Missing Numbers in k-Graceful Graphs

P. Pradhan
Department of Mathematics and Statistics,Gurukula Kangari University, Haridwar (UK), India

Kamesh Kumar
Department of Mathematics and Statistics,Gurukula Kangari University, Haridwar (UK), India

A. Kumar
Department of Mathematics Uttaranchal University, Dehradun (UK), India

ABSTRACT
The generalization of graceful labeling is termed as k-graceful labeling. In this paper it has been shown that C_n ,$n \equiv 0(mod4)$ is k-graceful for any $k \in N$ (set of natural numbers) and some results related to missing numbers for k-graceful labeling of cycle C_n , comb $P_n \odot 1K_1$, hairy cycle $C_n \odot 1K_1$ and wheel graph W_n have been discussed.

Keywords
k-Graceful labeling, k-graceful graphs, missing numbers.

1. INTRODUCTION
A labeling of vertices and edges of a graph G which are required to obey certain condition, have often been motivated by the labeling given by Rosa [10] in 1966. Let $G(V,E)$ be a simple undirected graph with order p and size q, if there exist an injective mapping $f: V(G) \rightarrow \{0,1, \dots \dots, q\}$ that induces a bijective mapping $f^*: E(G) \rightarrow \{1,2, \dots \dots, q\}$ and defined by

$$f^*(u,v) = |f(u) - f(v)| \; \forall \; (u,v) \in E(G) \text{and } u,v \in V(G),$$

Then Rosa [10] called the mapping f the β-labeling (valuation) of a graph G, Golomb [4] subsequently called such labeling to be graceful labeling and the graph is called a graceful graph, while f^* is called an induced edge's graceful labeling. k-graceful labeling is the generalization of graceful labeling that introduced by Slater [11] in 1982 and by Maheo and Thuillier [8] in 1982.

Let $G(V,E)$be a simple undirected graph with order p and size q, k be an arbitrary natural number, if there exist an injective mapping $f: V(G) \rightarrow \{0,1, \dots, q+k-1\}$ that induces bijective mapping $f^*: E(G) \rightarrow \{k, k+1, \dots, q+k-1\}$ and defined by

$$f^*(u,v) = |f(u) - f(v)| \; \forall \; (u,v) \in E(G) \text{ and } u,v \in V(G).$$

Then f is called k-graceful labeling, while f^* is called an induced edge's k-graceful labeling and the graph G is called k-graceful graph. Graphs that are k-graceful for all k are sometimes called arbitrarily graceful.

Maheo and Thuillier [8] proved that the cycle C_n is k-graceful if and only if either:

(i) $n \equiv 0(mod4)$, where k is even and $k \leq (n-1)/2$,
or
(ii) $n \equiv 1(mod4)$,where k is even and $k \leq (n-1)/2$,
or
(iii) $n \equiv 3(mod4)$, where k is odd and $k \leq (n^2-1)/2$.

They also proved that the wheel W_{2k+1} is k-graceful and conjectured that W_{2k} is k-graceful when $k \neq 3 \; or \; k \neq 4$. This conjecture was proved by Liang, Sun and Hu [7].

Bu, Zhang and He [3] has shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is k-graceful.

Lee and Wang [6] have shown that all combs are k-graceful.

Acharya [1] has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions:

(a) $q \equiv 0(mod4)$, where k is even.
(b) $q \equiv 1(mod4)$, where k is even.
(c) $q \equiv 3(mod4)$, where k is odd.

Maheo and Thuillier [8] has shown that C_n is k-graceful if $n \equiv 0(mod4)$ and k is even and $k \leq (n-1)/2$. We observe that k-gracefulness for the cycle C_n, $n \equiv 0(mod4)$ also holds for any $k \in N$ (set of natural numbers).

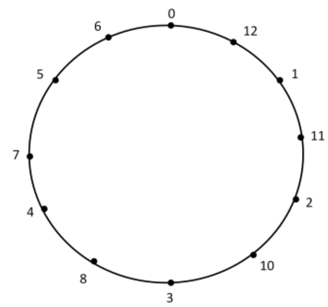
1.1 Corollary
For all $k \in N$ (set of natural numbers), the cycle C_n , $n \equiv 0(mod4)$ is k-graceful.

Proof: Let C_n be a cycle where $n \equiv 0(mod4)$ and $\{v_1, v_2, \dots, v_n\}$ be the vertices of cycle C_n . Consider the map $f: V(C_n) \rightarrow \{0,1, \dots, n+k-1\}$ defined as follows:

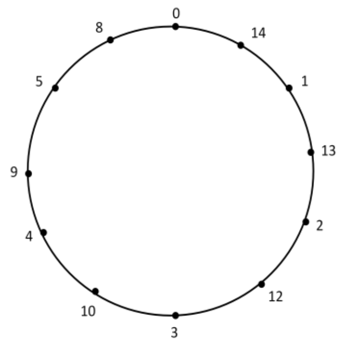
$$f(v_i) = \begin{cases} \dfrac{i-1}{2}, & i \; is \; odd, \\ n+k-\dfrac{i}{2}, & i \; is \; even \; and \; i \leq n/2, \\ n+k-1-\dfrac{i}{2}, & i \; is \; even \; and \; i > n/2. \end{cases}$$

It is clear that f is injective and the induced mapping $f^*: E(C_n) \rightarrow \{k, k+1, \dots, n+k-1\}$ is bijective, given as $f^*(u,v) = |f(u) - f(v)| \; \forall \; (u,v) \in E(C_n)$ and $u,v \in V(C_n)$. Thus, f is k-graceful labeling of cycle C_n.

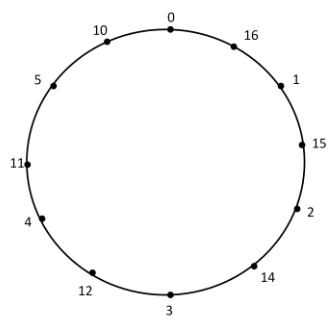
1.2 Example
Consider the figure (1) of k-graceful labeling of C_{12}, for $k = 1,3$ and 5 as given below:



$k = 1$



$k = 3$



$k = 5$

Figure (1): k-graceful labeling of C_{12}

1.3 Missing Numbers

In k-graceful labeling of a graph G, all the vertices $v \in V(G)$ are assigned distinct labels from the set of numbers $\{0, 1, \ldots, q + k - 1\}$ and some numbers of the set $\{0,1, \ldots, q + k - 1\}$ do not appear in the vertex labeling. Such numbers are called missing numbers.

Bagga, Heinz and Majumder [2] have given the range of values for the missing label m for graceful labeling of C_n as given below:

$$\left\lceil \frac{n}{4} \right\rceil \leq m \leq \left\lfloor \frac{3n}{4} \right\rfloor.$$

P. Pradhan and A. Kumar [9] have shown that

(i) Missing number in the graceful labeling of C_n, $n \equiv 0 \; or \; 3 (mod 4)$ is not unique.

(ii) Missing number in the graceful labeling of $C_n \odot 1K_1$ is $\frac{3n}{2}$, where $n \equiv 0 (mod 4)$.

(iii) The range of missing number m for graceful labeling of C_n, $n \equiv 0 \; or \; 3 \; (mod 4)$ is

$$\left\lceil \frac{n}{4} \right\rceil \leq m \leq \left\lfloor \frac{2n}{3} \right\rfloor.$$

2. MAIN RASULTS

There are k missing numbers in the k-graceful labeling of C_n, $n \equiv 0 (mod 4)$. Following theorem gives a way to find them.

2.1 Theorem

In k-graceful labeling of C_n ($n \equiv 0 (mod 4)$), one missing number is $\left(\frac{3n}{4} + k - 1\right) \; \forall \; k \in N$ (set of natural numbers) and remaining missing numbers will be from $\frac{n}{2}$ to $\left(\frac{n}{2} + k - 2\right) \; \forall \; k \geq 2$.

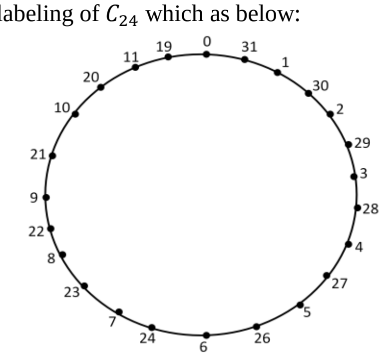
Proof: In k-graceful labeling of C_n ($n \equiv 0 (mod 4)$), (from corollary 1.1), numbers used in labeling to odd vertices of C_n are in increasing sequence beginning with 0 to $\left(\frac{n}{2} - 1\right)$, while numbers being used in labeling of even vertices are in decreasing sequence beginning with $(n + k - 1)$ to $\left(\frac{n}{2} + k - 1\right)$.Obviously, v_{n-1} is the last odd vertex and v_n is the last even vertex of C_n and numbers assigned to these two vertices are respectively,

$$f(v_{n-1}) = \frac{n}{2} - 1 \text{ and } f(v_n) = \frac{n}{2} + k - 1.$$

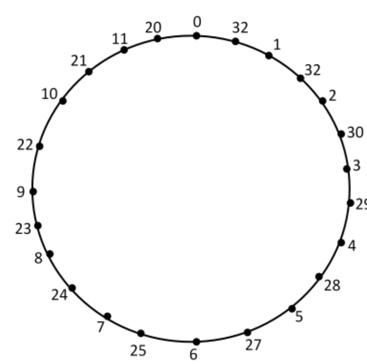
Sequence of $k - 1$ missing numbers lie between $(\frac{n}{2} - 1)$ and $(\frac{n}{2} + k - 1)$, so they will be from $\frac{n}{2}$ to $\left(\frac{n}{2} + k - 2\right)$, and one missing number will be $f(v_{\frac{n}{2}}) - 1$ i.e. $(\frac{3n}{4} + k - 1)$.

2.2 Example

Consider the following figure (2) of 8-graceful labeling and 9-graceful labeling of C_{24} which as below:



8-Graceful labeling of cycle C_{24}



9-Graceful labeling of cycle C_{24}

Figure (2): k-graceful labeling of C_{24}

Missing numbers in 8-graceful labeling of C_{24} are 12, 13, 14, 15, 16, 17, 18 and 25. Missing numbers in 9-graceful labeling of C_{24} are 12, 13, 14, 15, 16, 17, 18, 19 and 26.

In the following theorem, we have studied the missing numbers for k-graceful labeling of C_n, where $n \equiv 1$ or $3(mod 4)$.

2.3 Theorem

If there exists k-graceful labeling of cycle C_n (n is odd), one missing number in k-graceful labeling is $\frac{2k+n-1}{4}$ and remaining $k-1$ missing numbers will be from $\frac{n+3}{2}$ to $\frac{n+2k-1}{2}$, where $k \leq \frac{n-1}{2}$.

Proof: Let C_n be a cycle of odd length n and $V(C_n) = \{v_i : 1 \leq i \leq n\}$and $E(C_n) = \{e_i : 1 \leq i \leq n\}$. Now, the following two cases are arising for k-graceful labeling of cycle C_n of odd length.

Case I-In k-graceful labeling of C_n, if $n \equiv 3(mod 4)$, k is odd and $k \leq \frac{n-1}{2}$, then for $v_i \in V(C_n)$, we have

$$f(v_i) = \begin{cases} n + k - \frac{i}{2}, & i \text{ is even}, \\ \frac{i-1}{2}, & i \text{ is odd and } i \leq \left(k - 1 + \frac{n-1}{2}\right), \\ \frac{i+1}{2}, & i \text{ is odd and } i > \left(k - 1 + \frac{n-1}{2}\right). \end{cases}$$

Case II- In k-graceful labeling of C_n, if $n \equiv 1(mod 4)$, k is even and $k \leq \frac{n-1}{2}$, then for$v_i \in V(C_n)$, we have

$$f(v_i) = \begin{cases} n + k - \frac{i}{2}, & i \text{ is even}, \\ \frac{i-1}{2}, & i \text{ is odd and } i \leq \left(k - 1 + \frac{n-1}{2}\right), \\ \frac{i+1}{2}, & i \text{ is odd and } i > \left(k - 1 + \frac{n-1}{2}\right). \end{cases}$$

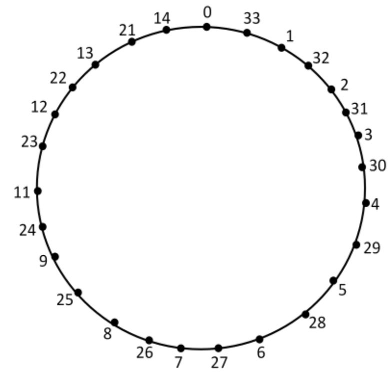
In k-graceful labeling of C_n ($n \equiv 1$ *or* 3 $(mod 4)$), numbers used in labeling to odd vertices of C_n are in increasing sequence beginning with 0 to $\frac{n+1}{2}$, while numbers being used in labeling of even vertices are in decreasing sequence beginning with $(n + k - 1)$ to $\frac{(n+2k+1)}{2}$. Obviously, v_n is the last odd vertex and v_{n-1} is the last even vertex of C_n and numbers assigned to these two vertices are respectively,

$$f(v_n) = \frac{n+1}{2} \text{ and } f(v_{n-1}) = \frac{n+2k+1}{2}.$$

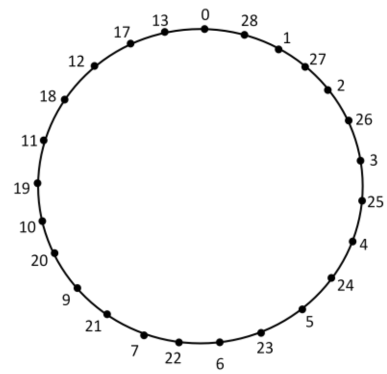
Sequence of $k-1$ missing numbers lie between $\frac{n+1}{2}$ and $\frac{(n+2k+1)}{2}$, so they will be from$\frac{n+3}{2}$ *to* $\frac{n+2k-1}{2}$, and one missing number is $f(v_i) + 1$; $i = \left(k - 1 + \frac{n-1}{2}\right)$ and i is odd i.e., one missing number is $\frac{n+2k-1}{4}$.

2.4 Example

Consider the figure (3) of 7-graceful labeling of C_{27} and 4-graceful labeling of C_{25} which are given below:



7-Graceful labeling of cycle C_{27}



4-Graceful labeling of cycle C_{25}

Figure (3): k-graceful labeling of C_{27} and C_{25}

Missing numbers in 7-graceful labeling of C_{27} are 10, 15, 16, 17, 18, 19, 20 and Missing numbers in 4-graceful labeling of C_{25} are 8, 14, 15, 16.

2.5 Theorem

In k-graceful labeling of comb $P_n \odot 1K_1$ have missing numbers from n to $(n + k - 2)$ $\forall$ $k \geq 2$.

Proof: The comb $P_n \odot 1K_1$has $2n$ vertices and $2n - 1$ edges. Let $\{v_1, v_2, \dots, v_n\}$ be the set of path vertices and$\{u_1, u_2, \dots, u_n\}$be the set of pendant vertices of comb $P_n \odot 1K_1$ such that v_i is adjacent to u_i; $i = 1,2, \dots, n$.

For k-graceful labeling of the comb $P_n \odot 1K_1$, consider a labeling map $f: V(P_n \odot 1K_1) \to \{0,1, \dots, 2n + k - 2\}$ defined as

$$f(v_i) = \begin{cases} i - 1, & i \text{ is odd}, \\ 2n + k - 1 - i, & i \text{ is even} \end{cases}$$

and $$f(u_i) = \begin{cases} i - 1, & i \text{ is even}, \\ 2n + k - 1 - i, & i \text{ is odd}. \end{cases}$$

Obviously f is injective and the induced labeling map$f^*: E(P_n \odot 1K_1) \to \{k, k + 1, \dots, n + k - 2\}$ defined as

$$f^*(u_i, v_i) = |f(u_i) - f(v_i)| \ \forall (u_i, v_i) \in E(P_n \odot 1K_1)$$

and $u_i, v_i \in V(P_n \odot 1K_1)$,

where u_i, v_i are adjacent vertices of $P_n \odot 1K_1$, is bijective.

Now for finding the missing numbers depending upon k, the vertex set $V(P_n \odot 1K_1)$ is partitioned into two disjoint sets say A and B in the following ways:

Case I- when n is odd,

A= $\{v_1, u_2, v_3, u_4, \ldots, u_{n-1}, v_n\}$

is the set of alternative sequence of path vertices and pendant vertices, beginning and ending with path vertices, where $v_1, v_3, \ldots, v_n$ are path vertices and $u_2, u_4, \ldots, u_{n-1}$ are pendant vertices,and

B= $\{u_1, v_2, u_3, v_4, \ldots, v_{n-1}, u_n\}$

is the set of alternative sequence of pendant vertices and path vertices, beginning and ending with pendant vertices, where $u_1, u_3, \ldots, u_n$ are pendant vertices and $v_2, v_4, \ldots, v_{n-1}$ are path vertices.

Case II- when n is even,

A= $\{v_1, u_2, v_3, u_4, \ldots, v_{n-1}, u_n\}$

is the set of alternative sequence of path vertices and pendant vertices, beginning with path vertex and ending with pendant vertex, and

B=$\{u_1, v_2, u_3, v_4, \ldots, u_{n-1}, v_n\}$

is the set of alternative sequence of pendant vertices and path vertices, beginning with pendant vertex and ending with path vertex.

In k-graceful labeling of $P_n \odot 1K_1$, we observe that in both cases, the numbers assigned to the vertices of one set A are in increasing order beginning with $0\ to\ n-1$, while the numbers assigned to the vertices of other set B are in decreasing order beginning with $(2n+k-2)\ to\ (n+k-1)$. So missing numbers will lie between $n-1$ and $n+k-1$, i.e. missing numbers will be from $n\ to\ n+k-2$.

2.6 Example

Case I- Consider the 4-graceful labeling of comb $P_9 \odot 1K_1$, where $n = 9$ (odd) and $k = 4$.

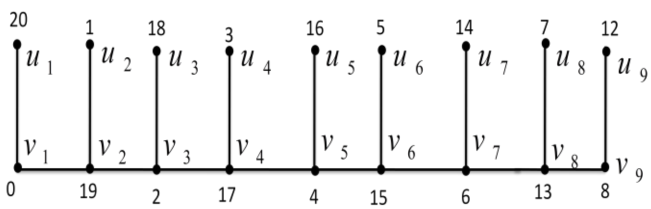


Figure (4):4-graceful labeling of comb $P_9 \odot 1K_1$.

Here we get;

$$f(A) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

and $$f(B) = \{20, 19, 18, 17, 16, 15, 14, 13, 12\}.$$

It is clear from above that the numbers assigned to the vertices of set A are in increasing order from 0 to 8 while the numbers assigned to the vertices of set B are in decreasing order from 20 to 12. So missing numbers are 9, 10 and 11.

Case II- Consider the 5-graceful labeling of comb $P_8 \odot 1K_1$, where $n = 8$ (even) and $k = 5$.

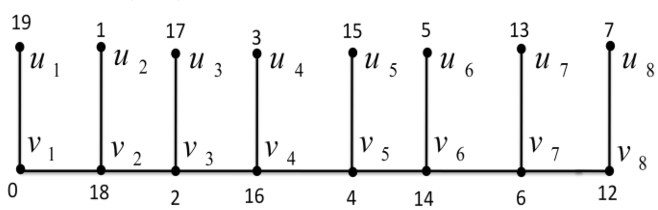


Figure (5): 5-graceful labeling of comb $P_8 \odot 1K_1$.

Here we get;

$$f(A) = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

and $$f(B) = \{19, 18, 17, 16, 15, 14, 13, 12\}.$$

Obviously, the numbers assigned to the vertices of set A are in increasing order from 0 to 7 while the numbers assigned to the vertices of set B are in decreasing order from 19 to 12. So missing numbers are 8, 9, 10 and 11.

2.7 Theorem

In $k-$graceful labeling of $C_n \odot 1K_1$ ($n \equiv 0(mod4)$), one missing number is $(\frac{3n}{2} + k - 1)$ and remaining missing numbers will be from n to$(n + k - 2)$.

Proof: Let $\{v_1, v_2, \ldots, v_n;\ u_1, u_2, \ldots, u_n\}$ be the set of vertices of hairy cycle $C_n \odot 1K_1$. The vertices on cycle C_n are $v_1, v_2, \ldots, v_n$ while $u_1, u_2, \ldots, u_n$ are pendant vertices such that $v_1, v_2, \ldots, v_n$ are adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For k-graceful labeling of the hairy cycle $C_n \odot 1K_1$, consider a labeling map $f: V(C_n \odot 1K_1) \to \{0,1, \ldots, 2n+k-1\}$ defined as follows:

$$f(v_i) = \begin{cases} i-1, & i\ is\ odd, \\ 2n+k-i, & i\ is\ even\ and\ i \le \dfrac{n}{2}, \\ 2n+k-1-i, & i\ is\ even\ and\ i > \dfrac{n}{2} \end{cases}$$

And $$f(u_i) = \begin{cases} i-1, & i\ is\ even, \\ 2n+k-i, & i\ is\ odd\ and\ i \le \frac{n}{2}, \\ 2n+k-1-i, & i\ is\ odd\ and\ i > \frac{n}{2}. \end{cases}$$

Obviously f is injective and the induced labeling map $f^*: E(C_n \odot 1K_1) \to \{k, k+1, \ldots,\ 2n+k-1\}$ defined as

$$f^*(u_i, v_i) = |f(u_i) - f(v_i)|\ \ \forall\ (u_i, v_i) \in E(C_n \odot 1K_1)$$

and $$u_i, v_i \in V(C_n \odot 1K_1),$$

where u_i, v_i are adjacent vertices of $C_n \odot 1K_1$, is bijective.

Co-domain of f contains $2n + k$ non-negative integers and only $2n$ non-negative integers are used for k-graceful labeling of $C_n \odot 1K_1$. We are left with k positive integers which are not used and we call them missing numbers. Thus, there are k missing numbers the k-graceful labeling of $C_n \odot 1K_1$.

For finding the missing numbers in the k-graceful labeling of $C_n \odot 1K_1$, the vertex set $V(\ C_n \odot 1K_1)$ is partitioned into two disjoint sets say A and B in the following way:

A = $\{v_1, u_2, v_3, u_4, \ldots, v_{n-1}, u_n\}$

is the set of an alternative sequence of cycle vertices and pendant vertices, beginning with cycle vertex v_1 and ending with pendant vertex u_n, and

B = $\{u_1, v_2, u_3, v_4, \ldots, u_{n-1}, v_n\}$

is the set of an alternative sequence of pendant vertices and cycle vertices, beginning with pendant vertex u_1 and ending with cycle vertex v_n.

Now the numbers to assign to vertices of set A by definition of f given in the beginning of proof of this theorem will be as below:

$f(v_1) = 0,\ \ f(u_2) = 1,$
$f(v_3) = 2,\ \ f(u_4) = 3,$
$f(v_5) = 4,\ \ f(u_6) = 5,$
………………………
………………………
………………………

$$f(v_{n-1}) = n-2, \quad f(u_n) = n-1.$$

Thus, there is an increasing sequence of numbers $0, 1, 2, \ldots, n-1$. i.e. an increasing sequence beginning with 0 and ending with $n-1$.

Similarly, for set B, we have a decreasing sequence of non-negative integers beginning with $(2n+k-1)$ and ending with $(n+k-1)$.

Last vertex of set A is u_n and last vertex of set B is v_n and their respectively labeling are

$$f(u_n) = n-1 \text{ and } f(v_n) = n+k-1.$$

Sequence of $k-1$ missing numbers must lie between $(n-1)$ and $(n+k-1)$. Since missing numbers are positive integers, therefore they must be from n *to* $(n+k-2)$, and one missing number will be $f(v_{\frac{n}{2}}) - 1$ i.e. $(\frac{3n}{2} + k - 1)$.

2.8 Example

Consider the figure of 4-graceful labeling of $C_8 \odot 1K_1$ which are given below:

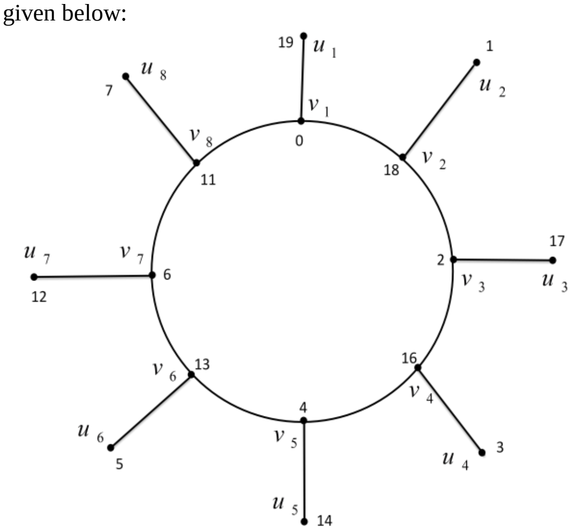


Figure (6): 4-graceful labeling of Hairy cycle $C_8 \odot 1K_1$.

Here we get;

$$f(A) = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

and $$f(B) = \{19, 18, 17, 16, 14, 13, 12, 11\}.$$

It is clear from above that the numbers used in labeling to vertices of set A are in increasing sequence beginning with 0 to 7, while numbers being used in labeling to vertices of set B are in decreasing sequence beginning with 19 to 11. So missing numbers are 8, 9, 10 and there is exactly one missing number $f(v_4) - 1$ i.e. 15.

A graph that obtained from the cycle $C_n, n \geq 3$ by adding a new vertex and edges joining it to all the vertices of the cycle is called a wheel graph W_n.

Maheo and Thuillier [8] have proved that the wheel graph W_{2k+1} is k-graceful. There are $3k$ missing numbers in the k-graceful labeling of the wheel graph W_{2k+1}. We have the following corollary for the range of missing numbers.

2.9 Corollary

Missing numbers in the k-graceful labeling of the wheel graph W_{2k+1} are from 1 *to* k and from $2k+1$ *to* $4k$.

Proof: Let v_0 be the labeling of centre vertex of W_{2k+1} and let its remaining vertices be labeled as $v_1, v_2, \ldots, v_{2k+1}$.

For $k-$ graceful labeling of the wheel graph W_{2k+1}, consider a labeling map $f : V(W_{2k+1}) \to \{0, 1, \ldots, 5k+1\}$ defined by

$$f(v_i) = \begin{cases} 0, & i = 0, \\ 5k + 2 - \dfrac{i+1}{2}, & i \text{ is odd}, \\ k + \dfrac{i}{2}, & i \text{ is even}. \end{cases}$$

Obviously f is injective and the induced labeling map $f^*: E(W_{2k+1}) \to \{k, k+1, \ldots, 5k+1\}$ defined as

$$f^*(u_i, v_i) = |f(u_i) - f(v_i)| \ \forall (u_i, v_i) \in E(W_{2k+1});$$

and $u_i, v_i \in V(W_{2k+1})$,

where u_i, v_i are adjacent vertices of W_{2k+1}, is bijective. After k-graceful labeling of the wheel graph W_{2k+1}, we observe that k missing numbers out of $3k$ lie from 1 to k and remaining $2k$ missing numbers lie from $2k+1$ to $4k$.

2.10 Example

Let 5-graceful labeling of W_{11} and missing numbers are from 1 to 5 and from 11 to 20.

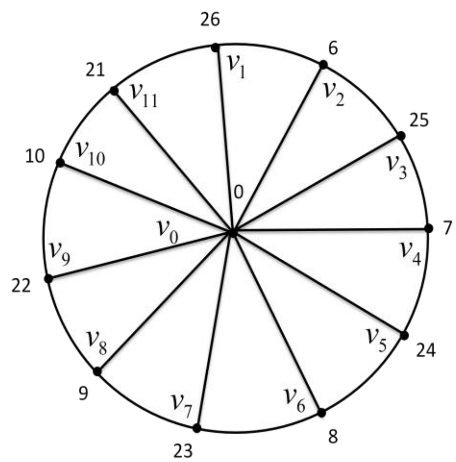


Figure (7): 5-graceful labeling of wheel graph W_{11}.

3. CONCLUSION

We have extended the result of Maheo and Thuillier [8] that C_n is k-graceful, $n \equiv 0 (mod 4)$ for every $k \in N$ (set of natural numbers) either even or odd. In k-graceful labeling of C_n, $n \equiv 0$ or 1 or $3\ (mod 4)$ and hairy cycle $C_n \odot 1K_1, n \equiv 0\ (mod 4)$, there are k missing numbers. A method to obtain k missing numbers for C_n and $C_n \odot 1K_1$ has been given. For comb $P_n \odot 1K_1$, there are $k-1$ missing numbers and for wheel graph W_{2k+1}, there are $3k$ missing numbers while performing k-graceful labeling of them.

4. ACKNOWLEDGEMENT

This research work is supported by Council of Scientific & Industrial Research and University Grant Commission (CSIR-UGC) New Delhi, India under the UGC Junior Research Fellowship (JRF) scheme to the second author. The authors are grateful to referee for his valuable suggestions to improve the manuscript of this paper.

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