Convergence of CR Iterative Scheme with Errors using Quasi-Contractive Operators

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ABSTRACT
The aim of this article is to introduce a new iterative scheme namely CR iterative scheme with errors and prove a general convergence theorem to approximate the unique common fixed point of three operators satisfying a certain contractive condition in an arbitrary Banach space using this newly introduced iterative scheme. An example showing the validity of our results is provided. Comparative analysis of new iterative scheme with already existing iterative schemes is also shown using programming in C++.

General Terms
Computational Mathematics

Keywords
CR Iterative Scheme; Quasi-contractive Operators.

1. INTRODUCTION AND PRELIMINARIES
In the recent years, fixed and common points of contractive type self operators have been approximated by various authors using different iterative schemes [2-5, 9, 10, 12-19]. Let (X, d) be a complete metric space and $T: X \longrightarrow X$ a selfmap of X. Suppose that $F_T = \{ p \in X,\ Tp = p\}$ is the set of fixed points of T. In a complete metric space, the Picard iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, \ldots \tag{1.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality

$$d(Tx, Ty) \le \alpha\, d(x, y), \text{ for all } x, y \in X \text{ and } \alpha \in [0, 1). \tag{1.2}$$

Condition (1.2) is called the Banach's contraction condition.

After consideration of errors terms in iterative schemes, in 1995, Liu[10] introduced iterative scheme with errors as follows :

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n$$

$$y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \qquad n \ge 0, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] and u_n, v_n are summable sequences in K, K being a closed, convex subset of a Banach space X.

But in succeeding years it was observed that the iterative process with errors introduced by Liu [10] was not satisfactory. The errors occur in a random way. The conditions imposed on the errors terms in (1.3) which say that they tend to zero as n tends to infinity are, therefore, unreasonable. In [19] Xu introduced Ishikawa iterative with errors with a more satisfactory error terms:

$$x_{n+1} = \alpha_n x_n + \alpha_n^1 T y_n + a_n u_n$$

$$y_n = \beta_n x_n + \beta_n^1 T x_n + b_n v_n, \quad n \ge 0, \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{\alpha_n^1\}, \{\beta_n^1\}$ are sequences in [0,1] with $\alpha_n + \alpha_n^1 + a_n = \beta_n + \beta_n^1 + b_n = 1$ and u_n, v_n, are summable sequences in K.

If $b_n = a_n = \alpha_n^1 = \beta_n^1 = 0$, then (1.4) reduces to Mann iterative scheme with errors:

$$x_{n+1} = \alpha_n x_n + \alpha_n^1 T x_n + a_n u_n,\ n \ge 0. \tag{1.5}$$

In 2002, Agarwal et al. [5] studied the following iterative scheme:

$$x_{n+1} = \alpha_n T x_n + \alpha_n^1 T y_n$$

$$y_n = \beta_n x_n + \beta_n^1 T x_n,\ n \ge 0, \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\alpha_n^1\}, \{\beta_n^1\}$ are sequences in [0,1] with $\alpha_n + \alpha_n^1 = \beta_n + \beta_n^1 = 1$.

In 2009, Khan [8] studied the convergence of following iterative scheme with errors to approximate the common fixed point of two Z operators:

$$x_{n+1} = \alpha_n x_n + \alpha_n^1 T_1 y_n + a_n u_n$$

$$y_n = \beta_n x_n + \beta_n^1 T_2 x_n + b_n v_n \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{\alpha_n^1\}, \{\beta_n^1\}$ are sequences in [0,1] with $\alpha_n + \alpha_n^1 + a_n = \beta_n + \beta_n^1 + b_n = 1$ and u_n, v_n, are summable sequences in K.

In 1972, Zamfirescu [20] obtained the following theorem:

Theorem 1.1. Let (X, d) be a complete metric space and T : X $\longrightarrow$ X a mapping for which there exists the real numbers *a*, *b* and *c* satisfying $a \in (0,1)$, b , $c \in (0, \frac{1}{2})$ such that for each pair $x, y \in$ X at least one of the following conditions hold

(*i*) $d(Tx, Ty) \le a\, d(x, y)$

(*ii*) $d(Tx, Ty) \le b\, [d(x, Tx) + d(y, Ty)]$

(*iii*) $d(Tx, Ty) \le c\, [d(x,Ty) + d(y, Tx)]$. (1.8)

Then *T* has a unique fixed point *p* and the Picard iterative scheme $\{x_n\}$ defined by (1.1) converges to *p* for any arbitrary but fixed $x_0 \in X$.

The operators satisfying the condition (1.8) are called Zamfirescu operators.

Berinde[3] introduced a new class of operators on an arbitrary Banach space satisfying

$$d(Tx, Ty) \le 2\, \delta\, d(x, Tx) + \delta\, d(x, y) \, , \quad (1.9)$$

$\forall$ x, y $\in$ X and $\delta \in [0,1)$. He proved that this class is wider than the class of Zamfirescu operators and used the Ishikawa iterative process to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem.

Theorem 1.2[3] Let *K* be a nonempty closed convex subset of an arbitrary Banach space *X* and $T : K \longrightarrow K$ a mapping satisfying (1.9). Let $\{s_n\}_{n=0}^{\infty}$ defined through the Ishikawa iterative (1.4) and $x_0 \in K$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers in [0,1] with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of *T*.

In 2012, for $\alpha_n, \beta_n, \gamma_n \in [0,1]$, Chugh and Kumar[5] introduced the CR iterative scheme as below:

$$x_{n+1} = (1-\alpha_n) y_n + \alpha_n T y_n$$

$$y_n = (1-\beta_n) T x_n + \beta_n T z_n$$

$$z_n = (1-\gamma_n) x_n + \gamma_n T x_n \quad , \quad \text{(CR)}$$

Inspired by the above mentioned works, for contractive type operators T_1,T_2 and T_3, we define the CR iterative scheme with errors as below:

$$x_{n+1} = a_n y_n + b_n T_1 y_n + c_n u_n$$

$$y_n = a_n^{'} T_3 x_n + b_n^{'} T_2 z_n + c_n^{'} v_n$$

$$z_n = a_n^{''} x_n + b_n^{''} T_3 x_n + c_n^{''} w_n \, , \; n \ge 0 \, , \quad (1.10)$$

where $\{a_n\}$,$\{b_n\}$,$\{c_n\}$,$\{a_n^{'}\}$,$\{b_n^{'}\}$,$\{c_n^{'}\}$,$\{a_n^{''}\}$,$\{b_n^{''}\}$,$\{c_n^{''}\}$ are sequences in [0,1] with $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$ and u_n, v_n, w_n, are summable sequences in *K*.

Remarks:

1. If $b_n = c_n = b_n^{'} = c_n^{'} = 0$ and T_3=T, then (1.10) reduces to Mann iterative scheme with errors:

$$x_{n+1} = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} w_n \, , \quad n \ge 0 \, . \quad (1.11)$$

2. If $b_n = c_n$ =0 and T_3=T_2=T, then (1.10) reduces to Agarwal et al. iterative scheme with errors:

$$x_{n+1} = a_n^{'} T x_n + b_n^{'} T z_n + c_n^{'} v_n$$

$$z_n = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} w_n \, , \quad n \ge 0 \, . \quad (1.12)$$

To prove our main results, we use the following Lemma:

Lemma 1.3.[2] If δ is a real number such that $0 \le \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying

$$u_{n+1} \le \delta u_n + \epsilon_n \, , \; n = 0, 1, 2, \ldots$$

we have $\lim_{n\to\infty} u_n = 0$.

The purpose of this paper to study the strong convergence of CR iterative scheme with errors (1.10) to the common fixed point of three quasi-contractive operators satisfying the following contractive condition: there exists $a \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \longrightarrow R^+$ with $\varphi(0) = 0$, such that

$$\| Tx - Ty \| \le \varphi(\| x - Tx \|) + a \| x - y \| \quad \forall \; x, y \in X. \quad (1.13)$$

2. MAIN RESULTS

Theorem 2.1. Let *K* be a nonempty closed convex subset of an arbitrary Banach space *X* and $T_i : K \longrightarrow K$ (i=1,2,3) be three quasi-contractive operators satisfying (1.13). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the CR iterative scheme with errors (1.10) and $x_0 \in K$, where $\{a_n\}$,$\{b_n\}$,$\{c_n\}$,$\{a_n^{'}\}$,$\{b_n^{'}\}$ $\{c_n^{'}\}$,$\{a_n^{''}\}$,$\{b_n^{''}\}$,$\{c_n^{''}\}$ are sequences in [0,1] are sequences in [0,1] satisfying $\lim c_n^{'} = 0 = \lim c_n^{''} = 0$. If $F(T_1) \cap F(T_2) \cap F(T_3) \ne \phi$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common fixed point of T_1, T_2 and T_3.

Proof : If $p \in F(T_1) \cap F(T_2) \cap F(T_3) \ne \phi$, then it follows from Theorem 1.1 that *p* is a unique common fixed point of T_1, T_2 and T_3.

Since K is bounded, we can choose a number M such that

M= $\text{Max}\{\sup_{n\geq 0}\|u_n - y_n\|, \sup_{n\geq 0}\|v_n - T_3x_n\|, \sup_{n\geq 0}\|w_n - x_n\|\}$,

therefore from (1.10), we have

$$||x_{n+1} - p|| = \| a_n y_n + b_n T_1 y_n + c_n u_n - (a_n + b_n + c_n) p \|$$

$$= \| (1-b_n)(y_n - p) + b_n (T_1 y_n - p) + c_n (u_n - y_n) \|$$

$$\leq (1-b_n)\,||y_n - p|| + b_n\,||T_1 y_n - p|| + c_n\,||u_n - y_n||$$

$$\leq (1-b_n)\,||y_n - p|| + b_n \delta\,||y_n - p|| + c_n M$$

$$= (1 - b_n(1-\delta))\,||y_n - p|| + c_n M.$$

(2.1) Now, we have the following estimates :

$$||y_n - p|| = \| a'_n T_3 x_n + b'_n T_2 z_n + c'_n v_n - (a'_n + b'_n + c'_n) p \|$$

$$= \| (1-b'_n)(T_3 x_n - p) + b'_n (T_2 z_n - p) + c'_n (v_n - T_3 x_n) \|$$

$$\leq (1-b'_n)\delta \| x_n - p \| + b'_n \delta \| z_n - p \| + c'_n M \qquad (2.2)$$

and

$$||z_n - p|| = \| a''_n x_n + b''_n T_3 x_n + c''_n w_n - (a''_n + b''_n + c''_n) p \|$$

$$= \| (1-b''_n)(x_n - p) + b''_n (T_3 x_n - p) + c''_n (w_n - x_n) \|$$

$$\leq (1-b''_n) \| x_n - p \| + b''_n \| T_3 x_n - p \| + c''_n \| w_n - x_n \|$$

$$\leq (1-b''_n) \| x_n - p \| + b''_n \delta \| x_n - p \| + c''_n M$$

$$= (1 - b''_n(1-\delta)) \| x_n - p \| + c''_n M \ .$$

(2.3)

Using estimates (2.2) and (2.3), (2.1) yields

$$\| x_{n+1} - p \| \leq (1 - b_n(1-\delta))[(1-b'_n)\delta \| x_n - p \|$$
$$+ b'_n \delta (1 - b''_n(1-\delta)) \| x_n - p \|]$$
$$+ (1 - b_n(1-\delta)) b'_n \delta c''_n M$$
$$+ (1 - b_n(1-\delta)) c'_n M + c_n M$$
$$\leq (1 - b_n(1-\delta)) \| x_n - p \| + M(c_n + c'_n + c''_n)$$

(2.4)

Since $\lim_{n\to\infty} c_n = \lim_{n\to\infty} c'_n = \lim_{n\to\infty} c''_n = 0$, hence using Lemma 1, (2.4) yields $\lim_{n\to\infty} \|x_n - p\| = 0$. Therefore $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Corollary 2.2. Let K be a nonempty closed convex subset of an arbitrary Banach space X and T_i: $K \longrightarrow K$ (i=1, 2) be two quasi-contractive operators satisfying (1.13). Let $\{x_n\}_{n=0}^{\infty}$ be *defined* through the iterative scheme with errors (1.12) and $x_0 \in K$, where $\{a'_n\}, \{b'_n\}\{c'_n\}, \{a''_n\}, \{b''_n\}, \{c''_n\}$ are sequences in [0,1] satisfying $\lim c'_n = 0 = \lim c''_n = 0$. If $F(T_1) \cap F(T_2) \neq \phi$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common fixed point of T_1 and T_2.

Corollary 2.3. Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T : K \longrightarrow K$ be a quasi-contractive operator satisfying (1.13). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the Mann iterative scheme with errors (1.11) and $x_0 \in K$, where $\{a''_n\}, \{b''_n\}, \{c''_n\}$ are sequences in [0,1] satisfying $\lim c''_n = 0$. If $F(T) \neq \phi$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

The following example supports the validity of our results.

Example 2.4. Let X be the real line with the usual norm and let $K = [0,1]$. Define T_i (i =1, 2, 3): $K \to K$ by

$$T_1(x) = \begin{Bmatrix} \frac{1}{6}, x \in (0.5, 1] \\ 0, x \in [0, 0.5] \end{Bmatrix}$$

$$T_2(x) = \begin{Bmatrix} \frac{1}{7}, x \in (0.5, 1] \\ 0, x \in [0, 0.5] \end{Bmatrix}$$

$$T_3(x) = \begin{Bmatrix} \frac{1}{8}, x \in (0.5, 1] \\ 0, x \in [0, 0.5] \end{Bmatrix}$$

Mappings T_1, T_2, T_3 and the set K fulfill the requirements of the Theorem 2.1. If we take, $c_n = c'_n = c''_n = \frac{1}{n+1}$, $b_n = b'_n = b''_n = \frac{1}{\sqrt{2n+1}}$, $u_n = v_n = w_n = \frac{1}{\sqrt{n^2+1}}$, then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (1.10) converges strongly to unique common fixed point 0 of T_1, T_2 and T_3.

3. EXPERIMENTS

In this section, with the help of computer programming in C++ , we study the nature of convergence of Mann, Ishikawa , Agarwal et al. and CR iterative schemes with errors to locate a common fixed point of operators T_i (i =1, 2, 3) as taken in Example 2.4. The outcome is listed in the form of

Tables1 by taking initial approximation x_0=1 for all iterative schemes.

4. CONCLUSIONS

1. Newly introduced iterative scheme namely CR iterative scheme with errors is indepengent of Ishikawa iterative scheme but more general than Mann and Agarwal et al. iterative schemes.

2. CR iterative scheme with errors have high convergence rate as compared to Mann, Ishikawa and Agarwal et al. iterative schemes with errors.

Table 1

No of iterati-ons (n)	CR iterative scheme with errors	Ishikawa iterative scheme with errors	Agarwal et.al. iterative scheme with errors	Mann iterative scheme with errors
0	0.25	0.02381	0.861111	0.166667
1	0.014008	0.01108	0.103652	0.086479
2	0.001127	0.00657	0.014396	0.05425
3	0.000106	0.00433	0.002114	0.037163
4	1.093763e-05	0.00303	0.00032	0.02684
5	1.197059e-06	0.00221	4.924172e-05	0.020096
6	1.368042e-07	0.00167	7.680866e-06	0.015451
7	1.615802e-08	0.00128	1.209025e-06	0.012127
8	1.958853e-09	0.00101	1.916265e-07	0.009676
9	2.425776e-10	0.00080	3.053698e-08	0.007826
10	3.057766e-11	0.00065	4.887532e-09	0.006403

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