The Connected Open Monophonic Number of a Graph

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Abstract

In this paper, we introduce and investigate the connected open monophonic sets and related parameters. For a connected graph G of order n, a subset S of vertices of G is a monophonic set of G if each vertex v in G lies on a x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is defined as the monophonic number of G, denoted by m(G). A monophonic set of cardinality m(G) is called a m–set of G. A set S of vertices of a connected graph G is an open monophonic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y monophonic path for some x, $y \in S$. An open monophonic set of minimum cardinality is a minimum open monophonic set and this cardinality is the open monophonic number, om(G). A connected open monophonic set of G is an open monophonic set S such that the subgraph $< S >$ induced by S is connected. The minimum cardinality of a connected open monophonic set of G is the connected open monophonic number, $om_c(G)$. Certain general properties satisfied by connected open monophonic sets are investigated. The connected open monophonic numbers of certain standard graphs are determined. A necessary condition for the connected open monophonic number of a graph G of order n to be n is determined. A graph with connected open monophonic number 2 is characterized. It is proved that for any k, n of integers with $3 \leq k \leq n$, there exists a connected graph G of order n such that $om_c(G) = k$.

Keywords

Distance, monophonic path, monophonic number, open monophonic number, connected open monophonic number.

1. INTRODUCTION

By a graph G = (V, E) we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [4]. The distance d(u,v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length d(u,v) is called an u-v geodesic. It is known that this distance is a metric on the vertex set V(G). The neighborhood of a vertex v is the set N(v) consisting of all vertices which are adjacent with v. The vertex v is an extreme vertex of G if the subgraph induced by its neighbors is complete. For a cutvertex v in a connected graph G and a component H of G – v, the subgraph H and the vertex v together with all edges joining v and V(H) is called a branch of G at v. A geodetic set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number g(G) of G is the cardinality of a minimum geodetic set. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P and x is called an internal vertex of P if $x \neq u, v$. A set S of vertices of a connected graph G is an open geodetic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y geodesic for some x, $y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number og(G). It is clear that every open geodetic set is a geodetic set so that $g(G) \leq og(G)$. The geodetic number of a graph was introduced and studied in [1, 2]. The open geodetic number of a graph was introduced and studied in [3, 5, 7] in the name open geodomination in graphs. A chord of a path $u_1, u_2, \ldots, u_n$ in G is an edge u_iu_j with $j \geq i + 2$. For two vertices u and v in a connected graph G, a u-v path is called a monophonic path if it contains no chords. A set S of vertices in a connected graph G is a monophonic set of G if every vertex of G is contained in a monophonic path joining some pair of vertices in S. The monophonic number m(G) of G is the cardinality of a minimum monophonic set. A set S of vertices in a connected graph G is an open monophonic set if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y monophonic path for some x, $y \in S$. An open monophonic set of minimum cardinality is a minimum open monophonic set and this cardinality is the open monophonic number om(G) of G.

The following theorems are used in the sequal.

Theorem 1.1. [8] Every open monophonic set of a graph G contains its extreme vertices. Also, if the set S of all extreme vertices of G is an open monophonic set, then S is the unique minimum open monophonic set of G.

Theorem 1.2. [8] If G is a non-trivial connected graph with no extreme vertices, then $om(G) \geq 3$.

2. CONNECTED OPEN MONOPHONIC NUMBER OF A GRAPH

Definition 2.1 Let G be a connected graph with at least two vertices. A connected open monophonic set of G is an open monophonic set S such that the subgraph $< S >$ induced by S is connected. The minimum cardinality of a connected open monophonic set of G is the connected open monophonic number of G and is denoted by $om_c(G)$ of G. A connected open monophonic set of cardinality $om_c(G)$ is called om_c-set of G.

Example 2.2. For the graph G given in Figure 1, $S = \{v_1, v_2, v_7, v_8\}$ is the unique minimum open monophonic set of G so that om(G) = 4. Also, $S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ is an $om_c(G)$-set so that $om_c(G) = 8$. The set $S_1 = \{v_1, v_2, v_3, v_6, v_7, v_8, v_9, v_{10}\}$ is another $om_c(G)$-set of G. Thus there can be more than one $om_c(G)$-set for a graph.

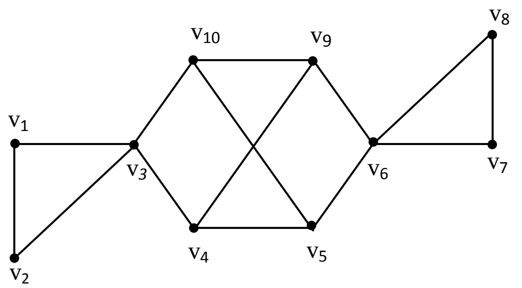

Fig 1: A graph with connected monophonic number 8.

Since every connected open monophonic set is also an open monophonic set and every open monophonic set is a monophonic set, the next result follows from Theorem1.1[8]

Theorem 2.3. Every extreme vertex of a connected graph G belongs to each connected open monophonic set of a graph G. In particular, every end vertex of G belongs to each connected open monophonic set of G.

A monophonic set needs at least two vertices and so $m(G) \geq 2$. Every open monophonic set is a monophonic set and it follows that $m(G) \leq om(G)$. Since every connected open monophonic set is also an open monophonic set, we have $om(G) \leq om_c(G)$. Also, since the set of all vertices of G forms a connected open monophonic set of G, we have $om_c(G) \leq n$. Thus we have the following theorem.

Theorem 2.4. For any connected graph G of order $n \geq 2$, $2 \leq m(G) \leq om(G) \leq om_c(G) \leq n$.

Remark 2.5. For the complete graph $G = K_n (n \geq 2)$, $m(G) = om(G) = om_c(G) = n$. Also, all the inequalities in Theorem 2.4 can be strict. For the graph G given in Figure 2, $S = \{v_1, v_2\}$ is the unique minimum monophonic set so that $m(G) = 2$. The set $S_1 = \{v_1, v_2, w_1, w_2, w_3, w_4\}$ is a minimum open monophonic set of G so that $om(G) = 6$. Also, $S_2 = \{v_1, v_2, w_1, w_2, w_3, w_4, x\}$ is a minimum connected open monophonic set of G so that $om_c(G) = 7$. Thus $m(G) < om(G) < om_c(G) < n$.

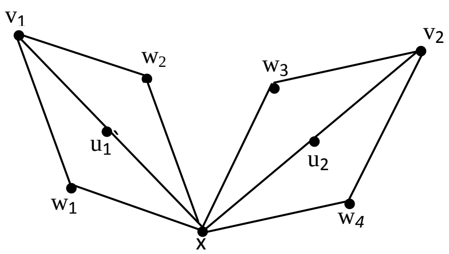

Fig 2: A graph with all the inequalities in Theorem 2.4 strict.

Corollary 2.6. If G is any connected graph such that $om_c(G) = 2$, then $m(G) = om(G) = 2$.

Corollary 2.7. If G is any connected graph of order n such that $m(G) = n$, then $om(G) = om_c(G) = n$.

The following result is a consequence of Theorems 2.3 and 2.4.

Corollary 2.8. For any connected graph G with k extreme vertices, $om_c(G) \geq \max\{2, k\}$.

Theorem 2.9. Let G be a connected graph with cutvertices and let S be a connected open monophonic set of G. If v is a cutvertex of G, then every component of G – v contains an element of S.

Proof. Let v be a cutvertex of G and let S be a connected open monophonic set of G. Suppose that there exits a component, say G_1 of G – v such that G_1 contains no vertex of S. By Theorem 2.3, S contains all the extreme vertices of G and hence it follows that G_1 does not contain any extreme vertex of G. Let u be a vertex of G_1. Since S is a connected open monophonic set of G, there exist vertices $x, y \in S$ such that u is an internal vertex of some x – y monophonic path P: $u_0, u_1, u_2, \ldots, u, \ldots, u_l = y$ in G. Since v is a cutvertex of G, the x – u subpath of P and u – y subpath of P both contain v, and it follows that P is not a path, contrary to assumption.

Corollary 2.10. Let G be a connected graph with cutvertices and let S be a connected open monophonic set of G. Then every branch of G contains an element of S.

Theorem 2.11. Each cutvertex of connected graph G belongs to every minimum connected open monophonic set of G.

Proof. Let S be a connected open monophonic set of G and let v be a cutvertex of G. Let $G_1, G_2, G_3, \ldots, G_r$ $(r \geq 2)$ be the components of G – v. By Theorem 2.9, S contains at least one vertex from each G_i $(1 \leq i \leq r)$. Since the subgraph induced by S is connected, it follows that $v \in S$.

As a consequence of Theorems 2.3 and 2.11, the next result follows.

Corollary 2.12. For any connected graph G with k extreme vertices and l cutvertices, $om_c(G) \geq \max\{2, k + l\}$.

Corollary 2.13. For any non-trivial tree T of order n, $om_c(T) = n$.

Corollary 2.14. Let G be a connected graph of order n. If every vertex of G is either an extreme vertex or a cutvertex of G, then $om_c(G) = n$.

Proof. This follows from Theorems 2.3 and 2.11.

The converse of Corollary 2.14 need not be true. For the graph G given in Figure 3, $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the only minimum connected open monophonic set of G so that $om_c(G) = 6 = n$.

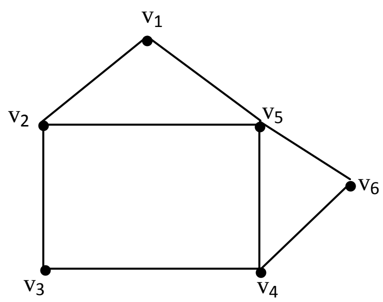

Fig 3: A graph illustrating the failure of converse of Corollary 2.14.

The following theorem characterizes graphs for which the connected open monophonic number is 2.

Theorem 2.15. For any connected graph G, $om_c(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $om_c(G) = 2$. Conversely, let $om_c(G) = 2$. Let S = {u, v} be a minimum connected open monophonic set of G. Then uv is an edge. If $G \neq K_2$, then there exists a vertex w different from u and v. Then w cannot lie as an internal vertex of any u – v monophonic path and so S is not a connected open monophonic set, which is a contradiction. Hence $G = K_2$.

3. CONNECTED OPEN MONOPHONIC NUMBER OF SOME STANDARD GRAPHS

It is proved in [8] that for the cycle $G = C_n$ ($n \geq 4$),

$$om(G) = \begin{cases} 3 & \text{if } n \geq 6 \\ 4 & \text{if } n = 4, 5. \end{cases}$$

Now, we give below the connected open monophonic number of the cycle C_n ($n \geq 4$).

Theorem 3.1 For the cycle $G = C_n (n \geq 4)$,

$$om_c(G) = \begin{cases} 5 & \text{if } n \geq 6 \\ 4 & \text{if } n = 4, 5. \end{cases}$$

Proof. Let the cycle $G = C_n$ ($n \geq 6$) be $C_n : v_1, v_2, \ldots, v_n, v_1$. Since G has no extreme vertices, it follows from Theorem 1.2[8] that $om(G) \geq 3$. Hence by Theorem 2.4, $om_c(G) \geq 3$. Now, $S = \{v_1, v_3, v_5\}$ is a minimum open monophonic set. However, the subgraph < S > induced by S is not connected and it is clear that $S_1 = \{ v_1, v_2, v_3, v_4, v_5\}$ is a minimum connected open monophonic set of G. Thus $om_c(G) = 5$. For $G = C_4$, it is clear that no 3-element subset of vertices is an open monophonic set of G. Hence it follows that $om_c(G) = 4$. For $G = C_5$, it is easily seen that no 3-element subset of vertices is an open monophonic set of G. Since $S = \{v_1, v_2, v_3, v_4\}$ is a connected open monophonic set of G, it follows that $om_c(G) = 4$. Thus the proof of the theorem is complete.

Theorem 3.2. For the complete bipartite graph $G = K_{r,s}$ ($2 \leq r \leq s$), $om_c(G) = 4$.

Proof. Let $G = K_{r,s}$ ($2 \leq r \leq s$). Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of G. Since G contains no extreme vertices, it follows from Theorems 1.2[8] and 2.4, $om_c(G) \geq 3$. It is easily verified that no 3-element subset of vertices of G is an open monophonic set of G so that $om_c(G) \geq 4$. Let S be any set of four vertices formed by taking two vertices from each of U and W. Then it is clear that S is a connected open monophonic set of G so that $om_c(G) = 4$.

Theorem 3.3 For any wheel $W_n = K_1 + C_{n-1}$ ($n \geq 5$),

$$om_c(W_n) = \begin{cases} 5 & \text{if } n \geq 7 \\ 4 & \text{if } n = 5, 6. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ ($n \geq 5$). Let $n \geq 7$. Since W_n has no extreme vertices, by Theorem 1.2[8], $om(G) \geq 3$. Hence by Theorem 2.4, $om_c(G) \geq 3$. The set $S = \{v_1, v_3, v_5\}$ is an open monophonic set of W_n such that the subgraph < S > induced by S is not connected. It is clear that $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimum connected open monophonic set so that $om_c(G) = 5$.

Let $W_n = K_1 + C_{n-1}$ (n = 5, 6). Since W_n has no extreme vertices, by Theorem 1.2[8], $om(W_n) \geq 3$. It is easily verified that no 3-element subset of vertices of W_n is an open monophonic set. Since $S = \{v_1, v_2, v_3, v_4\}$ is a connected open monophonic set of W_n, it follows that $om_c(W_n) = 4$. Thus the proof is complete.

Theorem 3.4 Let G be a connected graph of order $n \geq 3$. If G contains exactly one vertex of degree $n - 1$ such that it is not a cutvertex of G, then $om_c(G) \leq n - 1$.

Proof. Let x be the unique vertex of degree $n - 1$. It is clear that x is not an extreme vertex of G. Let $S = V - \{x\}$. We show that S is a connected open monophonic set of G. Since x is not an extreme vertex, there exist non-adjacent neighbors y and z of x. Hence x lies as an internal vertex of the y – z monophonic path y, x, z, where $y, z \in S$. Now, let $u \in S$. Suppose that u is not an extreme vertex of G. Then < N(u) > is not complete in < S >. This means that there exist non-adjacent neighbors v, w of u such that $v, w \in S$. This shows that u lies as an internal vertex of the v – w monophonic path v, u, w. Hence S is an open monophonic set of G. Since x is not a cutvertex of G, < S > is connected so that S is a connected open monophonic set of G. Thus $om_c(G) \leq |S| = n - 1$.

Remark 3.5 The bound in Theorem 3.4 is sharp . By Theorem 3.3, $om_c(W_5) = 4$ for the wheel $W_5 = K_1 + C_4$. The inequality can also be strict. For the graph G given in Figure 4, u_1 and u_3 are the only extreme vertices and so by Theorem 2.3, every connected open monophonic set contains the vertices u_1 and u_3. Now, it is clear that $S = \{ u_1, u_2, u_3 \}$ is a connected open monophonic set of G. Hence $om_c(G) = 3$.

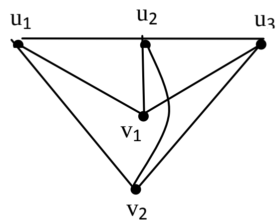

Fig 4: A graph with the inequality in Theorem 3.4 strict.

We now leave the following problem as an open question.

Problem 3.6 Characterize the class of graphs G of order n for which (i) $om_c(G) = n - 1$ and (ii) $om_c(G) = n$.

Theorem 3.7. For any pair k, n of integers with $3 \leq k \leq n$, there exists a connected graph G of order n such that $om_c(G) = k$.

Proof. We prove the theorem by considering three cases.

Case 1. k = 3. Let $P_3 : u_1, u_2, u_3$ be a path on three vertices. Add new vertices $v_1, v_2, \ldots, v_{n-3}$ and join each v_i ($1 \leq i \leq n - 3$) with u_1, u_2, u_3. Also, join each v_i ($1 \leq i \leq n - 3$) with v_j ($i+1 \leq j \leq n - 3$), thereby obtaining the graph G of order n, given in Figure 5. Note that u_1 and u_3 are extreme vertices. It is easily verified that $S = \{u_1, u_2, u_3\}$ is a minimum connected open monophonic set of G and so $om_c(G) = 3 = k$.

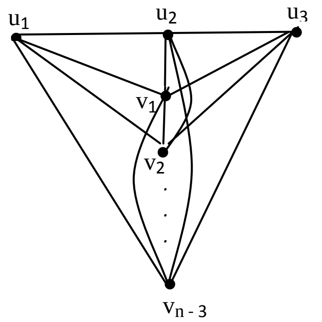

Fig 5: A graph in Case 1 of Theorem 3.7 with $om_c(G)$ =k.

Case 2. k = 4. Let $P_3 : u_1, u_2, u_3$ be a path on three vertices. Add new vertices, $v_1, v_2, \ldots, v_{n-3}$ and join each $v_i(1 \leq i \leq n - 3)$ with u_1 and u_3, thereby obtaining the graph G of order n given in Figure 6. The graph G has no extreme vertices and so by Theorems 1.2[8] and 2.4, $om_c(G) \geq 3$. It is clear that no 3-element subset of vertices of G is a connected open monophonic set of G. It is easily verified that for any v_i $(1 \leq i \leq n - 3)$, $S_1 = \{ u_1, u_2, u_3, v_i\}$ is a minimum connected open monophonic set of G so that $om_c(G) = 4 = k$.

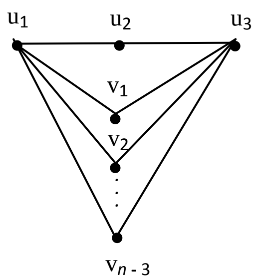

Fig 6:A graph in Case 2 of Theorem 3.7 with $om_c(G)$ =k.

Case 3. $k \geq 5$. Let $P_{k-1} : u_1, u_2, \ldots, u_{k-1}$ be a path on k – 1 vertices. Add new vertices $v_1, v_2, \ldots, v_{n-k+1}$ and join each v_i $(1 \leq i \leq n - k + 1)$ with u_1 and u_3, thereby obtaining the graph G of order n, given in Figure 7. Let $S_2 = \{u_3, u_4, \ldots, u_{k-1}\}$. By Theorems 2.3 and 2.11, $om_c(G) \geq k - 3$. It is clear that S_2 is not a connected open monophonic set of G and so $om_c(G) > k - 3$. Now, $S_2 \cup \{v\}$ is not an open monophonic set of G for any vertex $v \notin S_2$. It is easily verified that $S_2 \cup \{u, v\}$, where $u, v \notin S_2$ is also not a connected open monophonic set of G. Since $S_2 \cup \{u_1, u_2, u_i\}$ for any i $(1 \leq i \leq n - k + 1)$ is a connected open monophonic set of G, it follows that $om_c(G) = k$.

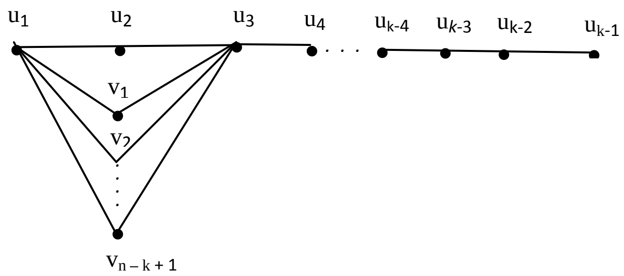

Fig 7: A graph in Case 3 of Theorem 3.7 with $om_c(G)$ =k.

4. CONCLUSION

The connected open monophonic number introduced in this paper has applications in security based communication network in real life applications. Hence the study of this concept is highly useful to human society.

5. FUTURE WORK

Various conditions like independence of vertices can be imposed on open monophonic sets and correspondingly related parameters can be investigated. Further, forcing open monophonic sets can be introduced and corresponding results can be investigated.

6. ACKNOWLEDGMENT

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