Generalization of Semi-Projective Modules

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ABSTRACT

In this paper characterization of pseudo M-p-projective modules and quasi pseudo principally projective modules are given and discussed the various properties of it. It is proved that a pseudo M-p-projective module is Hopfian iff M/N is Hopfian, for each fully invariant small submodule N of M. It is also provided the sufficient condition for pseudo M-p-projective module to be discrete. Finally several equivalent conditions are given for a quasi pseudo principally projective module to have the finite exchange property.

Keywords:

Pseudo M-p-projective module, Discrete module, Hollow module, Finite exchange property.

1. INTRODUCTION

The aim of this paper is to study quasi-pseudo principally projective modules. In 1999 Sanh et.al. [12], defined that N is M-principally injective, if every R-homomorphism from an M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. A module M is called quasi principally (or semi) injective, if it is M-principally injective. The dual notion of this is defined by Tansee and Wongwai in [14], that a module N is called M-principally projective, if every R-homomorphism from N to an M-cyclic submodule of M can be lifted to an R-homomorphism from N to M. A module M is called quasi principally (or semi) projective, if for any M-cyclic submodule N of M, any epimorphism $g : M \to N$ and any homomorphism $f : M \to N$, there exists an R-endomorphism h of M such that $f = g.h$. Motivated by this definition, authors have introduce the notion of quasi-pseudo principally projective module in [10] which is the dual notion of quasi-pseudo principally injective module defined by Chaturvedi et.al.[2]. In [11] T.C.Quynh have studied the same under the name Pseudo semi-projective Modules, Now authors are in position to prove the various property of such modules. It is easy to show that if M is quasi-pseudo principally projective, then every epimorphism in $EndM_R$ is an automorphism. Consequently proved (Proposition 2.17) that every quasi-pseudo principally projective module is Hopfian.

The paper is divided into three sections; In section 1, introduction, some definitions and notations are given.
Section 2, is devoted to the study of the properties of quasi-pseudo principally projective modules. Sufficient condition for quasi-pseudo principally projective to be quasi principally projective module is given. An example of a pseudo M-principally projective module which is not M-projective is given. Apart from this some results are proved related to Hopfian, co-Hopfian, and directly finite modules with Pseudo M-p-projective module.

Section 3, contains necessary and sufficient condition for Pseudo M-p-projective module to be discrete. Finally several equivalent conditions are given for a quasi-pseudo principally projective module to have the finite exchange property.

1.1 Preliminaries

Throughout this paper, by a ring R always mean an associative ring with identity and every R-module M is an unitary right R-module. Let M be an R-module; a module N is called M-generated, if there is an epimorphism $M^{(I)} \longrightarrow N$ for some index set I. If I is finite then N is called finitely M-generated. In particular, a submodule N of M is called an M-cyclic submodule of M if it is isomorphic to M/L for some submodule L of M. A submodule K of an R-module M is said to be small in M, written $K \ll M$, if for every submodule $L \subset M$ with $K + L = M$ implies $L = M$. A nonzero R-module M is called hollow if every proper submodule of it is small in M. A submodule N of M is called fully invariant submodule of M, if $f(N) \subset N$ for any $f \in S_M = EndM_R$. A module M is called indecomposable, if $M \neq 0$ and cannot be written as a direct sum of nonzero submodules.
Consider the following conditions for an R-module M:
(D_1) : For every submodule A of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll M$.
(D_2) : If $A \subseteq M$ such that M/A is isomorphic to a summand of M, then A is a summand of M.
(D_3) : If M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a summand of M.
An R-module M is called a lifting module if M satisfies (D_1), M is called discrete module if it satisfies (D_1) and (D_2) and quasi-discrete if it satisfies (D_1) and (D_3).

Given a cardinal number c, a module M is said to have the c-exchange property if for any module A and any internal direct sum decomposition of A given by

$$A = M' \oplus N = \bigoplus_I A_i.$$

for modules M', N, A_i where $M' \cong M$ and $card(I) \leq c$, there always exist submodules $B_i \subseteq A_i$ for each $i \in I$ such that

$$A = M' \oplus (\bigoplus_I B_i).$$

If M has the n-exchange property for every positive integer n, then M is said to have the finite exchange property.
For standard notations and terminologies refer to [3], [7] and [17].

2. PSEUDO M-P-PROJECTIVE AND QUASI-PSEUDO PRINCIPALLY PROJECTIVE MODULE

DEFINITION 2.1. *Let M be an R-module. An R-module N is called pseudo M-principally projective (pseudo M-p-projective, for short) if every epimorphism from N to an M-cyclic submodule of M can be lifted to an R-homomorphism from N to M. Equivalently, for any endomorphism f of M, every epimorphism from N to $f(M)$ can be lifted to an R-homomorphism from N to M. An R-module M is called quasi-pseudo principally projective, if it is pseudo M-principally projective module. N is called pseudo principally projective, if it is pseudo R-p-projective.*

REMARK 2.2. *Every M-projective module is pseudo M-p-projective but the converse is not necessarily true.*

An example of pseudo M-p-projective module which is not M-projective is given.

EXAMPLE 2.3. *$\mathbb{Z}/4\mathbb{Z}$ is pseudo $\mathbb{Z}$-p-projective module but not $\mathbb{Z}$-projective.*

Proof : Let $\mathbb{Z}$ denote the ring of integer. For any $n \in \mathbb{N}$ it is easily seen that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, n\mathbb{Z}) = 0$, thus $\mathbb{Z}/4\mathbb{Z}$ is pseudo $\mathbb{Z}$-p-projective. Now it require to show that $\mathbb{Z}/4\mathbb{Z}$ is not $\mathbb{Z}$-projective. Let $f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$, be defined by $f(1 + 4z) = 2 + 8z$. Clearly f is non zero $\mathbb{Z}$-homomorphism, but f can not be lifted to a $\mathbb{Z}$-homomorphism from $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}$, since $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, 1\mathbb{Z}) = 0$ so $\mathbb{Z}/4\mathbb{Z}$ is not $\mathbb{Z}$-projective.

It is well known that every quasi principally projective module is quasi-pseudo principally projective but the converse is not necessarily true (see [3], Exercise 4.45(8)). In the following proposition authors provide the sufficient condition in terms of hollow module on quasi-pseudo principally projective module to be quasi principally projective.

PROPOSITION 2.4. *Every hollow quasi-pseudo principally projective module is quasi principally projective.*

Proof : Let M be hollow quasi-pseudo principally projective and N be M-cyclic submodule of M, let $f : M \to N$ be any homomorphism implies that $Imf \subseteq N$, if $Imf = 0$, so case is trivial. If $Imf \neq 0$, means that f is not surjective homomorphism, since N is hollow then it is easily check that $\pi - f$ is surjective homomorphism from M to N where $\pi : M \to N$ be surjective homomorphism. Then by quasi-pseudo principally projectivity of M there exists an R-endomorphism $g : M \to M$ such that $\pi.g = \pi - f$ which implise that $f = \pi.(1 - g)$. Which shows that M is quasi principally projective module.

PROPOSITION 2.5. *If N is pseudo M-p-projective then any epimorphism $f : M \to N$ splits. In addition, if M is indecomposable, then f is an isomorphism.*

Proof : Let $f : M \to N$ be an epimorphism then $M/Kerf \cong N$ with an R-isomorphism $g : M/Kerf \to N$, and so $g^{-1} : N \to M/Kerf$ is also an R-isomorphism. Since N is pseudo M-p-projective then g^{-1} can be lifted to an R-homomorphism $f' : N \to M$ such that $g^{-1} = \pi f'$ where $\pi : M \to M/Kerf$ is natural epimorphism. Thus $gg^{-1} = g\pi f'$ implies that $I_N = ff'$ which gives identity map on N so f splits. Now if M is indecomposable that is M can not be written as direct sum of its nonzero submodules therefore $Kerf = 0$ which shows that f is an R-isomorphism.

COROLLARY 2.6. *If N is M-p-projective then any epimorphism $f : M \to N$ splits, and if M is indecomposable then f is an isomorphism.*

LEMMA 2.7. *Let M and N be an R-modules then the following statements are equivalent :*
(1) *N is pseudo M-p-projective;*
(2) *for each $f \in S_M = EndM_R$, the set of all epimorphism in $Hom_R(N, f(M))$ = the set of all epimorphism in $f.Hom_R(N, M)$ or $\{g \in Hom_R(N, f(M)) : g\ is\ epi\} = \{g \in f.Hom_R(N, M) : g\ is\ epi\}$;*
(3) *For every submodule L of M every epimorphism $f : M \to L$ and $g : N \to L$, there exists an R-homomorphism $h : N \to M$ such that $f.h = g$.*

Proof : Proof is straightforward.

REMARK 2.8. *Every N-cyclic submodule of a M-cyclic submodule N is M-cyclic.*

PROPOSITION 2.9. *N is pseudo M-p-projective if and only if N is pseudo K-p-projective for every M-cyclic submodule K of M. In particular, if K is direct summand of M then N is both pseudo K-p-projective and pseudo M/K-p-projective.*

Proof : Prove is given in [11] Proposition 2.4.

PROPOSITION 2.10. *For an R-module M, the following statements are equivalent :*
(1) *M is quasi-pseudo principally projective module,*
(2) *For submodules N, K of M, and epimorphisms $f : M/N \to M/K$ and $g : M \to M/K$ there exists an R-homomorphism $h : M \to M/N$ with $g = fh$,*
(3) *For any direct summand L and submodule K of M with epimorphisms $f : L \to M/K$ and $g : M \to M/K$ there exists an R-homomorphism $h : M \to L$ with $g = fh$.*

Proof : Proof is on the same line as proposition 2.2 of Tiwary et.al. [15].

It is known from Chaturvedi et. al.[2], if M-cyclic submodule K of M is pseudo M-p-injective then K is direct summand of M but it is not true in case of pseudo M-p-projective.

PROPOSITION 2.11. *If M-cyclic submodule K of M is pseudo M-p-projective then K is isomorphic to some direct summand B of M.*

Proof : Since K is an M-cyclic submodule of M, therefore $K = s(M)$ for some $s \in EndM_R$. By pseudo M-p-projectivity of K implies that the epimorphism $s : M \to s(M) = K$ splits by [proposition 2.5], so $M = Kers \oplus B$ for some summand B of M, therefore $B \cong M/Kers \cong s(M) = K$.
In general it can not conclude that K itself is a direct summand, which is proved by the following example :

EXAMPLE 2.12. *Let $\mathbb{Z}$ denote the ring of integer. Consider $2\mathbb{Z}$ as a $\mathbb{Z}$-cyclic submodule of $\mathbb{Z}$. Now it require to show that $2\mathbb{Z}$ is pseudo $\mathbb{Z}$-p-projective. Let $n\mathbb{Z} \subset \mathbb{Z}$, $f : \mathbb{Z} \to n\mathbb{Z}$ and $g : 2\mathbb{Z} \to n\mathbb{Z}$ are epimorphisms, now let $h : 2\mathbb{Z} \to \mathbb{Z}$ be defined by $h(2k) = g(2k)/n$. Clearly h is a $\mathbb{Z}$-homomorphism and $f.h = g$. Therefore $2\mathbb{Z}$ is pseudo $\mathbb{Z}$-p-projective but it is seen that $2\mathbb{Z} \cong \mathbb{Z} \subseteq^{\oplus} \mathbb{Z}$ and $2\mathbb{Z}$ is not a direct summand of $\mathbb{Z}$.*

In the following proposition it is proved that pseudo M-p-projective module is closed under direct summand. Thus it is clear that for any R-module M, $Soc(M)$ is pseudo M-p-projective if and only if each simple submodule of M is pseudo M-p-projective.

PROPOSITION 2.13. *$\bigoplus_{i\in I} N_i$ is pseudo M-p-projective if and only if each N_i is pseudo M-p-projective.*

Proof : Assume that $\bigoplus_{i\in I} N_i$ is pseudo M-p-projective. Let $f(M)$ be an M-cyclic submodule of M, $f : M \to f(M)$ and $g : N_i \to f(M)$ are an epimorphisms, now define the epimorphism $g^* = g.\pi_{N_i} : \bigoplus_{i\in I} N_i \to f(M)$ where π_{N_i} is natural projection from $\bigoplus_i N_i \to N_i$ then $g^*(n_1, n_2, ...n_i, ...) = g(n_i)$ for $n_i \in N_i, i \in I$. Since $\bigoplus_{i\in I} N_i$ is pseudo M-p-projective there exists an R-homomorphism $h : \bigoplus_{i\in I} Ni \to M$ such that $f.h = g^*$ then $h^* = h|_{N_i}$ is an R-homomorphism which lifts g that is $f.h^* = g$. Therefore N_i is pseudo M-p-projective.
Conversely, let N_i be pseudo M-p-projective. Let $g \in Hom_R(\bigoplus_{i\in I} N_i, f(M))$ be epimorphism where $f \in EndM_R$ then $g|_{N_i} : N_i \to f(M) \forall\ i$, is an epimorphism. Since N_i is pseudo M-p-projective so clearly $f.h_i = g|_{N_i}$ for some $h_i \in Hom_R(N_i, M)$ now set $h = \bigoplus_i hi$ then $h : \bigoplus_i N_i \to M$ and $f.h = g$ so that, the set of all epimorphism in $Hom_R(\bigoplus_i N_i, f(M)) \subset$ the set of all epimorphism in $f.Hom_R(\bigoplus_i N_i, M)$ hence the set of all epimorphism in $Hom_R(\bigoplus_i N_i, f(M)) =$ the set of all epimorphism in $f.Hom_R(\bigoplus_i N_i, M)$ by lemma 2.7 . Therefore, $\bigoplus_i N_i$ is pseudo M-p-projective module.
Thus it is seen that pseudo M-p-projectivity is inherited by direct summand.

COROLLARY 2.14. *Every direct summand of quasi principally projective module is also quasi principally projective.*

PROPOSITION 2.15. *If M is quasi projective module and K is fully invariant submodule of M then M/K is quasi-pseudo principally projective module.*

Proof : proof is straightforward and hence omit it.

An R-module M is called Hopfian(resp. co-Hopfian), if every surjective (resp. injective) R-homomorphism $f : M \to M$ is an automorphism. For example every noetherian R-modules are Hopfian and every artinian R-modules are co-Hopfian. A module M is called directly finite, if M is not isomorphic to a proper summand of itself.

LEMMA 2.16. *(Proposition 1.25, Mohamed and Muller [7]). An R-module M is directly finite if and only if $f.g = 1$ implies $g.f = 1$ for any $f, g \in EndM_R$.*

In the following propositions pseudo M-p-projective module related with with Hopfian, co-Hopfian and directly finite modules.

PROPOSITION 2.17. *Every quasi-pseudo principally projective module M is Hopfian.*

Proof : Let f be any surjective endomorphism of M, $I_M : M \to M$ be an identity map on M. Since M is quasi-pseudo principally projective then there exists an R-epimorphism $g : M \to M$ such that $f.g = I_M$. Which gives that g is an automorphism on M, therefore $f = g^{-1}$ is an automorphism on M. Hence M is Hopfian.

COROLLARY 2.18. *Every quasi-principally projective module M is Hopfian.*

Proof : Proof is easy.

PROPOSITION 2.19. *Let M be pseudo M-p-projective co-Hopfian, then it is Hopfian.*

Proof : Let f be surjective endomorphism on M, $I_M : M \to M$ be an identity map on M. By pseudo M-p-projectivity of M there exists an R-homomorphism $g : M \to M$ such that $f.g = I_M$, implies that g is monomorphism. Since M is co-Hopfian, then it follows that $f = g^{-1}$ is an automorphism on M. Therefore M is Hopfian.

COROLLARY 2.20. *If M be quasi-principally projective co-Hopfian module, then M is Hopfian.*

PROPOSITION 2.21. *Let M be pseudo M-p-projective and N is fully invariant small submodule of M. Then M is Hopfian if and only if M/N is Hopfian.*

Proof : Assume that M/N is Hopfian. Let $f : M \to M$ be any epimorphism, then pseudo M-p-projectivity of M implies that f splits, by proposition 2.5, hence $K = Kerf$ is direct summand of M. Since N is fully invariant implies $f(N) \subset N$, now induced a map $f' : M/N \to M/N$ which is clearly an epimorphism, the Hopficity of M/N implies that $f' : M/N \to M/N$ is an isomorphism. Now by $(f'.\pi)(K) = (\pi.f)(K) = 0$, where $\pi : M \to M/N$ be natural epimorphism, it is seen that $\pi(K) = 0$, it means $K \subset N$, but $K \subset N \ll M$ implies that $K \ll M$. Since M is pseudo M-p-projective there exist a spliting for f, i.e. $K = Kerf$ is direct summand of M. Therefore $K = Kerf = 0$, implies that M is Hopfian.
Conversely, assume that M is Hopfian and $N \ll M$ if $f : M/N \to M/N$ is an epimorphism. Thus $f.\pi : M \to M/N$, where π is natural epimorphism from $M \to M/N$. Then by pseudo M-p-projectivity of M there exists $g \in EndM_R$, such that $\pi.g = f.\pi$ implies that g is an epimorphism by 19.2, Wisbauer(1991) [17] as π is a small epimorphism. Since M is Hopfian then g is an isomorphism. Assume $Kerf \neq 0$, then there exists $x \in M$ such that $f(x + N) = N \Rightarrow f.\pi(x) = \pi.g(x) = g(x) + N = N$ gives that $g(x) \in N \Rightarrow x \in g^{-1}(N) \subseteq N$. It follows that $Kerf = N$, therefore M/N is Hopfian.

COROLLARY 2.22. *Let M be finitely generated pseudo M-p-projective module. Then M is Hopfian if and only if $M/J(M)$ is Hopfian.*

Proof : It is known that $J(M)$ is fully invariant submodule of M. If M is finitely generated then $J(M) \ll M$. Thus by above proposition proof is obvious.

COROLLARY 2.23. *Let M be pseudo M-p-projective, N and L are submodules of M such that $N + L = M$ and $N \cap L \ll M$. Then M/N and M/L are Hopfian.*

Proof : It is known that $M/N \cap L = N/N \cap L \oplus L/N \cap L$, by proposition 2.18 and 2.21 $M/N \cap L$ is Hopfian, hence so its direct summand, as $N/N \cap L \cong N + L/L = M/L$, similarly $L/N \cap L \cong N + L/N = M/N$ is Hopfian.
The next proposition is the just generalization of Pandeya and Pandey (proposition 2.8)[9], whose proof is straightforward and hence omit it.

PROPOSITION 2.24. *Let M be finitely generated pseudo M-p-projective hollow module then M is directly finite if and only if each homomorphic image is directly finite.*

PROPOSITION 2.25. *Let M be a hollow R-module and N be an R- module. Then N is pseudo M-p-projective and every M-cyclic submodule of M is N-injective if and only if M is N-injective and every submodule of N is pseudo M-p-projective.*

Proof : Assume that M is N-injective and every submodule of N is pseudo M-p-projective. Let $s(M)$ be an M-cyclic submodule of M for any $s \in S_M = EndM_R$, N' be any submodule of N and let $f : N' \to s(M)$ be any homomorphism. Since $s(M)$ is hollow then f is an epimorphism. By pseudo M-p-projectivity of N' there exists a homomorphism $h : N' \to M$ such that $s.h = f$. Also M is N-injective then h can be extended to a homomorphism $g : N \to M$ such that $g.i = h$. Now take $g' = s.g : N \to s(M)$ which is an extension of f to N, therefore $s(M)$ is N-injective. Since N is submodule of itself therefore it is pseudo M-p-projective module.
Conversely, assume that N is pseudo M-p-projective and every M-cyclic submodule of M is N-injective. Let N' be any submodule of N, $i : N' \to N$ be an inclusion map and $s(M)$ be an M-cyclic submodule of M. Since $s(M)$ is N-injective then for any homomorphism $f : N' \to s(M)$ there exists a homomorphism $g : N \to s(M)$ such that $g.i = f$. Since $s(M)$ is hollow so $Img = s(M)$ consequently, g is an epimomorphism. By pseudo M-p-projectivity of N, g can be lifted to a homomorphism $h : N \to M$ such that $s.h = g$. Now take $h' = h.i : N' \to M$ be homomorphism, which lifts f and $sh' = s.h.i = g.i = f$. Therefore N' is pseudo M-p-projective. M is an M-cyclic submodule of itself therefore it is N-injective.

The following lemma is the generalization of lemma 1.1, [5], which is useful to characterize semi-simple rings in terms of pseudo M-p-projective module in corollary 2.27.

LEMMA 2.26. *A sufficient condition for short exact sequence $0 \to K \to P \overset{\lambda}{\to} Q \to 0$, to splits is that $P \oplus Q$ is pseudo M-p-projective module.*

Proof : Proof is easily obtained in the light of lemma 1.1, [5].

COROLLARY 2.27. *A sufficient condition for R to be semi-simple is that $R \oplus M$ be pseudo M-p-projective for every simple module M.*

Proof : M is simple then there exists a short exact sequence $0 \to K \to R \to M \to 0$, which splits by above lemma (simple module being pseudo M-p-projective), therefore every simple module is projective, which implies that R is semi-simple.

PROPOSITION 2.28. *Every pseudo M-p-projective module satisfy (D_2) condition.*

Proof : Let A be a direct summand of M and B is a submodule of M with $M/B \cong A$ with an R-isomorphism $f : M/B \to A$. Now define $f^* = f.\pi_B : M \to A$ where π_B is natural epimorphism of M onto M/B then f^* is an epimorphism and $Kerf^* = B$. Since A is direct summand of M therefore A is pseudo M-p-projective, then f^* splits and $M = Kerf^* \oplus N$ for some direct summand N of M. It follows that $B = Kerf^*$ is a direct summand of M hence M satisfies (D_2) condition.

COROLLARY 2.29. *Let M be a pseudo M-p-projective module. Then the followings are equivalent :*
(1) M is discrete module;
(2) M is quasi discrete module;
(3) M is lifting module.

Proof : (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows from proposition 2.28.

In general the following implication is given :

$$\begin{aligned} projective &\Rightarrow quasi\,projective \\ &\Rightarrow semi\,projective \\ &\Rightarrow quasi\,pseudo\,principally\,projective \\ &\not\Rightarrow discrete. \end{aligned}$$

3. WHEN PSEUDO M-P-PROJECTIVE MODULES IS DISCRETE ?

It is provided that sufficient condition for pseudo M-p-projective module to be discrete. Infact a pseudo M-p-projective module does not satisfy (D_1) condition always. Thus pseudo M-p-projective module with (D_1) condition is discrete. In the following proposition authors provide a necessary and sufficient condition for a pseudo M-p-projective module to be discrete.

PROPOSITION 3.1. *An indecomposable pseudo M-p-projective module M is discrete if and only if M is hollow.*

Proof : Suppose M is discrete then it satisfies (D_1) and (D_2) conditions. By (D_1) condition, for any submodule N of M there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M$. Indecomposability of M implies either $M_1 = 0$ or $M_2 = 0$, then proof is done.
Conversely, assume that pseudo M-p-projective module is hollow, Since hollow module is indecomposable and satisfies (D_1) condition and by proposition 2.28, M satisfies (D_2) condition. Therefore M is discrete.

A right R-module M is called a duo module, if every submodule of M is fully invariant. A module M is called a self-generator, if it generates all of its submodules. The following lemma is helpful in the proof of corollary 3.4.

LEMMA 3.2. *(lemma 2.1, C. Somchit[13]) Let M be a duo right R-module and N be its direct summand. Then*
(1) N is itself a duo module.
(2) If M is self-generator, N is also a self generator.

LEMMA 3.3. *Let M be pseudo M-p-projective if $S_M = EndM_R$ is local. Then for any non-trivial fully invariant M-cyclic submodules M_1 and M_2 of M, $M \neq M_1 + M_2$.*

Proof : Let $M_1 = s(M) \neq 0$ and $M_2 = t(M) \neq 0$ for $s, t, \in S_M = EndM_R$. Assume that $M = M_1 + M_2$ define the map $f : (s + t)M = M \to M/M_1 \cap M_2$ by $f(s + t)(m) = s(m) + M_1 \cap M_2$. Clearly f is well defined R-epimorphism by pseudo M-p-projectivity there exists $g \in S_M$ such that $\pi.g = f$ where $\pi : M \to M/M_1 \cap M_2$ is natural epimorphism. It follows that $\pi.g(s+t)(m) = \pi(s(m))$ then $((1-g)s-g.t)(M) \subseteq M_1 \cap M_2$. Since S_M is local then g or $1 - g$ is invertible. If $1 - g$ is invertible then $(s - (1 - g)^{-1}g.t)(M) \subseteq (1 - g)^{-1}(M_1 \cap M_2)$ thus $M_1 \subseteq (s-(1-g)^{-1}g.t)(M) \subseteq (1-g)^{-1}(M_1 \cap M_2) \subseteq M_1 \cap M_2$. Then $M_1 \subseteq M_1 \cap M_2$ which is a contradiction. Similarly if g is invertible then $M_2 \subseteq (g^{-1}(1 - g)s - t) \subseteq g^{-1}(M_1 \cap M_2) \subseteq M_1 \cap M_2$. Then $M_2 \subseteq M_1 \cap M_2$ which is contradiction to our assumption $M = M_1 + M_2$ and hence $M \neq M_1 + M_2$.

COROLLARY 3.4. *If M is pseudo M-p-projective duo module which is self-generator with local endomorphism ring. Then M is hollow hence it is discrete module.*

Proof : For any $0 \neq m \in M, mR$ contains a nonzero M-cyclic submodule, since M is self-generator. It is clear from above lemma that M is hollow.

LEMMA 3.5. *(lemma 1.1, Birkenmeier et.al.[1]) Suppose $M = \bigoplus_{i\in I} M_i$ be duo module then every submodule N of M is $N = \bigoplus_{i\in I}(N \cap M_i)$.*

THEOREM 3.6. *Let $M = \bigoplus_{i\in I} M_i$ be pseudo M-p-projective where each M_i is hollow. If M is duo module with $Rad(M) \ll M$, then M is discrete module.*

Proof : Suppose for any submodule N of M from above lemma $N = \bigoplus_{j\in J}(N\cap M_j)$ where $J \subset I$ and $N\cap M_j \neq 0$, since $N\cap M_j$ is small in M_j then $N = \bigoplus_{j\in J}(N \cap M_j) \ll \bigoplus_{i\in I} M_i = M$ that is $N \ll M$ thus M is hollow and hence discrete module.

PROPOSITION 3.7. *Suppose M is semi-simple duo module and $Rad(M) \ll M$. If M is self-generator then M is discrete module.*

Proof : Since M is semi-simple then $M = \bigoplus_{i\in I} M_i$ such that each M_i is simple then $End(M_i)$ is local. Then by lemma 3.2 each M_i is duo and self-generator. Since every semi-simple module is pseudo M-p-projective and every direct summand of pseudo M-p-projective module is again pseudo M-p-projective. Then by corollary 3.4 each M_i is discrete so from theorem 3.6, M is discrete module.

Suppose $M = M_1 \oplus M_2$ be a decomposition of R-module M, now authors assign some condition on M_1 and M_2 so that M is a discrete module.

PROPOSITION 3.8. *Let M_1 be simple module and M_2 a pseudo M-p-projective uniserial module with unique composition series $0 \subseteq M_2' \subset M_2$. Then $M = M_1 \oplus M_2$ is discrete module.*

Proof : It is well known that every simple module is pseudo M-p-projective. Then $M = M_1 \oplus M_2$ is pseudo M-p-projective which satisfies (D_2) condition by proposition 2.28. Now for discreteness it remains to show that M satisfy (D_1) condition. Let L be non zero submodule of M, it require to show that there exists a submodule M_1 of M such that $M = M_1 \oplus M_2$ with $M_1 \subseteq L$ and $L \cap M_2 \ll M$ for some submodule M_2 of M. If $M_1 \cap (L + M_2) = 0$ then $L \subseteq M_2$. Since $M = M_1 \oplus M_2$ and M_2 has unique composition series hence L is small submodule of M or direct summand of M. Now assume that $M_1 \cap (L + M_2) \neq 0$, then $M_1 \subseteq L + M_2$ and $M = L + M_2$, if (i) $L \cap M_2 = M_2$ and $L \cap M_1 = M_1$, (ii) $L \cap M_2 = 0$ and $L \cap M_1 = M_1$, (iii) $L \cap M_2 = M_2'$ and $L \cap M_1 = M_1$. Then it can easily verified that proof is done. Assume that $L \cap M_2 = M_2'$ and $L \cap M_1 = 0$ then $M_2' \subseteq L$, hence $M = L \oplus M_1$. Thus M satisfies (D_1) condition and therefore M is discrete module.

Let M_1 and M_2 be an R-module then M_1 and M_2 are said to be relatively projective, if M_1 is M_2 projective and M_2 is M_1 projective.

LEMMA 3.9. *The following statements are equivalent for a module $M = M_1 \oplus M_2$;*
(i) For each submodule N of M with $M = M_1 + N$ there exists a submodule N' of N such that $M = M_1 \oplus N'$.
(ii) M_1 and M_2 are relatively projective.

Proof : See ([17] 41.14, $(3) \Leftrightarrow (4)$) and ([7] lemma 4.47).

PROPOSITION 3.10. *Let the pseudo M-p-projective module $M = M_1 \oplus M_2$ be a direct sum of relatively projective module M_1 and M_2, such that M_1 is semi-simple and M_2 is lifting module then M is discrete module.*

Proof : Let L be a nonzero submodule of M, now assume that $M_1 \cap (L + M_2) \neq 0$ and let $M_1 \cap (L + M_2) = N$ then for some submodule N' of M_1, then $M_1 = N \oplus N'$ and hence $M = N \oplus N' \oplus M_2 = L + (N' \oplus M_2)$. Then by ([7] prop. 4.31, prop. 4.32, prop. 4.33), N is $M_2 \oplus N'$ projective. Now from above lemma, there exists a submodule L' of L such that $M = L' \oplus (M_2 \oplus N')$. Assume $L \cap (M_2 \oplus N') \neq 0$, let K be any submodule of M_2. Since $L \cap (K + N') \subseteq K \cap (L + N') + N' \cap (L + K)$ and $N' \cap (L + K) = 0$, then $L \cap (K + N') \subseteq K \cap (L + N')$ similarly $K \cap (L + N') \subseteq L \cap (K + N')$, therefore $K \cap (L + N') = L \cap (K + N')$ for every submodule K of M_2. Since M_2 is lifting there exist a submodule A_1 of $M_2 \cap (L + N') = L \cap (M_2 \oplus N')$ such that $M_2 = A_1 \oplus A_2$ and $A_2 \cap (L + N') \ll A_2$ for some $A_2 \subseteq M_2$, thus $M = (L' \oplus A_1) \oplus (A_2 \oplus N'), L' \oplus A_1 \subseteq L$ and $L \cap (A_2 \oplus N') = A_2 \cap (L + N')$ is small in $A_2 \oplus N'$.
Now assume that $M_1 \cap (L + M_2) = 0 \Rightarrow L \subseteq M_2$. Since M_2 is lifting there exists a submodule A_1 of L such that $M_2 = A_1 \oplus A_2$ and $L \cap A_2 \ll A_2$ for some submodule A_2 of M_2. Hence $M = M_1 \oplus A_1 \oplus A_2 = A_1 \oplus (M_1 \oplus A_2)$ and $L \cap (M_1 \oplus A_2) = L \cap A_2$ is small in $M_1 \oplus A_2$. It follows that M satisfies (D_1) condition and by proposition 2.28, M is discrete module.

COROLLARY 3.11. *Let M_1 be semi simple and M_2 a module with $RadM_2 = M_2$. Then pseudo M-p-projective module $M = M_1 \oplus M_2$ is discrete if and only if M_1 and M_2 is relatively projective and M_2 is lifting.*

Proof : Sufficient part is clear from the above proposition. Conversely, assume $M = M_1 \oplus M_2$ is discrete, implies that M_2 has (D_1) and (D_2) condition, by lemma 4.7 [7], since M_1 is semi simple, M_2 is M_1-projective. Now it require to prove that M_1 is M_2-projective. Let N be a submodule of M with $M = N + M_2$, by prop. 4.8 [7], there exists a submodule N_1 of N such that $M = N_1 + M_2 = N_1 \oplus N_2$ and $N_1 \cap M_2 \ll N_1$ for some submodule N_2 of M. It follows easily that $RadN_1 = N_1 \cap M_2$. Since $RadM = RadN_1 \oplus RadN_2 = M_2$, then $N_1 \cap M_2$ is a direct summand of N_1. Hence $M = N_1 \oplus M_2$. By lemma 3.9, M_1 is M_2-projective.

An R-module M is called refinable (or suitable) if, for any submodules M_1, M_2 of M with $M_1 + M_2 = M$, there exist a direct summand M_1' of M with $M_1' \subseteq M_1$ and $M_1' + M_2 = M$.

Moreover if there exist a direct summand M_2' of M with $M_2' \subseteq M_2$ with $M = M_1' \oplus M_2'$, then M is said to be strongly refinable. For example semisimple modules, hollow modules are strongly refinable. A finitely generated module M in which every finitely generated submodule is a direct summand is strongly refinable, such modules are called regular module.

LEMMA 3.12. *(11.28, Clark et. al. [3]). Let M be a quasi-pseudo principally projective module with $S_M = EndM_R$. Then the following conditions are equivalent :*
(1) M is strongly refinable;
(2) $M/RadM$ is refinable and direct summands lift modulo $RadM$;
(3) S_M is left refinable.

LEMMA 3.13. *Let M be strongly refinable module;*
(1) If M_1 is a direct summand of M, then M_1 is strongly refinable.
(2) If $M = M_1 + M_2 + ... + M_n$, then $M = M_1' \oplus M_2' \oplus$

.... $\oplus M'_n$ where $M'_i \subseteq M_i$ for i=1,2,.....n. In particular, if M_i is cyclic then M'_i is cyclic. In other words if M is finitely generated then it is a direct sum of cyclic submodules.
(3) *If M has no infinite direct sum of submodules of M, then M is strongly $\oplus$-supplemented.*

Proof : Proof is straightforward.

Following lemmas generalize some results of Nicholson [8], for pseudo M-p-projective modules.

LEMMA 3.14. *If M is pseudo M-p-projective modules, $M = M_1 + M_2 + ... + M_n$ where M_i are submodules of M, then there exist $f_i \in EndM_R$ such that $f_i(M) \subseteq M_i$ for each i and $f_1 + f_2 + ... + f_n = I_M$.*

Proof : Proof is similar to Lemma 2.7 of [8].

LEMMA 3.15. *Let M be quasi-pseudo principally projective module and suppose $M = M_1 + M_2$ where M_1 is direct summand of M and M_2 is a submodule, then there exists $M'_2 \subseteq M_2$ such that $M = M_1 \oplus M'_2$.*

Proof : Proof is similar to Lemma 2.8 of [8].

PROPOSITION 3.16. *The following statements are equivalent for quasi-pseudo principally projective modules :*
(1) *M has finite exchange property;*
(2) *If $M = \sum_{i=1}^{n} M_i$ where M_i are submodules there exists a decomposition $M = \bigoplus_{i=1}^{n} M'_i$ with $M'_i \subseteq M_i$ for each $i = 1, 2, ...n$;*
(3) *If $M = M_1 + M_2$ where M_1 and M_2 are submodules there exists summand $M'_1 \subseteq M_1$ of M and $M = M'_1 + M_2$.*

Proof: (1) $\Rightarrow$ (2). If $M = \sum_{i=1}^{n} M_i$ by Lemma 3.14 there exists $f_i \in EndM_R$ such that $f_i(M) \subseteq M_i$ for each i and $f_1 + f_2 + ... + f_n = I_M$. By Prop. 1.11 of Nicholson [8] there exist orthogonal idempotents $g_i \in (EndM_R)f_i$ such that $g_1 + g_2 + ...g_n = I_M$ then (2) follows with $g_i(M) = M_i$.
(2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (1). Let $f_1, f_2 \in EndM_R$ be such that $f_1 + f_2 = 1$. Then $M = f_1(M) + f_2(M)$ therefore, by (3) and Lemma 3.15, let $M = M_1 \oplus M_2$ where $M_i \subseteq f_i(M)$ for each i. Let g_1, g_2 be idempotents in $EndM_R$ with $g_1 + g_2 = 1$ and $g_i(M) = M_i$. There exist $h_i \in EndM_R$ such that $f_i.h_i = g_i$. Hence $EndM_R$ is refinable and (1) follows by lemma 3.12 and 11.31 of Clark et.al. [3].

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