Optimal Wiring on Rectangular Structure

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ABSTRACT
In this paper we worked upon on optimal wiring on rectangular structure. Here we are given a rectangle partitioned into smaller rectangles by axis-parallel line segments. Find a subset of the segments such that the resulting structure from these segments is connected and it touches every smaller rectangle.

Here we reduce the problem of exact cover by 3-sets (X3C), which is known to be NP-complete, into this problem and thus claim wiring problem to be NP-hard. This problem carries a special importance because very few problems in the domain of geometry are known to be NP-hard.

Keywords
Wiring on rectangular structure, NP-hard, Computational geometry, Graph theory

1. INTRODUCTION
Given a rectangle partitioned into smaller rectangles by horizontal and vertical line segments, find a set of the line-segments which touches each rectangle at least at one point on its boundry and these segments are connected (i.e., there is a path between any two points of these segments). The optimization criterion is to minimize the sum of lengths of these segments.

An obvious application of this problem is to how to wire a building using minimum wire. Same applies to laying cooling or heating channels. Another application is in connecting modules of a VLSI chips.

2. NP_HARDNESS Of WIRING PROBLEM

2.1. Wiring Problem
A floorplan is a rectangle in a plane which is partitioned by horizontal and vertical line segments such that each region is also a rectangle. For convenience treat it as a graph where the vertex set is the collection of the corners of all the rectangles and edges are the line segments between the vertices in the floor plan. A side is a line segment which connects two corners of the same rectangle. In general a side may contain more than one edge.

The wiring problem is to compute a minimum length connected subgraph of a floorplan (i.e., total length of the edges of the subgraph be minimum) which contains at least one vertex on the boundry of every rectangle. Observe that it will always be a tree. See figure 1. In this section we shall show that this problem is NP-hard by reducing 3-set-exact-cover problem. The proof is adapted from the proof of hardness of steiner tree computation for geometric rectilinear graph by Garey, Graham, and Johnson[1].

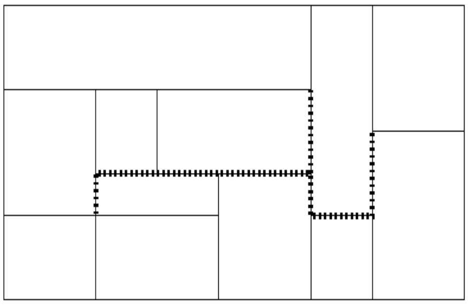

Figure 1: A floorplan and its solution

2.2 3-set-Exact-Cover (X3C) Problem
Given a family, F= {F1, F2,…..Ft}, of 3-element subsets of a universal set U of 3n elements, decide if there exists a subfamily F’ C F of pairwise disjoint sets such that the union of all members of F’ is equal to U. This problem is NP_complete[2]. We will prove the hardness of the wiring problem by transforming X3C into it.

2.3 The overall plan
Let F= {F1, F2,…..Ft} be an input to the X3C (3-set exact cover) problem, where the universal set is assumed to be the integer set {1, 2,……,3n}, we will construct a floorplan Pi associated with set Fi for each i. Each plan will be of the same dimensions. Then we shall join them side by side along the X-axis to form a single floor-plan P for the given F.

We will show that there is a polynomial L(n,t) such that the length of the wiring tree for P will be less than or equal to L(n.t) if and only if the given F has an exact cover. It will also be shown that an exact cover can be extracted from such a solution tree in O(L(n,t)) time.

2.4. Construction of Pi
We first introduce two gadgets, junction and crossover, which are the bulding blocks of the floor-plans Pi. Figure 1 shows the gadgets. There are two variants of crossover, standard and warped. The length parameters used in describing the gadgets are K = 162qnt + 2888n2t _ 9n + 1 and ϵ which will define later. It will be helpful to remember that ϵ <<1.

Symbol q is defined to be sum of (ai + bi +ci) for i= 1 to t, where Fi = {ai, bi, ci}.

Each junction has one and each crossover has two active regions which are highlighted by shading in figure 1.

The floorplan Pi associated with the set Fi = {4, 1, 2} is shown in figure 3, where a junction at the bottom is connected to three stacks of crossover of heights ai, bi, ci respectively.

Each stack has one warped crossover at the top and remaining crossovers are standard. The width of Pi is 24K and the height is 24nK + 8K + ϵ.

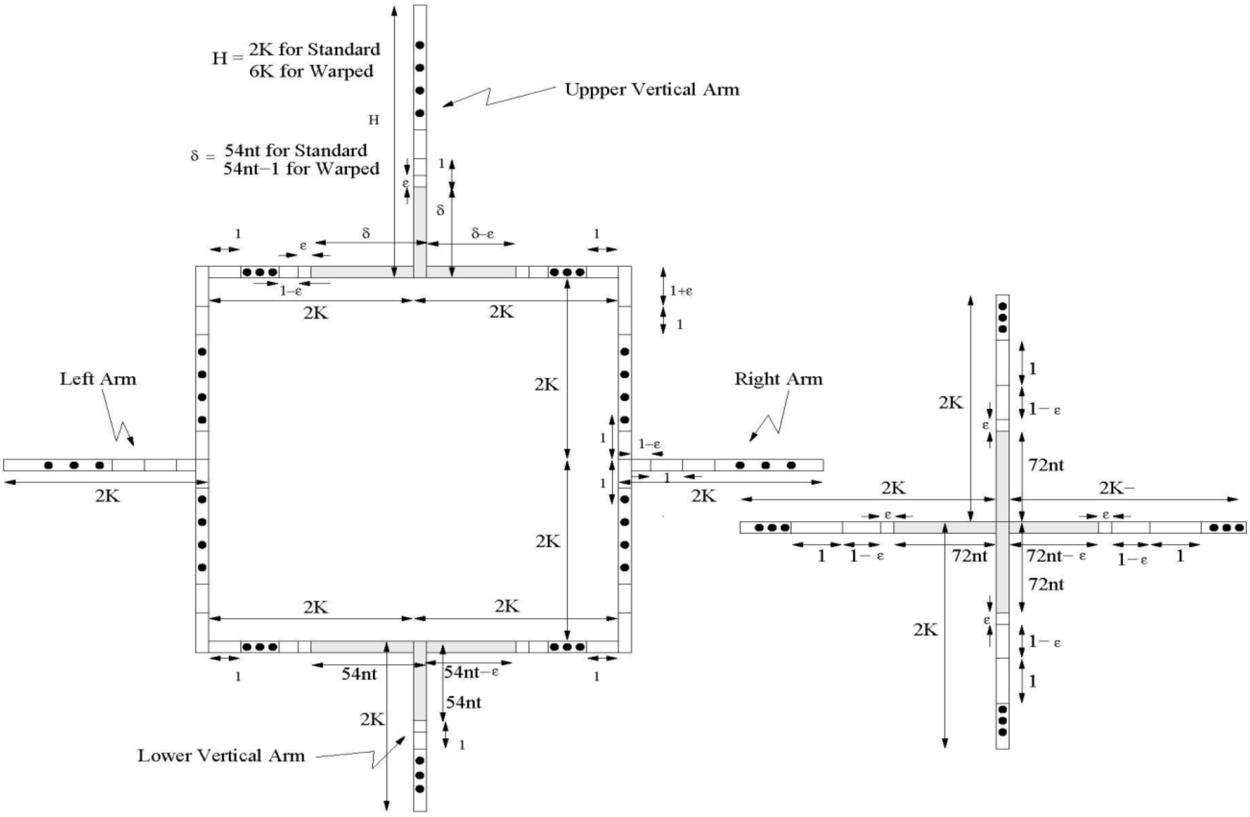

Figure 2: Crossover and Junction

The floorplan P associated with F is constructed by placing P_1, P_2,………,Pt side by side so that right side of P_i coincides with the left side of P_{i+1}. In addition, a stack is attached to the left of the figure consisting of 8K rectangles of size 1x ϵ and one 24nK xϵ rectangle at the top of the stack, see figure 4 for P_1 with the additional stack at the left wall. One way P_1 and P_t differ from other P_i is that the leftmost rectangles in the horizontal chain of small rectangles in P_1 end with an ϵ x ϵ rectangle. This is the case for all but the bottom two chains. In case of P_t the rightmost rectangles of all but the bottom chain is ϵ x ϵ. Figure 5 shows complete P. It uses q crossovers and t junctions.

2.5 Optimal wiring tree for P

In this section we will determine some properties of any optimal solution of the wiring on P which are crucial for the proof. In the following section we use these properties to show that the sum of the lengths of the edges in the optimal solution, will be less than or equal to L(n.t) if and only if the underlying X3C problem has a solution.

2.5.1 Coverage of the smaller rectangles

Let us partition the rectangles of P into 3 classes: R_0 which have longer side upto 1 + ϵ; R_1 which have longer sides in the range from 54nt -1 to 72nt; and R_2 have each side at least 2K-ϵ in length. This partitions all rectangles of P except the top rectangle at the left boundry. This ϵ x24nK rectangle is also included in class R_2. Observe that R_1 are precisely the rectangles in the active regions. We further partition R_0 into terminal and non terminal rectangles, where the former contains all the ϵ x ϵ sized rectangles. Observe that at least one vertex of each smaller side of a no-terminal rectangle is shared by other R_0 rectangles. We shall denote these subclass by R_{0t} and R_{0n} respectively. Terminal rectangles are attached to the left side of P_1, right side of P_t, and to each R_1 rectangle.

In the similar fashion as for the rectangles partition the edges also into 3 classes: E_0 consists of edges of length not exceeding 1 + ϵ; E_1 are the edged with length between 54nt-1 to 72nt; and E_2 have all the larger edges. Verify that each edge of E_2 is longer than K.

In order to establish polynomiality of this transformation from X3C, we need to determine some counts. Break the R_{0n} class into four subclasses, see figure 6, and denote the number of (i) 1 xϵ rectangles with 4 vertices by m_1, (ii) those with 5 vertices by m_2, (iii) (1- ϵ) x ϵ sized rectangles by m_3, and (iv) (1 + ϵ) x ϵ sized rectangles by m_4. The number of terminal rectangles will be denoted by m_0. A trivial but cumbersome exercise gives m_0 = 6q + 6n + 4t + 1, m_1 = 14Kq + 42Kt + 72Kn + 65K -324qnt – 288nt^2 -10t +q –n -1, m_2 = 2q + 5t + 1, m_3 = 6n +7t + 8q + 1, and m_4 = 4q + t + 1. Since q<9nt, each of these numbers is a polynomial in n, t. Each R_{0n} rectangle has two major edges which are parts of the longer sides and which have length between 1- ϵ and 1+ ϵ .

In this paper we adopt a convention in which the same symbol will represent a set of edges as well as the graph induced by those edges, depending on the context.

Symbol T will denote an optimal wiring tree on P, i.e. solution of the wiring problem on P. The limiting case of the floorplan P, where ϵ = 0, will be denoted by P', i.e., P' = $\lim_{\epsilon\to 0}$ P.

Henceforth P will denote the complete floorplan with ϵ = 1(20q + 23n +28t +9).

T_0 will denote T∩ E_0 and $T_1 = T - T_0$. Observe that the residue of each R_0 rectangle in P' is an edge of length 1. By T' denote $\lim_{\epsilon\to 0}$ T. Similarly by T'_0 and T'_1 denote the residue of T_0 and T_1 in P'.

Observation 1 Every cycle in P has at least one edge of size greater than ϵ.

Observation 2 Each cycle in P' contains an edge which is the residue of the longer side of an R_1 rectangle, i.e., an active region rectangle.

Observation 3 There are 4q + 8t + 9n + 2 rectangles and each of these rectangles has at least one R_{0n} rectangle adjacent to it.

Proposition 4 Let X be an R_0 rectangle and C be a cycle in P. Let C contain a major edge e of X which is contained in side s. Let s' be the side parallel to s in X. Then (i) s is a longer side of X, (ii) either C contain s' or it contains a longer side of an R_1 rectangle.

Proof (i) Major edges are contained in the longer sides of a rectangle.

(ii) As we take the limit $\lim_{\epsilon\to 0,}$ line segments e, s, and s' will coincide with a single segment, say s'' in P'. Let C reduce to C' in P'. Then the edges of C' can be partitioned into simple cycles and simple paths. If s'' is in a simple path, then s' must be on C. On the other hand, If s'' is in a cycle in C', then due to the previous observation the longer side of an R_1 rectangle must be on C.

Lemma 5 T contains at least one major edge from each R_{0n} rectangle.

Proof Let X be a non terminal rectangle of R_0 which does not satisfy the claim. By the definition,, X has two adjoining R_0 rectangle which are separated by more than ϵ. Label them by Y and z. Since T touches all three rectangles, $Y' = T \cap (Y \cup X)$ and $Z' = T \cap (Z \cup X)$ are non-empty where the X, Y, Z may be treated as the sets of the edges on their sides. Since T does not contain any major edge of X, Y' and Z' must be unconnected. Let y belongs to Y' and z belongs to Z' be a pair of vertices which are closest to each other.

Add the shortest path between y and z in P, to T. Let the resulting subgraph be called H. This added path must contain a major edge e of X and the length of the path canot exceed 3x(1 + ϵ) + ϵ since sides of each rectangle is at most 1 + ϵ and their width is ϵ. The subgraph H has a cycle (since T is a tree) and the cycle contains e. From proposition 4 either T contains one major edge of X or one longer side of a R_1 rectangle. The former is not possible from the assumption. Therefore T must contain a longer edge p of an R_1 rectangle. By deleting p from H we again get a tree, call it H', and it also touches all the rectangles. Therefore this is also a candidate of wiring solution. Since the length of p is at least 54nt – 1 and the added edges are at most 3 + 4c in length, H' has lesser length. This implies that T is not an optimal solution, which is contradiction.

In a wiring solution if a wire connects diagonally opposite vertices of a rectangle then it has two options of equal cost first horizontal then vertical side or its converse. This way we can delay a traversal along an ϵ edge on a non-terminal rectangle until a 5-vertex rectangle is reached. Call such wiring solution normalized, see figure 7. Using this observation and lemma 5 we have following result.

Corollary 6 The cost of T_0 is between $L_0 = m_1 + m_4 + (1-\epsilon)(m_2 + m_3)$ and $L_0 + \epsilon(28q + 21n + 28t + 8)$.

Proof The smaller major edge of five vertex 1 x ϵ is 1 - ϵ long. The smaller major edge of a (1 + ϵ) x ϵ rectangle has length 1. Using these fact and the lemma we directly get the lower bound. For the upper bound first delete all the ϵ edges which are pendants (having one vertex of degree 1) from T_0. As observed earlier, the reduced graph can be transformed into normalized from without any extra cost. So assume that the reduced T_0 is normalized.

Then the cost due to five vertex rectangles can be upto 1 + ϵ for 1 x ϵ rectangle and 1 + 2 ϵ for (1 + ϵ)x ϵ rectangles. In addition the solution may cover terminal rectangles. So the cost can increase upto 2 $\epsilon(m_0 + m_2 + m_4)$. To this we add the cost of the pendants. The only purpose for the pendants will be to touch the R_2 rectangles as all others are already in contact of T_0. This can add at most 4q + 8t + 9n + 2 ϵ edges to T_0.

Observation 7 The graph induced by R_0 rectangles has 2q + 3n +1 connected components.

This observation and lemma 5 imply that the subgraphs induced by T_0 must have at least 2q + 3n + 1 components.

Lemma 8 T_0 has exactly 2q + 3n +1.

Proof Assume that the number of components is greater than 2q + 3n +1. Then there must be at least two components of T_0 in the same component of R_0.

Consider two such T_0 components. If there is a R_{0n} rectangle in this component such that each of its major edge belongs to some T_0 component. Then these T_0 components are separated by ϵ distance. In case no R_{0n} rectangle contributes its major edge to more than one component, then there must be a rectangle whose one vertex is touched by one T_0 component and one major edge belongs to another, see figure 8. Sice the distance between a vertex and a major edge is at most 2 ϵ (in a 5 vertex rectangle), the two components are separated by a path of at most 2 ϵ length. In other words, their closest points are separated by a path of at most 2 ϵ length. Add this path to T, which will create a cycle. Delete a non- ϵ -edge from this cycle. The resulting tree connects the same set of vertices (or one more) as does T therefore it is also a candidate solution for the wiring problem. This tree costs lesser than T since 2 ϵ < 1- ϵ. but this is not possible since T is optimal.

Corollary 9 There is one T_0 component in each R_0 component.

Lemma 10 T' has no cycles.

Proof We know that T is a tree. Assume that T' has a cycle. Then T must have two vertices which are separated by a path S of ϵ-edges which is not a part of T. The longest such path in P has length 2 ϵ. Then T U S will have a cycle which contains S. Once again, as in the proof of Lemma 8 we can construct another solution of the wiring problem which costs less. Therefore the assumption must be wrong.

2.6 Connecting the components of T_0

In this section we will establish the underlying X3C problem has an exact cover if and only if T costs less than $L = L_0 + 162ntq + 288n^2t - 9n + 1$. Also if there is a solution of the wiring problem in this range, then a solution of X3C problem can be constructed in O(L) time. The immediate consequence of these claims is that if wiring problem has a polynomial time solution, then X3C can also be solved in polynomial time since L is a polynomial in n, t.

In order to construct the solution tree we need to include the edges in addition to T_0 so that the resulting graph becomes connected. Denote the set of these edges by T_1. Irrespective of any other criterion/consideration these edges must provide connections between 2q + 3n + 1 components of T_0 graph. The additional edges will be from E_1 U E_2. Each E_2 edge is at least K in length. We will show that this renders it too expensive to use. The T_1 edges belonging to an active region become responsible to connect the T_0 component surrounding the region. Therefore these edges from 1 or 2 connections, depending upon whether they connect 2 or 3 components. Figure 9 shows possible patterns of these T_1 edges. Table 1

shows the average cost of forming one connection under these patterns.

Table 1: Patterns for active region, number of connections and their costs

Gadget	Pat.	Con.	Range of cost	Range of cost/conn.
Warped	β_1	2	$162nt - 3 + .5\epsilon \pm .5\epsilon$	$81nt - 1.5 + .25\epsilon \pm .25\epsilon$
Crossover	β_2	1	$108nt - 2 - .5\epsilon \pm 1.5\epsilon$	$108nt - 2 - .5\epsilon \pm 1.5\epsilon$
	β_3	1	$108nt - 2 + \epsilon \pm \epsilon$	$108nt - 2 + \epsilon \pm \epsilon$
Standard	α_1	2	$162nt + .5\epsilon \pm .5\epsilon$	$81nt + .25\epsilon \pm .25\epsilon$
Crossover	α_2	1	$108nt - .5\epsilon \pm 1.5\epsilon$	$108nt - .5\epsilon \pm 1.5\epsilon$
	α_3	1	$108nt + \epsilon \pm \epsilon$	$108nt + \epsilon \pm \epsilon$
Junction	γ_1	3	$288nt + .5\epsilon \pm .5\epsilon$	$96nt + .166\epsilon \pm .166\epsilon$
	γ_2	2	$216nt + .5\epsilon \pm .5\epsilon$	$108nt + .166\epsilon \pm .166\epsilon$
	γ_3	2	$288nt - \epsilon \pm \epsilon$	$144nt - .5\epsilon \pm .5\epsilon$
	γ_4	1	$144nt - .5\epsilon \pm 1.5\epsilon$	$144nt - .5\epsilon \pm 1.5\epsilon$
	γ_5	1	$144nt$ to $144nt + 2\epsilon$	$144nt$ to $144nt + 2\epsilon$

Lemma 11 If T_1 C E_1 then (i) the number of α_1 and β_1 patterns cannot be more than q.

(ii) if the number of α_1 and β_1 patterns is equal to q, then the number of β_1 patterns cannot be more than 3n.

Proof (i) If there are more than q patterns of α_1 and β_1 type, then at least one crossover will have two α_1's or one α_1 and one β_1 (in case of top of the stack crossovers). Then T' will have a cycle, contradicting Lemma 10.

(ii) If q connection of the type α_1 and β_1 are used, then each crossover will have one of these connections, because as shown in part (i) same crossover cannot have two of these connections. This will connect all the chains of the same level. If more than 3n β_1-connections are used, then some level will have two or more connections. This will provide more than one connection between the horizontal chain of that level and that of the level immediately above. Once again it implies that T' has a cycle.

Lemma 12 Let T_1 C E_1. If in a wiring tree the cost of T_1 is less than $162qnt + 288n^2t - 9n + 1$, then the T_1 will have 3n patterns of β_1 type, q-3n patterns of α_1 type and n patterns of γ_1 type.

Proof Suppose in the given solution u connection (between a pair of T_0 components) are due to β_1 patterns, v connections are from α_1 patterns w connection from γ_1 patterns and remaining 2q + 3n –u-v-w connections due to other patterns. Then from the table 1 T_1 must cost off at least (81nt-1.5)u + (81nt)w + (108nt) (2q+3n-u-v-w). We are given that this cost is less than $162qnt + 288n^2t - 9n + 1$. Simplifying the inequality we get (2q + 3n –u –v-w)12nt + (2q-u-v)15nt + 9n – 1.5u -1 <0. Sice there can be at most 3t patterns of β_1 type, u≤6t. Thus 15nt > 12nt > 1.5u + 1. Lemmas 8 and 11 shows that 2q + 3n-u-v-w and 2q-u-v are non negative. If either of these quantities is positive then the left hand side expression will become positive. This requires that 2q + 3n –u –v –w =0 and 2q –u -v =0. Then inequality simplifies to 1.5(6n –u) -1 < 0. As 2q- u –v =0, from Lemma 11 we also know that 6n-u ≥ 0. Again observe that if 6n –u is positive then this inequality will be unsatisfiable. So we must have 6n – u =0. So u=6n, v=2q-6n and w= 3n.

Corollary 13 If T_1 C E_1 and T_1 costs less than $162qnt + 288 n^2t - 9n + 1$, then X3C problem has an exact cover.

Proof We have seen that T_1 consists of 3n patterns of β_1 type, q-3n patterns of α_1 type, and n patterns of γ_1 type. As we have seen in the proof of Lemma 12 that every crossover has one β_1 or α_1 pattern. This ensures that the T_0 components on horizontal arms of the same level are connected. If β_1 patterns are at the same level, say i, then there are two connections between levels i and i + 1 (above i). This will lead to a cycle in T' which is not possible. So each β_1 must be at a different level. This ensures that each level is connected to the upper level through the upper arm of a warped crossover.

Now we will argue that in any stack of crossovers either all active regions contributing to T_1 are upper ones or all are lower ones. Suppose crossover C_2 is directly above crossover C_1. First assume that upper active region of C_2 and lower active region of C_1 share edges with T_1. This will render the T_0 component I the vertical arm between the two disconnected. In case the lower active region of C_2 and the upper active region of C_1 contribute edges to T_1, then this will provide a connection between the two levels. In addition, the β_1 patterns at the level of C_1 crossover also provides a connection between the same levels. This implies that T' will have a cycle, which is not possible.

Next we will show that β_1 patterns can occur only on those stacks which are associated with the junction having γ_1 pattern. Suppose the top crossover of a stack has β_1 where the corresponding junction does not have γ_1. Then that junction does not contribute any edge to T_1 This implies that the vertical arm of the junction corresponding to the stack remains unconnected since the stack above it will have upper active region with α_1 (or β_1) pattern.

Therefore all β_1 pattern must be associated with only the stacks of junctions with γ_1 patterns. Since each β_1 pattern is at a different level, the sets corresponding the junctions having γ_1 patterns from an exact cover.

Now we are equipped to prove the main result.

Theorem 14 The underlying X3C problem has an exact cover iff the wiring tree of the corresponding floorplan has a solution costing less that $L_0 + 162\ qnt + 288n^2t - 9n + 1$.

Proof (only if) In Corollary 6 we have seen that a T_0 can be constructed which touches all the rectangles, has $2q + 3n + 1$ components and costs $L_0 + \epsilon(28q + 21n + 28t + 8)$. We construct T_1 as follows.

Let us denote the junction of P_i by J_i. Let the exact cover be $F' = \{ F_{a1},\ldots\ldots\ldots\ldots,Fa_n\}$.

Then a wiring tree can be formed by constituting T_1 with : β_1 pattern in every top crossover of all the three stacks of J_{ai} for all F_{ai} for all F_{ai} belongs to F' , α_1 pattern in the upper active regions of all the standard crossovers of these stacks; α_1 patterns in the lower active regions in all the other crossovers; and γ_1 pattern on every J_{ai} for all F_{ai} belongs to F'. Total cost of T_1 will be no more than $3n(162nt + \epsilon - 3) + (q - 3n)(162nt + \epsilon) + n(288nt + 2\epsilon)$. It simplifies to $162qnt + 288n^2t - 9n + \epsilon(q+2n)$. The total cost of T is at most $L_0 + 162qnt + 288n^2t - 9n + \epsilon(29q + 23n + 28t + 8)$ which is less than L.

(if) suppose the optimal wiring tree costs less than L. From corollary 6 we know that T_0 costs at least L_0 so T_1 must cost less than $162qnt + 288n^2t - 9n + 1$. From corollary 13 we know that the X3C problem must have an exact cover.

Corollary 15 If wiring problem is in class P, then X3C is also in class p.

Proof Since floorplan has a polynomial number of edges (in terms of n and t), it is possible to verify in O(|T|) time whether there are 3n patterns of β_1 type, 2q-3n patterns of α_1 type, and n patterns of γ_1 type. Also identify which junctions have the γ_1 patterns in the process. As we consequence we can construct an exact cover.

Theorem 16 The wiring problem on rectangular floorplan is NP-hard.

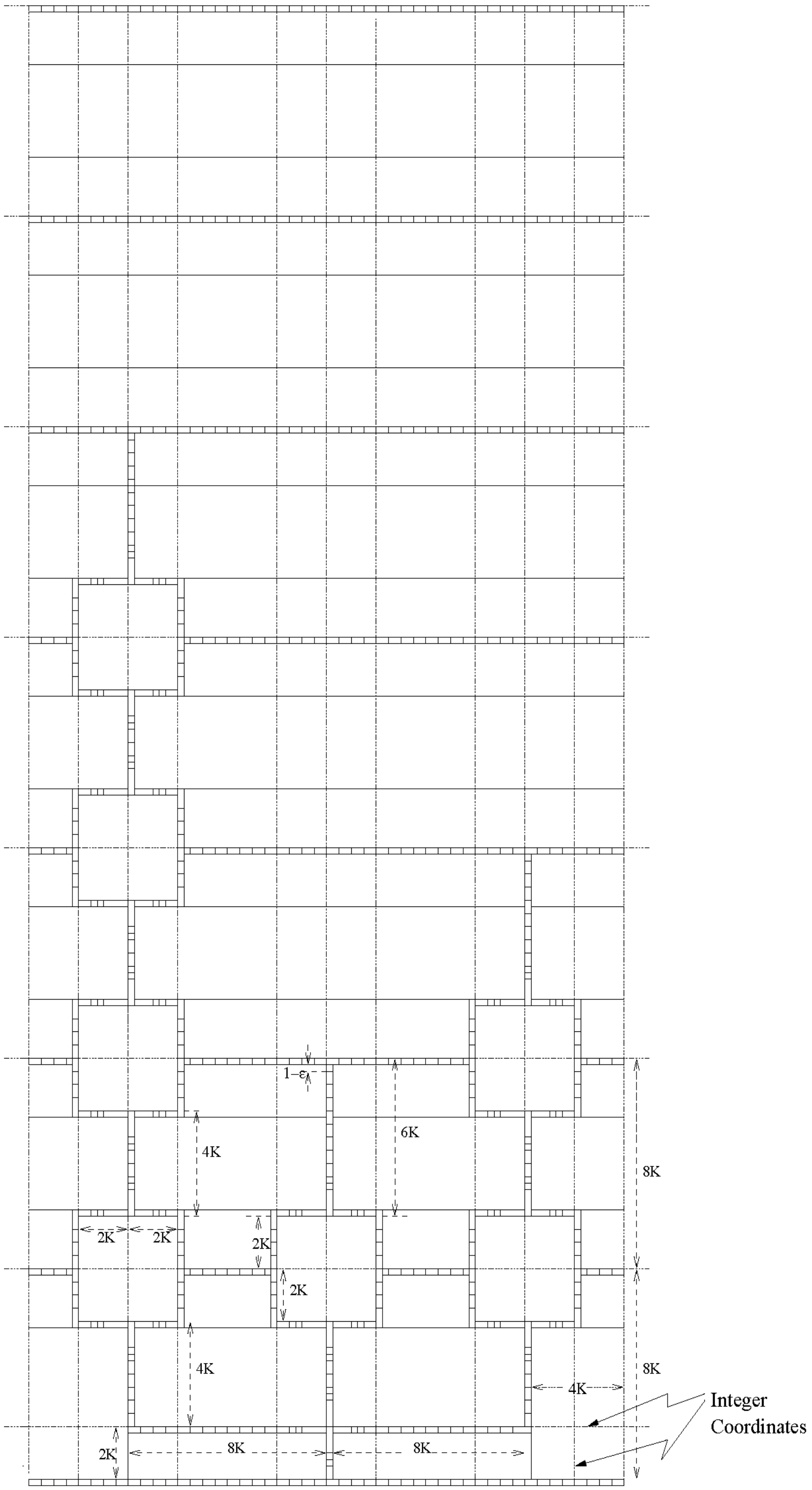

Figure 3: Floorplan of P_i with n=2 and F_i={ 4, 1, 2}

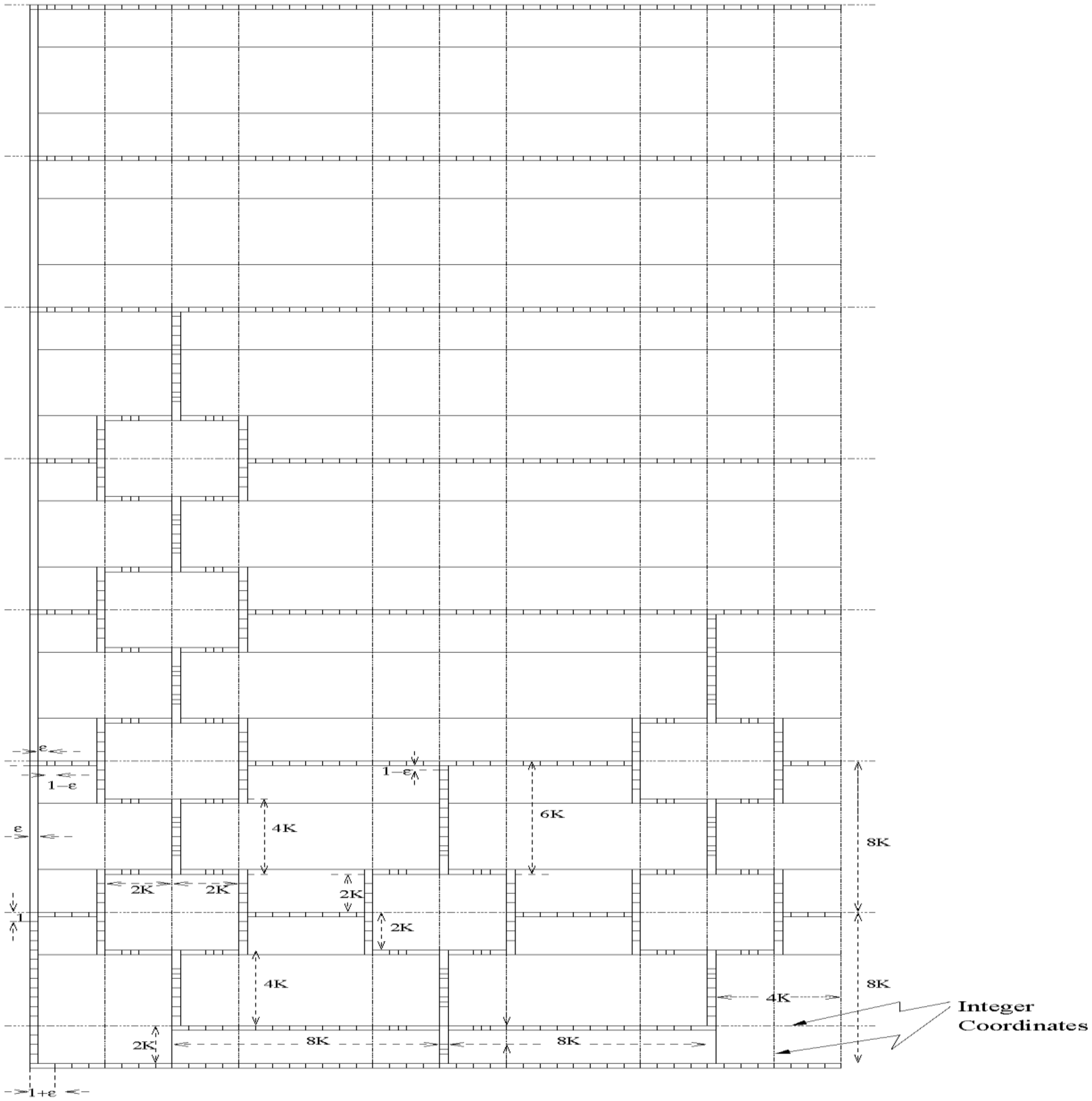

Figure 4: Floorplan of p1 with stack of rectangles and terminals at the left

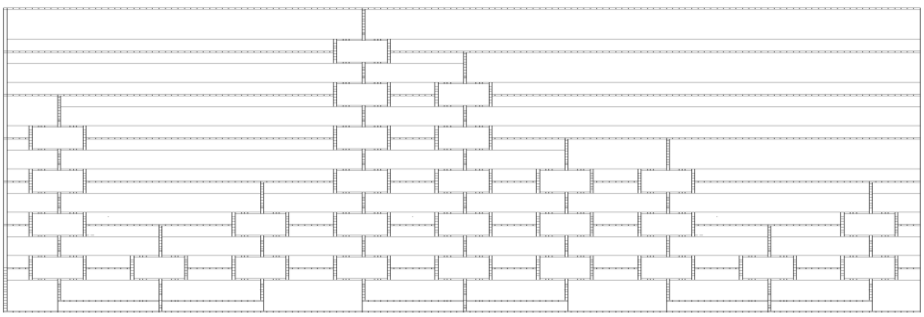

Figure 5: Floorplan of P for F={{4, 1 ,2},{6, 5, 3},{3, 1 ,2}}

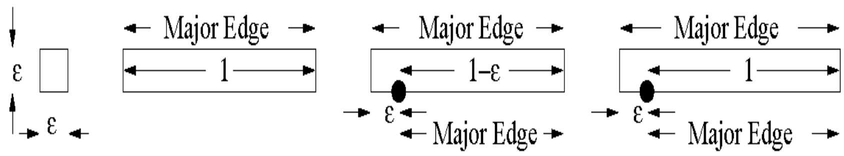

Figure 6: R0 Rectangles

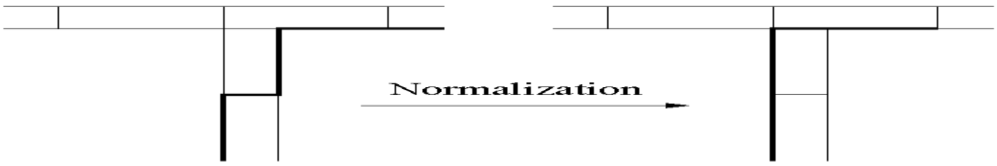

Figure 7: Tree Normalization

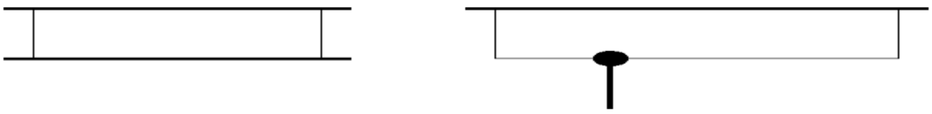

Figure 8: Two T0- components at the same R0 Rectangles

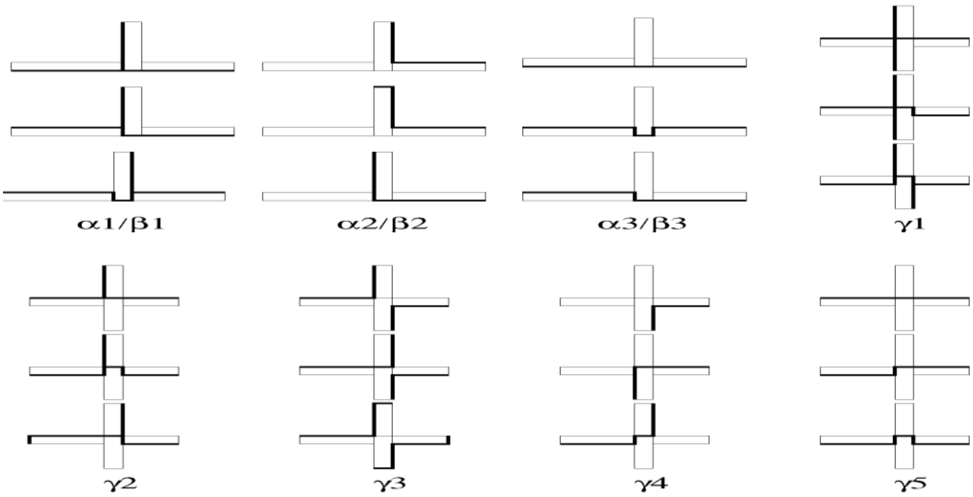

Figure 9 : Active region connection Patterns

3. CONCLUSION

In this paper we shows that wiring on rectangular floorplan is NP- hard problem by converting it into well known exact 3 cover (X3C). Due to wider application of this problem discuss above lead to think us, is there any good approximation algorithms that give good bound for this problem..

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