A Brief Note on Acyclic Coloring of Line Graph of Some Families

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ABSTRACT
The aim of this paper is to determine acyclic chromatic number of the line graph of some special graphs. We also present some structural properties of $L(K_{2,n})$ and $L(F_{2,k})$.

Keywords
Line graph, complete bipartite graph, firecracker graph, acyclic colouring, acyclic chromatic number, cartesian product.

1. INTRODUCTION
Graph considered in this paper are undirected, finite and contains neither loops nor multiple edges. Terms not defined here are used in the sense of Harary [6].

Graph colouring [2,4,6] is an assignment of labels traditionally called "colours" to elements of a graph subject to certain constraints. The most common types of colourings are vertex colouring, edge colouring and face colouring. The vertex colouring is proper, if no two adjacent vertices are assigned the same colour.

A proper vertex colouring of a graph is acyclic if every cycle uses at least three colours [8].The acyclic chromatic number of G, denoted by $a(G)$, is the minimum colours required for its acyclic colouring.

2. LINE GRAPH
2.1 Definition
Let G be a finite undirected graph with no loops and multiple edges, the line graph of G, denoted by $L(G)$, is the intersection graph $\Omega(G)$. Thus the points of $L(G)$ are the lines of G, with two points of $L(G)$ are adjacent whenever the corresponding lines of G are.

2.2 Definition
The cartesian graph product $G = G_1 \times G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with point set $V_1 \times V_2$ .Then $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G = G_1 \times G_2$, whenever [$u_1 = v_1$ and u_2 adj v_2] or [$u_2 = v_2$ and u_1 adj v_1].

2.3 Theorem
The acyclic chromatic number, $a(L[P_2 \times P_n]) = 3$ for $n \geq 2$.

Proof:

Let G be the given graph. Let $V = \{v_i, u_i, w_j$ for $1 \leq i \leq n, 1 \leq j \leq n-2\}$ be the vertex set, in which v_i and u_i are marked continuously in anti clock-wise direction and followed by w_j. Consider the colour class $C = \{c_1, c_2, c_3\}$. The vertices v_i and u_i are assigned the colour c_2 for even values of i and are assigned the colour c_3 for odd values of $i \neq 1$. The remaining vertices are assigned the colour c_1.

The colouring is acyclic because the sub graph induced by both $\langle c_1, c_2 \rangle$ and $\langle c_1, c_3 \rangle$ are the paths P_{2n-1}, sub graph induced by $\langle c_2, c_3 \rangle$ is the disjoint union of two paths P_{n-1}. Also the above said colouring is minimum, as G contains cycles, minimum three colours required for its acyclic colouring. For example see figure 1.

Example

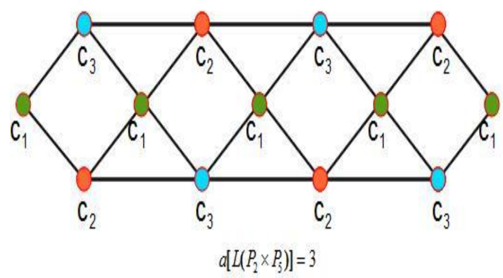


Fig. 1 Line graph of $P_2 \times P_5$

2.4 Theorem
The acyclic chromatic number, $a(L[P_3 \times P_n]) = 3$ for $n \geq 2$.

Proof:

Let G be the given graph. Consider the colour class $C = \{c_1, c_2, c_3\}$. The colour c_1 is assigned to all the vertices with even degree. The remaining all odd degree vertices are alternately assigned the colours c_2 and c_3 in anti clock wise direction so that the colouring is proper.
The colouring is acyclic because the sub graph induced by both $\langle c_1, c_2 \rangle$ and $\langle c_1, c_3 \rangle$ are trees with $4n-3$ vertices. Also

the sub graph induced by $\langle c_2, c_3 \rangle$ is the disjoint union of n paths P_2. Also the above said colouring is minimum, as G contains cycles, minimum three colours required for its acyclic colouring.

2.5 Some Structural properties of $L[P_4 \times P_n]$ for $n > 2$

- ❖ Number of vertices $p = 7n - 4$.
- ❖ Number of edges $q = 16(n-1)$.
- ❖ Maximum Degree of the vertices Δ = 5, for $n > 2$
- ❖ Minimum Degree of vertices δ = 3, for $n > 2$

2.6 Theorem

The acyclic chromatic number, $a(L[P_4 \times P_n]) = \begin{cases} 4, \text{ for } n \geq 3 \\ 3, \text{ for } n = 2 \\ 2, \text{ for } n = 1 \end{cases}$

Proof:

Let $G = a(L[P_4 \times P_n])$ be the given graph. The proof is obvious for n=1 and 2. Now for n>2, consider the colour class $C = \{c_1, c_2, c_3, c_4\}$ and assign the colours to the vertices so that it does not induce a bi-chromatic cycle.

Now by [7, 10], suggested by Robert E. Jamison, the acyclic chromatic number of a graph G satisfies $a(G) > \frac{|E(G)|}{|V(G)|} + 1$.

So in our case it is $a(G) > \frac{16(n-1)}{7n-4} + 1$

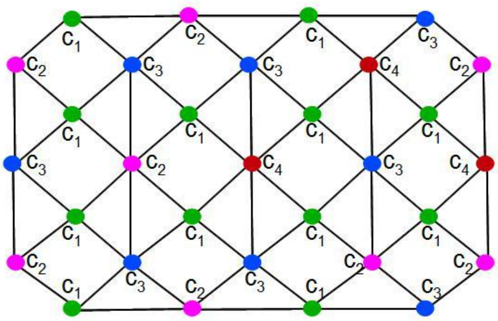
Hence $a(G) > 3.28$

This implies,

$a(G) \geq 4$. Also by the way of arrangement there does not exist any bi-chromatic cycle, hence the colouring is acyclic.

Thus $a(G) = 4$ for $n > 2$.

Example



$a(L[P_4 \times P_5]) = 4$

Fig. 2 Line graph of $P_4 \times P_5$

3. ACYCLIC CHROMATIC NUMBER OF SOME BIPARTITE GRAPH

3.1 Structural properties of $K_{2,n}$ and its Line graph

- The maximum degree in the graph $K_{2,n}$, $\Delta(K_{2,n}) = n$.
- The minimum degree in the graph $K_{2,n}$, $\delta(K_{2,n}) = 2$.
- The number of vertices in $K_{2,n}$, $p(K_{2,n}) = n + 2$.
- The number of edges in $K_{2,n}$, $q(K_{2,n}) = 2n$.
- The maximum and minimum degree in the graph $L(K_{2,n})$, $\Delta(L(K_{2,n})) = \delta(L(K_{2,n})) = n$.
- Number of edges in the graph $L(K_{2,n})$, $q(L(K_{2,n})) = 2\ {}^{n}C_2 + n$.
- The number of vertices in $L(K_{2,n})$, $p(L(K_{2,n})) = 2n$.

3.2 Theorem

The acyclic chromatic number, $a(L[K_{1,n}]) = n$

3.3 Theorem

For the graph $K_{2,n}$ the acyclic chromatic number, $a(L[K_{2,n}]) = n$ for $n \geq 3$.

Proof:

Consider the graph $K_{2,n}$ with vertex set $V(K_{2,n}) = \{u, v, v_1, v_2, v_3, ..., v_n\}$. In the line graph $L(K_{2,n})$, by the definition each edge uv_i and vv_i for $1 \leq i \leq n$ of $K_{2,n}$ is transformed to the vertex e_i and e'_i respectively in $L(K_{2,n})$ and the vertices $\{e_1, e_2, e_3, ..., e_n, e'_1, e'_2, e'_3, ..., e'_n\}$ induce two cliques of order n in $L(K_{2,n})$ .i.e. $V(L(K_{2,n})) = \{e_i / 1 \leq i \leq n\} \cup \{e'_i / 1 \leq i \leq n\}$. Now assign a proper colouring to these vertices as follows. Consider a colour class $C = \{c_1, c_2, c_3, ..., c_n\}$. For $i = 2, 3, ..., n$ assign the colour c_i to the vertex e_i and e'_{i-1} and the colour c_1 to the remaining two vertices, (See figure 3). The colouring is minimum, as $L(K_{2,n})$ contains clique of order n, minimum n colours are required for its proper colouring.

Next we have to prove that the above said coloring is acyclic. The subgraph induced by $\langle c_i, c_j \rangle$ is P_4 for consecutive values of i and j. Otherwise it is the disjoint union of two paths P_2.

Hence the above said colouring is acyclic.

Thus $a(L[K_{2,n}]) = n$ for $n \geq 3$.

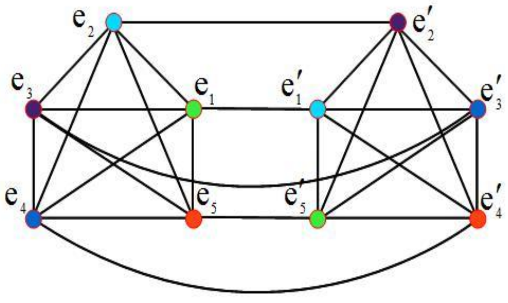


Fig. 3 Line graph of $K_{2,5}$

Note:

$a(L[K_{2,n}]) = 3$ for $n = 2$

4. FIRECRACKER GRAPH

4.1 Definition

An (n,k)-firecracker is a graph obtained by the concatenation of n, k - stars by linking one leaf from each and it is denoted by $F_{n,k}$.

4.2 Some Structural properties of $F_{2,k}$, $k > 2$ [11]

- The number of vertices in $F_{2,k}$ is $2k$
- The number of edges in $F_{2,k}$ is $2k-1$
- The maximum degree in $F_{2,k}$ is $k-1$
- The minimum degree in $F_{2,k}$ is 1

4.3 Some Structural Properties of $L(F_{2,k})$, $k > 2$ [11]

- The number of vertices in $L(F_{2,k})$ is $2k-1$
- The number of edges in $L(F_{2,k})$ is $2[{}^{k-1}C_2 + 1]$
- The maximum degree in $L(F_{2,k})$ is $k-1$
- The minimum degree in $L(F_{2,k})$ is 2, for $k > 3$.

4.4 Theorem [11]

The number of edges in $L(F_{2,k})$ is $2[{}^{k-1}C_2 + 1]$ for $k > 2$.

4.5 Theorem [11]

For the Firecracker Graph $F_{2,k}$ the acyclic chromatic number, $a(L[F_{2,k}]) = k-1$, for $k > 2$.

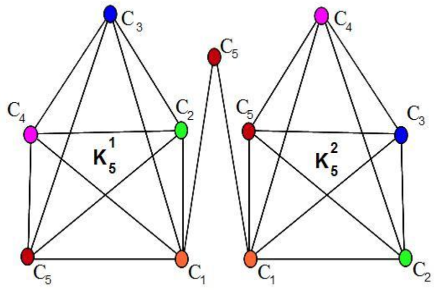


Fig. 4 Line graph of $F_{2,6}$

Note:

$a(L[F_{2,k}]) = 2$, for $k = 2$.

4.6 Observations

- For $k > 3$,

$a(L[F_{m,k}]) = k-1$, for $m = 2,3,...$

- For $k = 2$ and 3,

$a(L[F_{m,k}]) = $ 3 for $m = 3,4,5,...$

2 for $m = 2$.

5. CONCLUSION

In this paper the authors derived the exact values of the acyclic chromatic number of line graph of some families. In future it can be formulated for the line graph of entire families also the formulae and some properties of line graph can be correlated.

6. ACKNOWLEDGMENTS

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7. REFERENCES

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