On The Effect of Random Variations in the Parameter of Arrival and Service Time Distributions on Various Queue Characteristics in (M/M/1): (∞/FIFO) Queue System Model

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ABSTRACT
The present study deal with the development of statistical methodology for updating the basic arrival and service time distributions in respect of prior variations. These updated arrival and services time distributions provide us the modified form of traffic intensity as the ratio of updated mean service time to updated mean inter-arrival time. As such all the basic queue characteristics will be developed in the changed scenario.

Keywords
Traffic Intensity, Arrival and Service Rates, Prior Distribution, Compound Distribution.

1. INTRODUCTION
The characteristics of various queue systems have been analyzed by using their respective probability mass functions (p.m.f.). With exponential arrival and service time distributions, the traffic intensity is defined as the ratio of the arrival rate to the service rate and various queue's system have been defined using this parameter in studies like (Ackoff and Sasiei, 1968; Taha, 1976). In reliability theory, this ratio is also known as availability ratio. Sharma and Bhutani (1992, 1994) studied the posterior analysis of the system availability and developed the Bay’s point and interval estimators for availability by assuming the prior distribution for the parameter involved in failure and repair time distributions. Following the concept, a study regarding statistical inferences on various important performance measures has been done by Sharma and Kumar (1999) for a (M/M/1) (∞ / FIFO) queue system model. In this continuation, Maurya (2004, 2005), Gupta et. al. (2009) have confined their considerable attention to analyze different types of queue system models regarding statistical inferences on its useful characteristics. However, while recording arrival and service information for a large interval of time, it is reasonable to assume random variations in the parameter involve in the arrival and service time distributions. Here it should be recognized that prior do have an impact on the basic distribution and therefore updating these basic distributions in its parameter is another important aspect for analyzing the queue characteristics of the model in changed scenario. The variations in the parameter can be neutralized by averaging as we do in the case of compound distribution of the concern variable (Johnson and Kotz, 1969).

In the light of the above discussion, the present study deals with the development of the methodology for updating the basic distributions in respect of prior variations in arrival and service time distributions. These updated distributions have been used to study the robust character of various queue characteristics in (M / M / 1): (∞ / FIFO) queue system model.

2. PRELIMINARY IDEAS AND ASSUMPTIONS USED
(a) The probability mass function (p.m.f) of the system size distribution of the typical queue in (M/M/1):(∞/FIFO) queue system model is given by

$$P(X=x)=(1-\rho)\rho^{x} \quad ; \quad 0<\rho<\infty \;, x=0,1,2...; \qquad \text{……(1)}$$

where the notations used in (1) have their significance as following -

- $P(X = x)$: Stationary probability that there are x customers in the system
- ρ : The traffic intensity i.e. $\rho = \dfrac{\lambda}{\mu}$ i.e. the ratio of the mean service time to mean inter-arrival time .
- λ , μ : Poisson parameter representing respectively current requirement rate and current supply rate.

(b) Based on p.m.f. in (1), some important characteristics of the system are

(i) Average queue length, say L_s, is

$$L_s = E(X) = \frac{\rho}{(1-\rho)} = \frac{\lambda}{(\mu-\lambda)}$$

(ii) Average length of waiting line L_q, will be

$$L_q = E(X-1) = \frac{\rho^2}{(1-\rho)} = \frac{\lambda^2}{\mu(\mu-\lambda)^2}$$

(iii) The expected length of non empty queue (L / L > 0) will be

$$(L/L>0) = \frac{1}{(1-\rho)} = \frac{\mu}{(\mu-\lambda)}$$

(iv) The probability of minimum queue size being n_0, i.e.

$$Q_m = P(x \geq n_0) = \rho^{n_0} = \left(\frac{\lambda}{\mu}\right)^{n_0}$$

(v) The variance of the queue length will be

$$V_s = E(X-E(X))^2 = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

(vi) The Co-efficient of variation for x, say CV, will be

$$CV_s = \frac{\sqrt{V_s}}{L_s} * 100$$

3. STATISTICAL BACKGROUND

For developing the procedure, it is assumed that

(a) The arrival time distribution for the system is exponential with p.d.f.

$$f(t_1, \lambda) = \lambda\, e^{-\lambda t_1} \;\; ; \; (t_1, \lambda) > 0 \qquad ..(2)$$

Here λ is the arrival rate and

$$E(T_1) = \text{Mean Inter-arrival time} = \frac{1}{\lambda},$$

$$V(T_1) = \frac{1}{\lambda^2}.$$

(b) The service time distribution for the system is also exponential with p.d.f.

$$f(t_2, \mu) = \mu\, e^{-\mu t_2} \qquad ; \; (t_2, \mu) > 0 \qquad ..(3)$$

Here μ is the survival rate and

$$E(T_2) = \text{Mean service time} = \frac{1}{\mu},$$

$$V(T_2) = \frac{1}{\mu^2}.$$

(c) The designer's prior belief about the arrival rate λ of the system is represented by a gamma distribution having p.d.f.

$$g\;(\lambda, m, n) = \frac{m^n}{\Gamma n}\, \lambda^{n-1}\, e^{-m\lambda} \;\; ; (\lambda, m, n) > 0 \; ..(4)$$

Here, $E(\lambda) = n / m$, $V(\lambda) = \frac{n}{m^2}$ Similarly,

the prior distribution for the service rate μ is taken as

$$g\;(\mu, u, v) = \frac{u^v}{\Gamma v}\, \mu^{v-1}\, e^{-u\mu} \;\; ; \; (\mu, u, v) > 0 \;\; ..(5)$$

Here, $E(\mu) = v / u$, $V(\mu) = \frac{v}{u^2}$

(d) In view of (2) and (4) the compound distribution of T_1 is

$$f_1(t_1, a, b) = \int_o^\infty f(t_1, \lambda)\; g\,(\lambda, m, n)\; d\lambda$$

$$= \frac{n\,(m)^n}{(t_1 + m)^{b+1}}$$

...(6)

Here, $E(T_1)$ = updated the mean inter-arrival time = $\frac{m}{n-1}$, $V(T_1) = \frac{m^2 n}{(n-1)^2 (n-1)}$

In the compounding process, the variations in λ get neutralized on taking expectation over the function $f_1(t_1, \lambda)$ in respect of λ for fixed T_1. Consequently, the compound distribution in (6) is interpreted as an updated basic arrival time distribution when the variation in λ have been condensed or neutralized.

(e) Similarly, In view of (3) and (5) the compound distribution of T_2 is

$$f_1(t_2, u, v) = \int_o^\infty f(t_2, \mu)\; g\,(\mu, u, v)\; d\mu$$

$$= \frac{v\,u^v}{(t_2 + u)^{v+1}}$$

...(7)

Here, $E(T_2)$ = updated mean service time = $\frac{u}{v-1}$ and

$$V(T_2) = \frac{u^2 v}{(v-1)^2 (v-1)}$$

As per earlier argument, the compound distribution in (7) is also interpreted as an updated basic service time distribution. Thus, in the process one gets a basic and an updated basic service time distributions in (3) and (7) respectively. Similarly, a basic and updated basic arrival time distributions are given in (2) and (6) respectively. These distributions enable us to analyse the queue characteristics of the model in the following two specific situations.

(i) When the basic arrival and service time distributions as given in (2) and (3) respectively used in the analysis, where the parameters involved are treated as constant

(ii) When the updated basic arrival and service time distributions as given in (6) and (7) respectively used in the analysis which accounts for variations in the parameters of basic distributions in (4) and (5).

4. QUEUE CHARACTERISTICS OF THE MODEL WHEN THE PARAMETERS INVOLVED IN ARRIVAL AND SERVICE TIME DISTRIBUTIONS ARE TAKEN TO BE RANDOM VARIABLE

In this case, the respective updated arrival (T_1) and service time (T_2) distributions, as given in (6) and (7) respectively, are used for getting expressions for various queue characteristics of the model with identical conditions in changed setup. On using these distributions, initially, the traffic intensity (ρ) can be modified as the ratio of the updated mean service time to updated mean inter-arrival time. Mathematically

$$\rho^* = \frac{E(T_2)}{E(T_1)} = \frac{u(n-1)}{m(v-1)}$$

Now, when ρ is modified as above, various queue characteristics of the present model are updated as under :

(i) Average queue length, say L^*_s, is

$$L^*_s = \frac{\rho^*}{(1-\rho^*)} = \frac{u\,(n-1)}{m\,(v-1)-u\,(n-1)}$$

(ii) Average length of waiting line L^*_q, will be

$$L^*_q = \frac{\rho^{*2}}{(1-\rho^*)} = \frac{u^2\,(n-1)^2}{m(v-1)[m\,(v-1)-u\,(n-1)]}$$

(iii) The probability of minimum queue size being n_0, i.e.

$$Q^*_m = (\rho^*)^{n_0} = \left(\frac{u(n-1)}{m(v-1)}\right)^{n_0}$$

(iv) The expected length of non empty queue, ($L^*/L^*>0$), will be

$$\left(L^*/L^*>0\right) = \frac{1}{(1-\rho^*)} = \frac{m(v-1)}{[m\,(v-1)-u\,(n-1)]}$$

(v) The variance of the queue length will be

$$V^*_s = \frac{\rho^*}{(1-\rho^*)^2} = \frac{mu(n-1)(v-1)}{[m\,(v-1)-u\,(n-1)]^2}$$

(vi) The Co-efficient of variation for X, say CV^*_s, will be

$$CV^*_s = \frac{\sqrt{V^*_s}}{L^*_s} * 100$$

5. DISCUSSION

For analyzing the effect of random variations in arrival and service time distributions, we compare the queue characteristics as given in subsection 2 (b) with those given in section 4.0. Choosing suitability of the parameters, the corresponding value of the $\lambda = E(\lambda) = \frac{n}{m}$ and $\mu = E(\mu) = \frac{v}{u}$ may be used for analyzing the effect of random variations on various queue characteristics. The estimates of the various queue characteristics with variations in λ [or E(λ)], (while μ is fixed) are shown in [Table-1]and [Table-2] respectively. Similarly, for fixed λ, and with variations in μ [or E(μ)] , the estimates of the various queue characteristics are shown in Table-3 [Table-4] respectively **.**

6. ANALYSIS

From Table-1 and 2, we observe that when λ = 0.50, the corresponding value of L_s is 1.667. However, for [m = 10, n = 5 or λ = E (λ) = 0.50], [m = 10, n = 6 or λ = E (λ) = 0.60], and [m = 10, n = 7 or λ = E (λ) = 0.70] the corresponding values of L_s, say L^*_s are 1.333, 2.5000 and 6.000 respectively. The trends in Table-1and 2 values clearly reveals that these updated values are tend to be smaller up to a certain point, but thereafter, these values tend to be uniformly higher when E(λ) ≥λ. On the other hand, for E(λ) < λ, the

By the same way from Table-3 and 4, we observe that when μ = 0.60, the corresponding value of Ls is 0.5000. However, for [u = 10, v = 6 or μ = E (μ) = 0.60], [u = 10, v = 7 or μ = E (μ) = 0.70], and [u = 10, v = 8 or μ = E (μ) = 0.80] the corresponding values of Ls, say L^*_s are 0.2500, 0.1667 and 0.1429 respectively. These trends clearly reveals that all these updated values are tend to be smaller up to a certain point, but thereafter, these values tend to be uniformly higher when $E(\mu) \le \mu$. On the other hand, for E(λ) > λ, the updated values of queue characteristics are tend to be uniformly smaller.

On comparing the variations in CV^*_s in respect of λ and E(λ) in Table-1 and 2 respectively, we observe that the estimates tend to be more and more consistent as either λ or E(λ) increases. Similarly, on comparing the variations in CV^*_s in respect of μ and E(μ) in Table-3 and 4 respectively, we observe that the estimates tend to be less and less consistent as either μ or E(μ) increases. Finally, the conclusion is that, the queue characteristics of a (M/M/1):(∞/FIFO) queue system model are observed to be non-robust and as such, this model should be very cautiously used whenever we suspect variations in arrival and service rates or in ρ.

Table-1

Estimates of queue characteristics when basic arrival and service time distributions are used (μ = E(μ) = 0.8)

λ	L_s	L_q	Q_m	V_s	$(L/L>0)$	CV_s
0.2	0.3333	0.0833	0.0156	0.4444	1.3333	200.01
0.3	0.6000	0.2250	0.0527	0.9600	1.6000	163.29
0.4	1.0000	0.5000	0.1250	2.0000	2.0000	141.42
0.5	1.6667	1.0417	0.2441	4.4444	2.6667	126.49
0.6	3.0000	2.2500	0.4219	12.0000	4.0000	115.47
0.7	7.0000	6.1250	0.6699	56.0000	8.0000	106.90

Table-2

Estimates of queue characteristics when updated basic arrival and service time distributions are used (m=10, μ = E(μ)=0.8, v=8, u=10)

n	λ=E(λ)	L^*_s	L^*_q	Q^*_m	V^*_s	$(L^*/L^*>0)$	CV^*_s
2	0.2	0.1667	0.0238	0.0029	0.1944	1.1667	264.49
3	0.3	0.4000	0.1143	0.0233	0.5600	1.4000	187.08
4	0.4	0.7500	0.3214	0.0787	1.3125	1.7500	152.75
5	0.5	1.3333	0.7619	0.1865	3.1111	2.3333	132.29
6	0.6	2.5000	1.7857	0.3644	8.7500	3.5000	118.32
7	0.7	6.0000	5.1428	0.6297	42.0000	7.0000	108.01

Table-3

Estimates of queue characteristics when basic arrival and service time distributions are used(λ = E(λ) = 0.2)

μ	L_s	L_q	Q_m	V_s	$(L/L>0)$	CV_s
0.3	2.0000	1.3333	0.2963	6.0000	3.0000	122.47
0.4	1.0000	0.5000	0.1250	2.0000	2.0000	141.42
0.5	0.6667	0.2667	0.0640	1.1111	1.6667	158.11
0.6	0.5000	0.1667	0.370	0.7500	1.5000	173.21
0.7	0.4000	0.1143	0.0233	0.5600	1.4000	187.08
0.8	0.3333	0.0833	0.0156	0.4444	1.3333	200.01
0.9	0.2857	0.0127	0.0109	0.3673	1.2857	212.13

Table-4

Estimates of queue characteristics when updated basic arrival and service time distributions are used(u=10, λ = E(λ) = 0.2, m=10, n=2)

v	μ=E(μ)	L^*_s	L^*_q	Q^*_m	V^*_s	$(L^*/L^*>0)$	CV^*_s
3	0.3	1.0000	0.5000	0.1250	2.0000	2.0000	141.42
4	0.4	0.5000	0.1667	0.0370	0.7500	1.5000	173.21
5	0.5	0.3333	0.0833	0.0156	0.4444	1.3333	200.01
6	0.6	0.2500	0.0500	0.0080	0.3125	1.2500	223.61
7	0.7	0.2000	0.0333	0.0046	0.2400	1.2000	244.95
8	0.8	0.1667	0.0238	0.0029	0.1944	1.1667	264.49
9	0.9	0.1429	0.0178	0.0019	0.1633	1.1428	282.79

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