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Some Variations of Y – Dominating Functions of Corona Product Graph of a Cycle with a Complete Graph

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ABSTRACT
Domination in graphs has been studied extensively and at present it is an emerging area of research in graph theory. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [1, 2]. Dominating sets have applications in diverse areas such as logistics and networks design, mobile computing, resource allocation and telecommunication etc.

Product of graphs occurs naturally in discrete mathematics as tools in combinatorial constructions. In this paper we present some results on minimal Y- dominating functions of corona product graph of a cycle with a complete graphs..

Key Words
Corona Product, Signed dominating function, Minus dominating function, Roman dominating function.

Subject Classification: 68R10

1. INTRODUCTION
Among the variations of domination, there is an extensive study of Y-domination and its variations. A Y – dominating function of a graph $G(V,E)$ is a function $f : V \to Y$ such that $\sum_{u \in N_G[v]} f(u) \geq 1$, for each $v \in V$. Then the Y – domination problem is to find a Y – dominating function of minimum weight for a graph.

Recently, dominating functions in domination theory have received much attention. A purely graph – theoretic motivation is given by the fact that the dominating function problem can be seen, in a clear sense, as a proper generalization of the classical domination problem.

Product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structural problems. There are four main products that have been studied in the literature: the Cartesian product, the strong product, the direct product and the Lexicographic product of finite and infinite graphs. A new and simple operation on two graphs G_1 and G_2 called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G_1 and of G_2 is constructed [3].

Some results on the minimal dominating functions of corona product graph of a cycle with a complete graph are presented in [4]. In this paper, we study the variations of Y – domination such as signed domination, minus domination, efficient minus domination and Roman domination for these graphs.

2. CORONA PRODUCT OF C_n AND K_m
The **corona product** of a cycle C_n with a complete graph K_m is a graph obtained by taking one copy of a n – vertex graph C_n and n copies of K_m and then joining the i^{th}vertex of C_n to every vertex of i^{th} copy of K_m and it is denoted by $C_n \odot K_m$.
By the definition of the corona product of a cycle with a complete graph the proof of the following theorem is immediate.

Theorem 2.1: The degree of a vertex v in G $= C_n \odot K_m$ is given by

$$d(v) = \begin{cases} m+2, & if\ v \in C_n, \\ m, & if\ v \in K_m. \end{cases}$$

3. SIGNED DOMINATING FUNCTIONS
In this section we present some results on minimal signed dominating functions of the graph $G = C_n \odot K_m$. Let us define the signed dominating function and minimal signed dominating function of a graph $G(V,E)$.

Definition: Let $G(V,E)$ be a graph. A function $f : V \to \{-1,1\}$ is called a **signed dominating function** (SDF) of G if $f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1,\ for\ each\ v \in V.$

A signed dominating function f of G is called a **minimal signed dominating function** (MSDF) if for all $g < f, g$ is not a signed dominating function.

Theorem 3.1: A function $f : V \to \{-1,1\}$ defined by
$f(v) =$

$$\begin{cases} -1, & \text{for } \left\lfloor \frac{m}{2} \right\rfloor \text{ vertices in each copy of } K_m \text{ in } G, \\ 1, & \text{otherwise.} \end{cases}$$

is a MSDF of $G = C_n \odot K_m$.

Proof: Let f be a function defined as in the hypothesis.

Case I: Suppose m is even. Then $\left\lfloor \frac{m}{2} \right\rfloor = \frac{m}{2}$.

By the definition of the function, -1 is assigned to $\frac{m}{2}$ vertices in each copy of K_m in G and 1 is assigned to $\frac{m}{2}$ vertices in each copy of K_m in G. Also 1 is assigned to the vertices of C_n of G.

Case 1: Let $v \in C_n$ be such that $d(v) = m+2$ in G.

Then $N[v]$ contains m vertices of K_m and three vertices of C_n in G.

So $\sum_{u\in N[v]} f(u) = 1+1+1+\left[\frac{m}{2}(-1)+\frac{m}{2}(1)\right] = 3.$

Case 2: Let $v \in K_m$ be such that $d(v) = m$ in G.

Then $N[v]$ contains m vertices of K_m and one vertex of C_n in G.

So $\sum_{u\in N[v]} f(u) = 1+\left[\frac{m}{2}(-1)+\frac{m}{2}(1)\right] = 1.$

Therefore for all possibilities, we get $\sum_{u\in N[v]} f(u) \geq 1,$

$\forall \; v \in V.$

This implies that f is a SDF. Now we check for the minimality of f.

Define $g : V \to \{-1,1\}$ by

$$g(v) = \begin{cases} -1, & \text{for any one vertex } v_k \text{ of } C_n \text{ in } G, \\ -1, & \text{for } \left\lfloor \frac{m}{2} \right\rfloor \text{ vertices of } K_m \text{ in each copy in } G, \\ 1, & \text{otherwise.} \end{cases}$$

Case (i): Let $v \in C_n$ be such that $d(v) = m+2$ in G.

Sub case 1: Let $v_k \in N[v]$.

Then

$$\sum_{u\in N[v]} g(u) = (-1)+1+1+\left[\frac{m}{2}(-1)+\frac{m}{2}(1)\right] = 1.$$

Sub case 2: Let $v_k \notin N[v]$.

Then

$$\sum_{u\in N[v]} g(u) = 1+1+1+\left[\frac{m}{2}(-1)+\frac{m}{2}(1)\right] = 3.$$

Case (ii): Let $v \in K_m$ be such that $d(v) = m$ in G.

Sub case 1: Let $v_k \in N[v]$.

Then $\sum_{u\in N[v]} g(u) = (-1)+\left[\frac{m}{2}(-1)+\frac{m}{2}(1)\right] = -1.$

Sub case 2: Let $v_k \notin N[v]$.

Then $\sum_{u\in N[v]} g(u) = 1+\left[\frac{m}{2}(-1)+\frac{m}{2}(1)\right] = 1.$

This implies that $\sum_{u\in N[v]} g(u) < 1,$ for some $v \in V$.

So g is not a SDF.

Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is a SDF.

Thus f is a MSDF.

Case II: Suppose m is odd. Then $\left\lfloor \frac{m}{2} \right\rfloor = \frac{m-1}{2}$.

By the definition of the function, -1 is assigned to $\frac{m-1}{2}$ vertices in each copy of K_m in G and 1 is assigned to $\frac{m+1}{2}$ vertices in each copy of K_m in G. Also 1 is assigned to the vertices of C_n in G.

Case 1: Let $v \in C_n$ be such that $d(v) = m+2$ in G.

Then $N[v]$ contains m vertices of K_m and three vertices of C_n in G. So

$$\sum_{u\in N[v]} f(u) = 1+1+1+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(1)\right]$$

$$= 3+\left[-\frac{m}{2}+\frac{1}{2}+\frac{m}{2}+\frac{1}{2}\right] = 3+1 = 4.$$

Case 2: Let $v \in K_m$ be such that $d(v) = m$ in G.

Then $N[v]$ contains m vertices of K_m and one vertex of C_n in G.

So $\sum_{u\in N[v]} f(u) = 1+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(1)\right]$

$$= 1+\left[-\frac{m}{2}+\frac{1}{2}+\frac{m}{2}+\frac{1}{2}\right] = 1+1 = 2.$$

Therefore for all possibilities, we get $\sum_{u\in N[v]} f(u) > 1$, $\forall\ v \in V$.

This implies that f is a SDF. Now we check for the minimality of f.

Define $g : V \to \{-1,1\}$ by

$$g(v) = \begin{cases} -1, & \text{for any one vertex } v_k \text{ of } C_n \text{ in } G, \\ -1, & \text{for } \left\lfloor \frac{m}{2} \right\rfloor \text{ vertices in each copy of } K_m \text{ in } G, \\ 1, & \text{otherwise.} \end{cases}$$

Case (i): Let $v \in C_n$ be such that $d(v) = m + 2$ in G.

Sub case 1: Let $v_k \in N[v]$. Then

$$\sum_{u\in N[v]} g(u) = (-1)+1+1+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(1)\right]$$

$$= 1+\left[-\frac{m}{2}+\frac{1}{2}+\frac{m}{2}+\frac{1}{2}\right] = 1+1 = 2.$$

Sub case 2: Let $v_k \notin N[v]$. Then

$$\sum_{u\in N[v]} g(u) = 1+1+1+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(1)\right]$$
$$= 3+1 = 4.$$

Case (ii): Let $v \in K_m$ be such that $d(v) = m$ in G.

Sub case 1: Let $v_k \in N[v]$. Then

$$\sum_{u\in N[v]} g(u) = (-1)+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(1)\right]$$
$$= -1+1 = 0.$$

Sub case 2: Let $v_k \notin N[v]$. Then

$$\sum_{u\in N[v]} g(u) = 1+\left[\left(\frac{m-1}{2}\right)(-1)+\left(\frac{m+1}{2}\right)(1)\right]$$
$$= 1+1 = 2.$$

This implies that $\sum_{u\in N[v]} g(u) < 1$, for some $v \in V$.

So g is not a SDF.

Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is a SDF.

Thus f is a MSDF. ■

4. MINUS DOMINATING FUNCTIONS

In this section we study the concepts of minimal minus dominating functions and efficient minus domination functions of the corona product graph $C_n \odot K_m$ and some results on these functions are obtained. We now define the minus dominating function and efficient minus dominating function of a graph $G(V,E)$ as follows.

Definition: Let $G(V,E)$ be a graph. A function $f : V \to \{-1,0,1\}$ is called a **minus dominating function** (Minus DF) of G if

$$f(N[v]) = \sum_{u\in N[v]} f(u) \ge 1, \text{ for each } v \in V.$$

A minus dominating function f of G is called a **minimal minus dominating function** (M Minus DF) if for all $g < f$, g is not a minus dominating function.

Definition: Let $G(V,E)$ be a graph. A function $f : V \to \{-1,0,1\}$ is called an **efficient minus dominating function** (E Minus DF) of G if

$$f(N[v]) = \sum_{u\in N[v]} f(u) = 1, \text{ for each } v \in V.$$

Theorem 4.1: A function $f : V \to \{-1,0,1\}$ defined by

$$f(v) = \begin{cases} -1, & \text{for any one vertex } v = v_i \text{ in each copy of } K_m \text{ in } G, \\ 1, & \text{for any one vertex } v = v_j,\ i \neq j \text{ in each copy of } K_m \\ & \text{and for the vertices of } C_n \text{ in } G, \\ 0, & \text{for the vertices } v \text{ of } K_m \text{ where } v \neq v_i, v_j. \end{cases}$$

is a minimal minus dominating function of $G = C_n \odot K_m$.

Proof: Let f be a function defined as in the hypothesis.

Case 1: Let $v \in C_n$ be such that $d(v) = m + 2$ in G.

Then $N[v]$ contains m vertices of K_m and three vertices of C_n in G.

So

$$\sum_{u\in N[v]} f(u)=1+1+1+\left[(-1)+1+\underbrace{0+0+......+0}_{(m-2)-times}\right]=3.$$

Case 2: Let $v \in K_m$ be such that $d(v) = m$ in G.

Then $N[v]$ contains m vertices of K_m and one vertex of C_n in G.

So

$$\sum_{u\in N[v]} f(u)=1+\left[(-1)+1+\underbrace{0+0+........+0}_{(m-2)-times}\right]=1.$$

Therefore for all possibilities, we get $\sum_{u\in N[v]} f(u)\geq 1$,

$\forall \quad v \in V$.

This implies that f is a minus dominating function.

Define $g: V \to \{-1,0,1\}$ by

$$g(v)=\begin{cases}-1, & \text{for any one vertex } v=v_i \text{ in each copy of } K_m \text{ and one vertex } v_k \text{ of } C_n \text{ in } G,\\ 1, & \text{for any one vertex } v=v_j,\ i\neq j \text{ in each copy of } K_m \text{ and } (n-1) \text{ vertices of } C_n \text{ in } G,\\ 0, & \text{for the vertices } v \text{ of } K_m \text{ where } v\neq v_i,\ v_j.\end{cases}$$

Case (i): Let $v \in C_n$ be such that $d(v) = m + 2$ in G.

Sub case 1: Let $v_k \in N[v]$.

Then

$$\sum_{u\in N[v]} g(u)=(-1)+1+1+\left[(-1)+1+\underbrace{0+0+........+0}_{(m-2)-times}\right]$$
$$=1.$$

Sub case 2: Let $v_k \notin N[v]$. Then

$$\sum_{u\in N[v]} g(u)=1+1+1+\left[(-1)+1+\underbrace{0+0+........+0}_{(m-2)-times}\right]$$
$$=3.$$

Case (ii): Let $v \in K_m$ be such that $d(v) = m$ in G.

Sub case 1: Let $v_k \in N[v]$. Then

$$\sum_{u\in N[v]} g(u)=(-1)+\left[(-1)+1+\underbrace{0+0+........+0}_{(m-2)-times}\right]=-1.$$

Sub case 2: Let $v_k \notin N[v]$. Then

$$\sum_{u\in N[v]} g(u)=1+\left[(-1)+1+\underbrace{0+0+........+0}_{(m-2)-times}\right]=1.$$

This implies that $\sum_{u\in N[v]} g(u)<1$, for some $v \in V$.

So g is not a Minus DF.

Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is a Minus DF.

Thus f is a M Minus DF. ∎

Theorem 4.2: A function $f: V \to \{-1,0,1\}$ defined by

$$f(v)=\begin{cases}-1, & \text{for any one vertex } v=v_i \text{ in each copy of } K_m \text{ in } G,\\ 1, & \text{for any two vertices } v_j,\ v_k,\ i\neq j\neq k \text{ in each copy of } K_m \text{ in } G,\\ 0, & \text{otherwise.}\end{cases}$$

is an efficient minus dominating function of $G = C_n \odot K_m$ where $m \geq 3$.

Proof: Let f be a function defined as in the hypothesis.

The summation value taken over N[v] of $v \in V$ is as follows:

Case 1: Let $v \in C_n$ be such that $d(v) = m + 2$ in G.

Then $N[v]$ contains m vertices of K_m and three vertices of C_n in G. So

$$\sum_{u\in N[v]} f(u)=0+0+0+\left[(-1)+1+1+\underbrace{0+0+...+0}_{(m-3)-times}\right]$$
$$=1.$$

Case 2: Let $v \in K_m$ be such that $d(v) = m$ in G.

Then $N[v]$ contains m vertices of K_m and one vertex of C_n in G. So

$$\sum_{u\in N[v]} f(u)=0+\left[(-1)+1+1+\underbrace{0+0+........+0}_{(m-3)-times}\right]=1.$$

Therefore for all possibilities, we get $\sum_{u \in N[v]} f(u) = 1$,

$\forall \; v \in V$.

This implies that f is an efficient minus dominating function of $G = C_n \odot K_m$ where $m \geq 3$. ■

5. ROMAN DOMINATING FUNCTIONS

In this section we prove results on minimal Roman dominating functions of $G = C_n \odot K_m$. First we define Roman dominating function of a graph $G(V, E)$.

Definition: Let $G(V, E)$ be a graph. A function $f : V \to \{0, 1, 2\}$ is called a **Roman dominating function** (RDF) of G if

$$f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1, \text{ for each } v \in V$$

and satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$.

A Roman dominating function f of G is called a **minimal Roman dominating function** (MRDF) if for all $g < f$, g is not a roman dominating function.

Theorem 5.1: A function $f : V \to \{0, 1, 2\}$ defined by

$$f(v) = \begin{cases} 2, & \text{for any one vertex } v = v_i \text{ in each copy of } K_m \text{ in } G, \\ 0, & \text{otherwise.} \end{cases}$$

is a Minimal Roman Dominating Function of $G = C_n \odot K_m$.

Proof: Let f be a function defined as in the hypothesis.

Case 1: Let $v \in C_n$ be such that $d(v) = m + 2$ in G.

Then $N[v]$ contains m vertices of K_m and three vertices of C_n in G. So

$$\sum_{u \in N[v]} f(u) = 0 + 0 + 0 + \left[2 + \underbrace{0 + 0 + \dots\dots + 0}_{(m-1)-times} \right] = 2.$$

Case 2: Let $v \in K_m$ be such that $d(v) = m$ in G.

Then $N[v]$ contains m vertices of K_m and one vertex of C_n in G.

So $$\sum_{u \in N[v]} f(u) = 0 + \left[2 + \underbrace{0 + 0 + \dots\dots + 0}_{(m-1)-times} \right] = 2.$$

Therefore for all possibilities, we get

$$\sum_{u \in N[v]} f(u) > 1, \;\; \forall \;\; v \in V.$$

Let u be any vertex in G such that $f(u) = 0$ and $v \neq u$ be a vertex in G such that $f(v) = 2$.

Then $u \in C_n$ or $u \in K_m$ and $v \in K_m$.

Obviously if $u \in C_n$ is adjacent to v, since every vertex in C_n is adjacent to every vertex in the corresponding copy of K_m. Also if $u \in K_m$ then u is adjacent to $v \in K_m$, since K_m is a complete graph.

This implies that f is a RDF.

Now we check for the minimality of f.

Define $g : V \to \{0, 1, 2\}$ by

$$g(v) = \begin{cases} 1, & \text{for any one vertex } v = v_k \text{ in one copy of } K_m \text{ in } G, \text{ say } i^{th} \text{ copy} \\ 2, & \text{for any one vertex } v = v_k \text{ in } (n-1) \text{ copies of } K_m \text{ of } G, \\ 0, & \text{otherwise.} \end{cases}$$

Case (i): Let $v \in C_n$ be such that $d(v) = m + 2$ in G.

Sub case 1: Let $v_k \in N[v]$, where v_k is in the i^{th} copy of K_m in G. Then

$$\sum_{u \in N[v]} g(u) = 0 + 0 + 0 + \left[1 + \underbrace{0 + 0 + \dots\dots + 0}_{(m-1)-times} \right] = 1.$$

Sub case 2: Let $v_k \notin N[v]$. Then

$$\sum_{u \in N[v]} g(u) = 0 + 0 + 0 + \left[2 + \underbrace{0 + 0 + \dots\dots + 0}_{(m-1)-times} \right] = 2.$$

Case (ii): Let $v \in K_m$ be such that $d(v) = m$ in G.

Sub case 1: Let $v_k \in N[v]$, where v_k is in the i^{th} copy of K_m in G.

Then $$\sum_{u \in N[v]} g(u) = 0 + \left[1 + \underbrace{0 + 0 + \dots\dots + 0}_{(m-1)-times} \right] = 1.$$

Sub case 2: Let $v_k \notin N[v]$.

Then $\sum_{u \in N[v]} g(u) = 0 + \left[2 + \underbrace{0 + 0 + \ldots\ldots + 0}_{(m-1)-times} \right] = 2.$

Therefore $\sum_{u \in N[v]} g(u) \geq 1, \ \forall \ v \in V.$

i.e. g is a DF. But g is not a RDF, since the RDF definition fails in the i^{th} copy of K_m in G because the vertex u in the i^{th} copy of K_m in G for which $f(u) = 0$ is adjacent to the vertex v_k for which $f(v_k) = 1$. Therefore f is a MRDF. ∎

6. CONCLUSION

It is interesting to study the dominating functions of the corona product graph of a cycle with a complete graph. This work gives the scope for the study of total Y – dominating functions of these graphs and the authors have also studied this concept.

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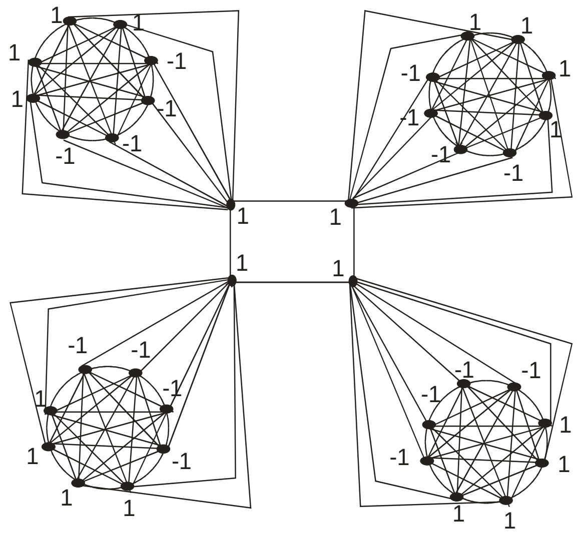
ILLUSTRATION



$C_4 \odot K_8$

The function f takes the value 1 for vertices of C_n and value -1 for vertices of K_m in each copy.

REFERENCES

[1] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J. 1998, Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York.

[2] Haynes, T.W., Hedetniemi, S.T. and Slater, P.J. 1998, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York .

[3] Frucht, R. and Harary, F, On the corona of Two Graphs, Aequationes Mathematicae, Volume 4, Issue 3, (1970), 322 – 325.

[4] M. Siva Parvathi and B.Maheswari, Minimal Dominating Functions of Corona Product Graph of a Cycle with a Complete Graph - International Journal of Computer Engineering & Technology, Volume 4, Issue 4, (July – August. 2013), 248 – 256.