Convergence of Deficient Discrete Quartic Spline Interpolation

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ABSTRACT
The present paper is devoted for investigating the existence, uniqueness and convergence properties of deficient discrete quartic spline interpolation over uniform mesh which matches the given functional values at mesh points, interior points and second difference at boundary points.

Key words & Phrases
Discrete, Quartic Spline, Interpolation, Deficient, Error Bound, Uniform mesh..

1. INTRODUCTION

Discrete splines have been introduced by Mangasarian and Schumaker [3] in connection with certain studies of minimization problem involving differences. Astor and Duris [5] studied discrete L splines. Total positivity of the discrete spline callocation matrix given by Rong Qing Jia [6]. To compute non-linear splines iteratively Malcolm [2] used discrete splines. Discrete splines are useful for best summation formula (see Mangasarian and Schumaker [4]). Rana and Dubey [7] have obtained local behaviour of discrete cubic spline interpolation which is some time used to smooth histogram. For some constructive aspects of discrete splines reference may be made to Schumaker [1]. In this paper, we have generalized the result of Dubey and Shukla [10] for discrete deficient quartic spline interpolation, over uniform mesh and obtained existence uniqueness and convergence properties of deficient discrete quartic spline interpolation over uniform mesh which matches the given functional values at mesh points, interior points with boundary condition of second difference.

Let us consider a mesh P on [a b] which is defined by

$$P: a = x_0 < x_1 < ...., < x_n = b.$$

For i = 1, 2,.....n, pi shall denote the length of the mesh interval $[x_{i-1}, x_i]$ p is said to be a uniform mesh if p_i is constant for all i. Throughout, h will represent a given positive real number. Consider a real function $s(x,h)$ such that s_i is the restriction of $s(x,h)$ on $[x_{i-1}, x_i]$ a polynomial of degree 4 or less i=1, 2,.....n. Then $s(x,h)$ defines a deficient discrete quartic splines with deficiency 1 as,

$$D_h^{\{j\}} s_i(x_i,h) = D_h^{\{j\}} s_{i+1}(x_i,h)$$
$$j = 0,1,2. \tag{1.1}$$

whereas the difference operator $D_h^{\{j\}}$ for a function f is defined by

$$D_h^{\{o\}} f(x) = f(x),$$
$$D_h^{\{1\}} f(x) = \frac{f(x+h) - f(x-h)}{2h},$$
$$D_h^{\{2\}} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2},$$

and $D_h^{(m+n)} f = D_h^{\{m\}} D_h^{\{n\}} f(x),\ m,n \geq 0$.

Let S(4, 1, P, h) be the class of all deficient discrete quartic splines with deficiency I satisfying the boundary conditions,

$$D_h^{\{2\}} s(x_0,h) = D_h^{\{2\}} f(x_0,h),$$
$$D_h^{\{2\}} s(x_n,h) = D_h^{\{2\}} f(x_n,h). \tag{1.2}$$

Now writing $\alpha_i = x + \theta p, 0 < \theta < 1$, introducing the following interpolatory conditions for given function f,

$$s(x_i,h) = f(x_i,h) \quad i = 0, 1......n \tag{1.3}$$
$$s(\alpha_i,h) = f(\alpha_i,h) \quad i = 0, 1,......n-1,$$

and pose the following.

PROBLEM 1.1 : Given h > 0, for what restriction on P does there exist a unique $s(x,h) \in S(4,1,P,h)$ which satisfies the conditions (1.2) and (1.3)?

2. EXISTENCE AND UNIQUENESS

Let P(z) be a discrete quartic spline polynomial on [0, 1], then we can show that

$$P(z) = P(0)q_1(z) + P(1)q_2(z) + P(\theta)q_3(z)$$
$$+ D_h^{\{2\}} P(0)q_4(z) + D_n^{\{2\}} P(1)q_5(z), \tag{2.1}$$

whereas

$$q_1(z)=1+\frac{1}{6A}[\{-6\theta^4+12\theta^3-6+6h^2(\theta^2-1)\}z$$
$$-6(\theta-1)h^2z^2-12(\theta-1)z^3+6(\theta-1)z^4],$$
$$q_2(z)=\frac{1}{6A}[6\theta^4-12\theta^3-6\theta^2h^2)z ,$$
$$+6h^2\theta z^2+12\theta z^3-6\theta z^4]$$
$$q_3(z)=\frac{-1}{6A}[6(1+h^2)-6h^2z^2-12z^3+6z^4] ,$$
$$q_4(z)=\frac{1}{6A}[-2\theta^4+5\theta^3-3\theta^2+(\theta^3-\theta^2)h^2\}z$$
$$+\{3(\theta^4+\theta-2\theta^3)+(\theta-\theta^3)h^2\}z^2+\{(-\theta^4-5\theta+$$
$$+(\theta^3-3\theta^2+2\theta)z^4] ,$$
$$q_5(z)=\frac{1}{6A}[\{-\theta^4+\theta\ \ -h^2(\theta^3-\theta^2)\}z+h^2$$
$$(\theta^3-\theta)z^2+\{\theta^4-\theta+h^2(\theta-\theta^2\}z^3-(\theta^3-\theta)z^4]$$

where

$$A=[(\theta^4+\theta-2\theta^3+(\theta-\theta^2)h^2] .$$

Now we are set to answer problem 1.1 in the following.

THEOREM 2.1. For h > 0 and p > h there exist a unique deficient discrete quartic spline $s(x,h)\in S(4,1,P,h)$ which satisfies interpolatory conditions (1.3) and boundary conditions (1.2).

PROOF OF THEOREM 2.1. Denoting $(x-x_i)/p$ by t, $0 \le t \le 1$ writing (2.1) in the form of restriction $s_i(x,h)$ of the quartic spline s(x, h) on $[x_i, x_{i+1}]$ as follows :

$$s_i(x,h)=f(x_i)q_1(z)+f(x_{i+1})q_2(z)+f(\alpha_i)q_3(z)+$$
$$D_h^{\{2\}}s_i(x_i,h)q_4(z)+D_h^{\{2\}}s_i(x_i,h)q_5(z) \qquad (2.2)$$

where

$$q_1(z)=\frac{1}{6Ap^4}[6Ap^4+\{-6\theta^4-6+12\theta^3+6h^2(\theta-1)\}$$
$$p^3(x-x_i)-6(\theta-1)h^2(x-x_i)^2p^2-12(\theta-1)(x-x_i)^3p+6(\theta-1)(x-x_i)^4],$$
$$q_2(z)=\frac{1}{6Ap^4}[(6\theta^4-12\theta^3-\theta^2h^2)(x-x_i)p^3+6h^2\theta(x-x_i)^2$$
$$p^2+12\theta(x-x_i)^3p-6\theta(x-x_i)^4],$$
$$q_3(z)=\frac{1}{6Ap^4}[6(1+h^2)(x-x_i)p^3-6h^2(x-x_i)^2p^2$$
$$-12(x-x_i)^3p+6(x-x_i)^4],$$
$$q_4(z)=\frac{1}{6Ap^4}[\{-2\theta^4+5\theta^3-3\theta^2+(\theta^3-\theta^2)h^2\}$$
$$(x-x_i)p^3+\{3(\theta^4+\theta-2\theta^3)+(\theta-\theta^2)h^2\}(x-x_i)^2p^2$$
$$+\{(-\theta^4-5\theta+6\theta^2-(\theta-\theta^2)h^2\}(x-x_i)^3p$$
$$+(\theta^3-3\theta^2+2\theta)(x-x_i)^4],$$
$$q_5(z)=\frac{1}{6Ap^4}[\{\theta^4+\theta^3-h^2(\theta^3-\theta^2)\}(x-x_i)$$
$$p^3+h^2(\theta^3-\theta)(x-x_i)^2p^2$$
$$+((\theta^4-\theta)+(\theta-\theta^2)h^2)(x-x_i)^3p-(\theta^3-\theta)$$
$$(x-x_i)^4].$$

Clearly $s(x,h)$ is a discrete quartic on $[x_i, x_{i+1}]$ for i=0,1,....n and satisfies (1.2) and (1.3). Now applying continuity of first difference of $s_i(x,h)$ at x_i given by (1.1), we get the following system of equations.

$$[\{(\theta^4-\theta+3\theta^2-3\theta^3)+(\theta^3-\theta)h^2\}p^2+$$
$$\{(-\theta^4+3\theta-6\theta^2+4\theta^3)$$
$$-h^2(\theta-\theta^2)\}h^2D_h^{\{2\}}s_{i-1}(x,h)+[[\{(2\theta^4-3\theta^3+\theta)$$
$$+h^2(\theta^3+\theta^2-2\theta)\}p^2$$
$$+\{(\theta^4+3\theta-4\theta^3)+(\theta-\theta^2)h^2\}h^2]-[\{(-2\theta^4+$$
$$5\theta^3-3\theta^2+(\theta^3-\theta^2)h^2\}p^2$$
$$+\{(-\theta^4-5\theta)+6\theta^2-(\theta-\theta^2h^2\}h^2]]$$
$$D_h^2s_i(x_{i,}h)+[\{\theta^4-\theta^3+h^2(\theta^3-\theta^2)\}p^2$$
$$+\{\theta-\theta^4+(\theta^2-\theta)h^2\}h^2D_h^{\{2\}}s_{i+1}(x,h)=F_i$$

(say) (2.3)

where

$$F_i=\frac{1}{p^2}[(-6\theta^4+6+12\theta^3-12\theta)+(6\theta^2+6-12\theta)h^2)$$
$$p^2+12(\theta-1)h^2\}f(x_{i-1})$$
$$+\{-6(1+h^2)p^2+12h^2\}f(\alpha_{i-1})+\{12\theta-6)(1+h^2)p^2$$
$$+12(1-2\theta)h^2\}f(x_i)$$
$$+\{(6\theta^4-12\theta^3-6\theta^2h^2)p^2+12\theta h^2\}f(x_{i+1})$$

$+\{6(1+h^2)p^2-12h^2\}f(\alpha_i)]]$.

Writing $D_n^{\{2\}}s(x_i,h)=m_i(h)=m_i$ for all i (say),

we can easily see that excess of the absolute value of the coefficient of m_i over the sum of the absolute values of the coefficients of m_{i-1} and m_{i+1} in (2.3) under the condition of Theorem 2.1 is given by

$$\theta(1-\theta)p[\{(1+2\theta-2\theta^2)+h^2(1-2\theta)\}p^2+4h^2\{1+\theta-\theta^2+h^2\}]=\text{Ci(h)}$$

which is clearly positive under the condition of Theorem 2.1. Therefore the coefficient matrix of the system of equations are diagonal dominant and hence invertible. Thus, the system of equations has a unique solution. This completes the proof of Theorem 2.1.

3. ERROR BOUNDS

It may be seen that system of equations (2.3) may be written as

$$A(h),m(h)=F(h)$$

where A(h) is coefficient matrix and m(h) = mi(h) for all i. However, as already shown in the proof of Theorem 2.1, A(h) is invertible. Denoting the inverse of A(h) by A-1(h), we note that row max norm A-1(h) satisfies the following inequality,

$$\|A^{-1}(h)\|\le y(h) \quad (3.1)$$

where $y(h)=\max\{l_i(h)\}^{-1}$.

For a given $h>0$, we introduce the set $R_h=\{jh:j$ is an integer} and define a discrete interval as follows -

$$[0,1]_h=[0,1]\cap R_h.$$

For a function f and three distinct points x_1,x_2,x_3 in the domain, the first and second divided difference are defined by

$$[x_1,x_2]f=\frac{f(x_1)-f(x_2)}{(x_1-x_2)}$$

and $[x_1,x_2,x_3]f=\dfrac{[x_2,x_3]f-[x_1,x_2]f}{(x_3-x_1)}$ respectively.

(3.2)

For convenience, we write $f^{\{2\}}$ for $D_h^{\{2\}}f$ and f_i^{2} for $D_n^{\{2\}}f(x_i)$ and w(f,p) is the modulus of continuity of f and

$$\|f\|=\max_{x\in[0,1]_h}|f(x)|$$

is the discrete norm of a function f over the interval [0, 1]h. Without assuming any smoothness condition on the data f, we shall obtain in the following the bounds for the error function $e(x)=s(x,h)-f(x)$ over the discrete interval [0, 1]h.

THEOREM 3.1. Suppose $s(x,h)$ is the discrete quartic spline interpolant of Theorem 2.1. Then

$$\|e_i^{\{2\}}\|\le C_1(h)K(p,h)w(f,p), \quad (3.3)$$

$$\|e_i^{\{1\}}\|\le C_2(h)K_1(p,h)w(f,p) \quad (3.4)$$

and $\|e(x)\|\le p^2K^*(p,h)w(f,p)$ (3.5)

where K (p, h), K1 (p, h) and K* (p,h) are positive function of p and h.

Proof of Theorem 3.1 : Equation (3.1) may be written as

$$A(h)e^{(2)}(x_i)=F_i-A(h)f_i^{(2)}=M_i(h) \quad \text{(say)} \quad (3.6)$$

When we replace

$m_i(h)$ by $e^{\{2\}}(x_i)=D_h^{\{2\}}s(x_i,h)-f_i^{\{2\}}$ in (2.3).

To estimate row max norm of the matrix Mi(h) in (3.6), we shall need following Lemma due to Lyche [8, 9].

Lemma 3.1 : Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ be a given sequence of non-negative real numbers such that $\sum a_i=\sum b_j$. Then for any real valued function f defined on a discrete interval $[0,1]_h$ we have

$$\left|\sum_{i=1}^m a_i[x_{i0},x_{i1},......x_{ik}]_f-\sum_{j=1}^n b_j[y_{j0},y_{j1},----y_{jk}]_f\right!$$

$$\le w(f^{\{k\}},|1-p|)\sum a_i|k$$

where $x_{ik},y_{jk}\in[0,1]_h$ for relevant values of i, j and k. It may be observed that the r.h.s. of (3.6) is written as for m=8 and n=7.

$$|(M_i(h)|=\sum a_i[x_{i0},x_{i1}]_f-\sum b_j[y_{j0},y_{j1}]_f| \quad (3.7)$$

where $a_1=(6\theta^4-12\theta^3-6\theta^2h^2)p^2=b_1$,

$a_2=p^2 6\theta(1+h^2)=a_3$,

$b_2=12p^2\theta(1+h^2)$,

$a_4=12\theta h^2=b_3=a_5=b_4$,

$$a_6=\frac{p}{2h}[\{(\theta^4-\theta+3\theta^2-3\theta^2)+(\theta^3-\theta)h^2\}p^2$$

$$+\{(-\theta^4+3\theta-6\theta^2+4\theta^3)-(\theta-\theta^2)h^2\}h^2=b_5,$$

$$a_7=\frac{p}{2h}[\{(4\theta^4-8\theta^3+3\theta^2+\theta)+2h^2(\theta^2-\theta)\}p^2$$

$$+\{(2\theta^4+8\theta-6\theta^2-4\theta^3)+2h^2(\theta-\theta^2)\}h^2]=b_6$$
,

$$a_8=\frac{p}{2h}[\{\theta^4-\theta^3+h^2(\theta^3-\theta^2)\}p^2+\{(\theta-\theta^4)$$

$$+h^2(\theta^2-\theta)\}h^2]=b_7,$$

$$y_{10}=x_{i-1}=x_{21}=y_{20}=x_{40}=x_{61}=y_{50},$$

$$x_{10}=x_i=y_{11}=x_{30}=y_{21}=x_{41}=y_{31}=x_{50}=y_{40}=x_{71}=y_{60}$$

$$x_{11}=x_{i+1}=x_{51},$$

$$x_{20}=\alpha_{i-1}=x_{i-1}+\theta p=y_{30},$$

$$x_{31}=\alpha_i=x_1+\theta p=y_{41},$$

$$y_{5_1}=x_{i-1}+h,$$

$$x_{60}=x_{i-1}-h,$$

$$y_{6_1}=x_i+h,$$

$$x_{70}=x_i-h,$$

$$y_{7_1}=x_{i+1}+h,$$

$$x_{80}=x_{i-1}-h,\ x_{81}=x_{i+1}=y_{70}.$$

Hence $\sum_{i=1}^{8}a_i=\sum_{j=1}^{7}b_j$ = N (θ, p, h) (say).

Thus applying Lemma 3.1 in (3.7) for m_i=8, n=7 and k= 1, we get

$$\|(M(h))\|\le N(\theta,p,h)\ w(f^{\{1\}},p), \qquad (3.8)$$

Now using equation (3.1) and (3.7) in (3.6) we get

$$\|e^{\{2\}}(x_i)\|\le C_1(h)K(p,h)w(f^{\{1\}},p). \qquad (3.9)$$

where K (p, h) is some positive function of p and h.

We now proceed to obtain an upper bound for *e(x)*. Replacing $m_1(h)$ by $e_i^{\{1\}}$ in equation (2.2), we obtain

$$e(x,h)=p^2[Q_4(t)e^{(2)}(x_i)+Q_5(t)e^{\{2\}}(x_{i+1})]+L_i(f) \qquad (3.10)$$

Now we write $L_i(f)$ in term of divided difference as follows :-

$$L_i(f)=\left|\sum_{i=1}^{3}u_i[x_{i0},x_{i1}]f-\sum_{j=1}^{4}v_j[y_{j0},y_{j1}]f\right|$$

where

$$u_1=p\{(6^4\theta-12\theta^3-6\theta^2h^2)t+6\theta h^2t^2+12\theta t^3-6\theta t^4\}$$

$$v_1=(\theta p)\{-6(h^2)t\ +6h^2t^2+12t^3-6t^4\},$$

$$u_2=\frac{p^2}{2h}Q_4(t)=v_2\ ,$$

$$u_3=\frac{p^2Q_5(t)}{2h}=v_3,$$

$$v_4=6Apt.$$

and

$$x_{10}=x_i=x_{21}=y_{11}=y_{20}=y_{41},$$

$$x_{11}=x_{i+1}=y_{30},$$

$$y_{21}\,x_i+h,$$

$$x_{20}=x_i-h,$$

$$y_{31}=x_{i+1}+h,$$

$$x_{30}=x_{i+1}-h,x_{31}=x_{i+1},y_{40}=x,$$

$$y_{10}=\alpha_i=x_i+\theta p,$$

Thus $\sum_{i=1}^{3}u_i=\sum_{j=1}^{4}v_j=6\theta tp[\{\theta^3-2\theta^2-\theta h^2)$

$$+h^2+2t^2-t^3\}+\frac{p^2}{2h}\{Q_4(t)+Q_5(t)\}]$$

= N* (p, h) (say)

We again apply Lemma 3.1 in (3.10) for i=3, j=4 and k=1, we see that

$$|L_i(f)|\le N^*(p,h)w(f^{\{1\}},p) \qquad (3.11)$$

Thus, using (3.10) and (3.11) in (3.9) we get the following

$$\|e(x)\|\le p^2\,K^*(p,h)w(f^{\{1\}},p) \qquad (3.12)$$

where K* (p, h) is a positive constant of p and h. This is the inequality (3.5) of Theorem 3.1.

We now proceed to obtain an upper bound of $e_i^{\{1\}}$. From equation (2.2) we get,

$$s_i^{\{1\}}(x,h)=f_i\,q_1^{\{1\}}(t)+f_{i+1}\,q_2^{\{1\}}(t)+f_{\alpha i}\,q_3^{\{1\}}(t)$$

$$+p^2\,s_i^{\{2\}}(x,h)q_4^{\{1\}}(t)+p^2\,s_i^{\{2\}}(x,h)q_5^{\{1\}}(t) \qquad (3.13)$$

Thus

$$e_i^{\{1\}}(x,h) = p^2[e_i^{\{2\}} q_4^{11}(t) + e_{i+1}^{\{2\}} q_5(t)] + U_i(f) \quad (3.14)$$

where $U_i(f)$

$$= f_i q_i^{(1)}(t) + f_{i+1} q_2^{\{1\}}(t) + f_{\alpha_i} q_3^{\{1\}}(t)$$

$$+ p^2[f_i^{\{2\}} q_4^{(1)}(t) + f_{i+1}^{\{2\}} q_5^{\{1\}}(t)] - f_i^{\{1\}}(x,h).$$

By using Lemma 3.1 and first and second divided difference in $U_i(f)$ we see that :

$$|U_i(f)| \le w(f^{\{1\}}, p) \sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j$$

where

$$a_1 = p\{16\theta^4 - 12\theta^3 - 6\theta^2 h^2) + 12h^2\theta t,$$

$$+12\theta(h^2 + 3t^2) - 24\theta t(t^2 + h^2)\}$$

$$a_2 = p\{6\theta(1 + h^2) - 12\theta h^2 t - 12\theta(h^2 + 3t^2)$$

$$+24\theta t(t^2 + h^2)\}, \; a_3 = \frac{p^2}{2h} q_4^{\{1\}}(t) = b_1$$

$$a_4 = \frac{p^2}{2h} q_5^{\{1\}}(t) = b_2$$

$b_3 = a_1 + a_2 = p$ and also

$$x_{10} = x_i = x_{20} = x_{31} = y_{30} = y_{11},$$

$$x_{21} = x_{i+1} = x_{40} = y_{21}$$

$$y_{31} = x_{30} = x_i + h$$

$$x_{41} = x_{i+1} + h$$

$$y_{10} = x_i - h, \; y_{20} = x_{i+1} - h$$

$$\sum_{i=1}^{4} a_i = \sum_{j=1}^{3} b_j = \frac{p^2}{2h}\left[q_4^{\{1\}}(t) + q_5^{(1)}(t)\right]$$

From equation (3.8) putting the value of $e_i^{\{2\}}$ in equation (3.13) we get upper bound of $e_i^{(1)}$. This is inequality (3.4) of Theorem 3.1.

4. CONCLUSION AND FUTURE SCOPE

Existence, uniqueness and convergence properties of deficient discrete quartic spline interpolant matching the given function values at mesh points and interior points of the given mesh with appropriate boundary conditions have been investigated in the present paper. These results can be generalized in future for general mean averaging condition.

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