ with the topology $\tau' = \{\phi, X, \{1\}\}$, $\tau'_f = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$, $F'_f = \{\phi, X, \{2,3\}, \{3\}, \{2\}\}$. We define a map $f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f')$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It can be easily check that, the function f is strongly $f-\theta-$semi-continuous.

3. 19 Definition A function $f: X \to Y$ is said to be strongly $f-\theta-$ precontinuous if for each $x \in X$ and each fine open set V of Y containing $f(x)$, there exists a fine-preopen set U of X containing x such that $f(f_{pcl}(U)) \subset V$.

3.20 Example Consider the example 3.7 and it can be easily check that, the function f is strongly $f-\theta-$precontinuous.

3.21 Definition A function $f: X \to Y$ is said to be strongly $f-\theta-b-$ continuous if for each $x \in X$ and each fine open set V of Y containing $f(x)$, there exists a fine-b-open set U of X containing x such that $f(f-b-cl(U)) \subset V$.

3. 22 Example Consider the example 3.7 and it can be easily check that, the function f is strongly $f-\theta-b-$continuous.

3.23 Definition A function $f: X \to Y$ is said to be fine-weakly $BR-$continuous at $x \in X$ if for each fine-open set U of Y containing f(x), there exists a f-$b-$regular set V containing x such that $f(V) \subset f_{cl}(U)$.

3. 24 Definition A function $f: X \to Y$ is said to be fine weakly $BR-$continuous if for each $x \in X$, f is fine –weakly-BR-continuous.

3. 25 Definition A function $f: X \to Y$ is said to be fine$-BR-$continuous if $f^{\{-1\}}(U) \in BR(X)$ for each fine-open set U of Y.

3. 26 Example Let $X = \{a,b,c\}$ be a topological space with the topology $\tau = \{\phi, X, \{b\}, \{a,b\}, \{b,c\}\}$, $\tau_f = \{\phi, X, \{a\}, \{a,b\}, \{a,c\}, \{b\}, \{b,c\}\}$, $F_f = \{\phi, X, \{b,c\}, \{c\}, \{b\}, \{a,c\}, \{a\}\}$ and let $Y = \{1,2,3\}$ with the topology $\tau' = \{\phi, X, \{1\}\}$, $\tau'_f = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$, $F'_f = \{\phi, X, \{2,3\}, \{3\}, \{2\}\}$. We define a map $f: (X, \tau, \tau_f) \to (Y, \tau', \tau_f')$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It can be easily check that, the function f is strongly $f-\theta-$semi-continuous.

3. 27 Definition A function $f: X \to Y$ is said to be fine-contra $b-\theta-$continuous if $f^{-1}(V)$ is $f-b-\theta-$closed in X for each fine-open set V of Y.

3.28 Theorem The following properties are equivalent for a function $f: X \to Y$

1) F is fine weakly BR-continuous at $x \in X$.
2) $x \in f-b-\theta-f_{int}(f^{-1}(f_{cl}(V)))$ for each fine - neighborhood V of f(x).

Proof:

1) ⇒2) Let V be any fine-neighborhood of f(x). Then there exists a $f-b-$regular set U containing x such that $f(U)\subset f_{cl}(V)$ and $U\subset f^{-1}\big(f_{cl}(V)\big)$. Since, U is $f-b-$regular, then $x\in U\subset f-b-\theta-int(U)\subset f-b-\theta-int\left(f^{-1}\big(f_{cl}(U)\big)\right)$.
2) ⇒ 1) Let $x\in f-b-\theta-int\left(f^{-1}\big(f_{cl}(V)\big)\right)$ for each fine-neighborhood V of f(x). There exists a $f-b-$regular set U containing x such that $f(U)\subset f_{cl}(V)$. Hence, f is fine-weakly BR-continuous at $x\in X$.

3.29 Theorem If $f:X\to Y$ is weakly BR-continuous at $x\in X$, then there exists a non-empty $f-b-$open set $U\subset H$ such that $U\subset f-b-\theta-cl(f^{-1}\,f_{cl}(V))$ for each fine-neighborhood V of f(x) and each $f-\alpha-$open neighborhood H of x.

Proof: Let V be any neighborhood of f(x) and H be an $f-\alpha-$open set of X containing x. Since $x\in f-b-\theta-int\left(f^{-1}\big(f_{cl}(V)\big)\right)$, then there exists a $f-b-$regular set G containing x such that $G\subset f-b-\theta-int\ \left(f^{-1}\big(f_{cl}(V)\big)\right)$.

We have $H\cap G\neq\phi$. Take $U=H\cap G$. Hence U is a non-empty $f-b-$open set and $U\subset H$and $U\subset f-b-\theta-int\ \left(f^{\{-1\}}\big(f_{cl}(V)\big)\right)\subset f-b-\theta-cl\big(f^{-1}(V)\big)$.

3.30 Theorem Let $f:X\to Y$ be a function. If $f^{-1}(f-\theta-cl(U))$ is $f-b-\theta-$closed in X for every subset U of Y, then f is f-weakly BR-continuous.

Proof: Let $U\subset Y$. Since $f^{-1}(f-\theta-cl(U))$ is $f-b-\theta-$closed in X, then $f-b-\theta-cl\big(f^{-1}(U)\big)\subset f-b-\theta-cl(f^{-1}(f-\theta-cl(U)))=f^{-1}\big(f-\theta-cl(U)\big)$. By theorem 3.1, f is fine-weakly BR-continuous.

3.31 Theorem Let $f:X\to Y$ be a function. If Y is fine-regular , then following are equivalent:

1) f is fine- weakly BR-continuous.
2) F is fine-strongly $f-\theta-b-$continuous.

Proof:

⇒2) Let $x\in X$ and V be a fine-open set of Y containing f(x). Since Y is fine-regular, then there exists a fine-open set H of Y containing f(x) such that $H\subset f_{cl}(H)\subset V$. Since f is fine-weakly BR-continuous, there exists a f-b-regular set U of X containing x such that $f(U)\subset f_{cl}(H)\subset V$. Thus, f is strongly $f-\theta-b-$continuous.

1) ⇒ 1) By definition, it is obvious.

4. CONCLUSION

New generalizations of strongly $\theta-b-f-$contnuous functions and fine-weakly clopen functions are investigated and obtained some characterizations of fine weakly BR-continuous functions and the relationships among fine weakly BR-continuous functions, fine-weakly-clopen functions. This concept may have an extensive applicational value in quantum physics.

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