Infinite Time Exact Ruin Probabilities in a Stochastic Economic Environment

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ABSTRACT

This article investigates the infinite-time ruin probabilities in a discrete-time stochastic economic environment platform under the assumption that the insurance risk-the total net loss within one time period is absolute-repeatedly-varying or suddenly-varying tailed, a different accurate estimates for the ruin probabilities are derived. In particular, some estimates found are standardized with respect to the time horizon, and so utilize in the case of infinite-time ruin.

Keywords

End point, extended regular variation, financial risk, insurance risk, rapid variation, and ruin probability.

1. INTRODUCTION

In examining the nature of the risk associated with a portfolio of business, it is often of interest to assess how the portfolio may be expected to perform over an extended period of time. One approach concerns the use of ruin theory introduced by Panjer and Willmot in the year of 1992. Ruin theory is concerned with the excess of the income (with respect to a portfolio of business) over the outgo, or claims paid. This quantity, referred to as insurer's surplus, varies in time. Specifically, ruin is said to occur if the insurer's surplus reaches a specified lower bound, e.g. minus the initial capital. One measure of risk is the probability of such an event, clearly reflecting the instability natural in the business. In addition, it can serve as a useful tool in long range planning for the use of insurer's funds. We recall now a definition of the standard mathematical model for the insurance risk has been addressed [1] and [2]. The initial capital of the insurance company is denoted by u, the Poisson process N_t with intensity (rate) λ describes the number of claims in (0, t) interval and claim severities are random, given by i.i.d. non-negative sequence $\{X_K\}_{K=1}^{\infty}$ with mean value μ and variance σ^2, independent of Nt. The insurance company receives a premium at a constant rate c per unit time, where $c = (1+\theta)\lambda\mu$ and $\theta \geq 0$ is called the relative safety loading. The classical risk process $\{R_t\}$ $t \geq 0$ is given by

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i . \quad (1)$$

Define a claim surplus process $\{S_t\} t \geq 0$ as

$$S_t = u - R_t = \sum_{i=1}^{N_t} X_i - ct \quad (2)$$

The time to ruin is defined as

$$\tau(u) = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : S_t > u\}. \quad (3)$$

Let $L = \sup_{0 \leq t < \infty}\{S_t\}$, and $L_T = \sup_{0 \leq t < T}\{S_t\}$. The ruin probability in infinite time, i.e. the probability that the capital of an insurance company ever drops below zero can be then written as

$$\psi(u) = P(\tau(u) < \infty) = P(L > u) \quad (4)$$

Note that the above definition implies that the relative safety loading θ has to be positive, otherwise c would be less than $\lambda\mu$ and thus with probability 1 the risk business would become negative in infinite time. The ruin probability in finite time T is given by

$$\psi(u,T) = P(\tau(u) \leq T) = P(L_T > u) \quad (5)$$

Also note that obviously $\psi(u,T) < \psi(u)$. However, the infinite time ruin probability may be sometimes also relevant for the finite time case. From a practical point of view, $\psi(u,T)$, where T is related to the planning horizon of the company, may perhaps sometimes be regarded as more interesting than $\psi(u)$. Most insurance managers will closely follow the development of the risk business and increase the premium if the risk business behave badly. The planning horizon may be thought of as the sum of the following: the time until the risk business is found to behave ``badly", the time until the management reacts and the time until a decision of a premium increase takes effect. Therefore, in non-life insurance, it may be natural to regard T equal to four or five years as reasonable. Also note that the situation in infinite time is markedly different from the finite horizon case as the ruin probability in finite time can always be computed directly using Monte Carlo simulations. Also remark that generalizations of the classical risk process, where the occurrence of the claims is described by point processes other than the Poisson process (i.e., non-homogeneous, mixed Poisson and Cox processes) do not alter the ruin probability in infinite time. This stems from the following fact. Consider a risk process $\tilde{R}_t$ driven by a Cox process $\tilde{N}_t$ with the intensity process $\tilde{\lambda}(t)$, namely

$$\tilde{R}_t = u + (1+\theta)\mu\int_0^t \tilde{\lambda}(s)ds - \sum_{i-1}^{\tilde{N}(t)} X_i .$$

Define now $\Lambda_t = \int_0^t \tilde{\lambda}(s)ds$ and $R_t = \tilde{R}(\Lambda_t^{-1})$. Then, the point process $N_t = \tilde{N}(\Lambda_t^{-1})$ is a standard Poisson process with intensity 1. Therefore,

$\tilde{\psi}(u) = P(\inf_{t\geq 0}\{\tilde{R}_t\} < 0 = P(\inf_{t\geq 0}\{R_t\} < 0) = \psi(u)$

The time scale defined by Λ_t^{-1} is called the operational time scale. It naturally affects the time to ruin, hence the finite time ruin probability, but not the ultimate ruin probability. The ruin probabilities in infinite and finite time can only be calculated for a few special cases of the claim amount distribution. Thus, finding a reliable approximation, especially in the ultimate case, when the Monte Carlo method cannot be utilized, is really important from a practical point of view. Note that ruin theory has been also recently employed as an interesting tool in operational risk. In the view of the data already available on operational risk, ruin type estimates may become useful [3]. Finally note that all presented explicit solutions and approximations are implemented in the Insurance library of explore. All figures and tables were created with the help of this library.

Therefore, in this paper it is to propose to investigate the infinite-time ruin probabilities in a discrete-time stochastic economic environment platform under the assumption that the insurance risk-the total net loss within one time period is unmitigated-repeatedly-varying or suddenly-varying tailed at different conditions.

The organization of this paper is as follows. Section 2 presents detailed derivation and various comparisons of the light and heavy tailed distributions. The conclusions are listed in section 3.

2. LIGHT AND HEAVY TAILED DISTRIBUTIONS

First distinguish here between light and heavy tailed distributions. A distribution $F_X(x)$ is said to be light tailed, if there exist constants $a>0, b>0$ such that $\tilde{F}_X(x) = 1 - F_X(x) \leq ae^{-bx}$ or equivalently, if there exist $z>0$, such that $M_X(z) < \infty$, where $M_X(z)$ is the moment generating function, . Distribution $F_X(x)$ is said to be heavy-tailed, if for all $a>0, b>0 : \tilde{F}_X(x) > ae^{-bx}$, or equivalently, if $\forall\, z>0\ M_X(z) = \infty$. We study here claim size distributions as in **Table 1**.

In the case of light-tailed claims the adjustment coefficient (called also the Lundberg exponent) plays a key role in calculating the ruin probability. Let $\gamma = \sup_z\{M_X(z)\} < \infty$ and let R be a positive solution of the equation.

Table 1. Typical claim size distributions in all cases $x \geq 0$

Light-tailed distributions

Name	Parameters	pdf
Exponential	$\beta > 0$	$f_X(x) = \beta\exp(-\beta x)$
Gamma	$\alpha > 0$, $\beta > 0$	$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x)$
Weibull	$\beta > 0$, $\tau \geq 1$	$f_X(x) = \beta\tau x^{\tau-1}\exp(-\beta x^\tau)$
Mixed exp's	$\beta_i > 0$, $\sum_{i=1}^n a_i = 1$	$f_X(x) = \sum_{i=1}\{\alpha_i\beta_i\exp(-\beta_i x)\}$

Heavy-tailed distributions

Name	Parameters	pdf
Weibull	$\beta > 0$, $0 < \tau < 1$	$f_X(x) = \beta\tau x^{\tau-1}\exp(-\beta x^\tau)$
Log-normal	$\mu \in \mathbb{R}$, $\alpha > 0$	$f_X(x) = \frac{1}{\sqrt{2\pi\sigma\tau}}\exp\left\{-\frac{(\ln\tau-\mu)^2}{2\sigma^2}\right\}$
sPareto	$\alpha > 0$, $\lambda > 0$	$f_X(x) = \frac{\alpha}{\lambda+x}\left(\frac{\lambda}{\lambda+x}\right)^\alpha$
Burr	$\alpha > 0$, $\lambda > 0$, $\tau > 0$	$f_X(x) = \frac{\alpha\tau\lambda^\alpha x^{\tau-1}}{(\lambda+x^{\tau)^{\alpha+1}}}$

:

$$1 + (1+\theta)\mu R = M_X(R), \qquad R < \gamma. \tag{6}$$

If there exists a non-zero solution R to the above equation, we call it an adjustment coefficient. Clearly, $R = 0$ satisfies the equation (6), but there may exist a positive solution as well (this requires that X has a moment generating function, thus excluding distributions such as Pareto and the log-normal). To see the possibility of this result, note that

$M_X(0) = 1, M'_X(z) < 0, M''_X(z) > 0$

and $M'_X(0) = -\mu$. Hence, the curves $y = M_X(z)$ and $y = 1 + (1+\theta)\mu z$ may intersect, as shown in Fig.1.

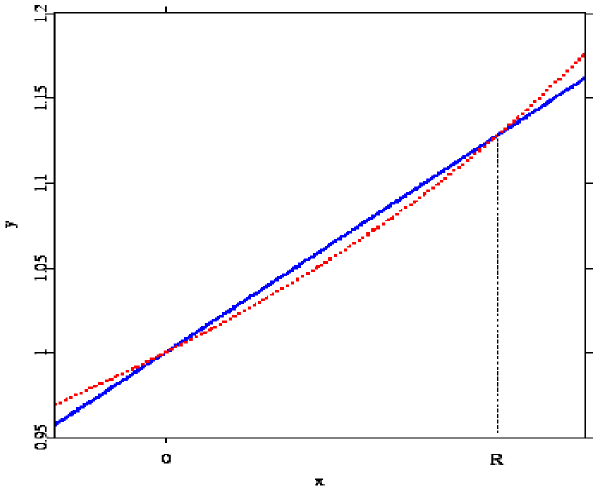

Fig 1: **Illustration of the existence of the adjustment coefficient. The solid blue line represents the curve** $y = 1 + (1+\theta)\mu z$ **and the dotted red one** $y = M_X(z)$.

An analytical solution to equation (6) exists, only for few claim distributions. However, it is quite easy to obtain a numerical solution. The coefficient R satisfies the inequality:

$$R < \frac{2\theta\mu}{\mu^{(2)}}, \tag{7}$$

where $\mu^{(2)} = E(X_i^2)$, see Asmussen (2000). Let $D(z) = 1 + (1+\theta)\mu z - M_X(z)$. Thus, the adjustment coefficient $R > 0$ satisfies the equation $D(R) = 0$.In order to get the solution one may use the Newton-Raphson formula

$$R_{j+1} = R_j - \frac{D(R_j)}{D'(R_j)}, \tag{8}$$

with the initial condition $R_0 = 2\theta\mu/\mu^{(2)}$, where $D'(z) = (1+\theta)\mu - M'_X(z)$.
Moreover, if it is possible to calculate the third raw moment $\mu^{(3)}$, we can obtain a sharper bound than (1.4),Panjer and Willmot (1992) .

$$R < \frac{12\mu\theta}{3\mu^{(2)} + \sqrt{9(\mu^{(2)})^2 + 24\mu\mu^{(3)}\theta}},$$

and use it as the initial condition in (8).

In order to present a ruin probability formula we first use the relation (1) and express L as a sum of so-called ladder heights. Let L_1 be the value that the process {Si} reaches for the first time above the zero level. Next, let L_2 be the value which is obtained for the first time above the level $L_1, L_3, L_4, \ldots$ are defined in the same way. The values L_k are called ladder heights. Since the process $\{S_t\}$ has stationary and independent increments, $\{L_k\}_{k=1}^{\infty}$ is a sequence of independent and identically distributed variables with the density

$$f_{L_1}(x) = \bar{F}_X(x)/\mu. \tag{9}$$

One may also show that the number of ladder heights K is given by the geometric distribution with the parameter $q = \theta/(1+\theta)$. Thus, the random variable L may be expressed as

$$L = \sum_{i=1}^{k} L_i \tag{10}$$

and it has a compound geometric distribution. The above fact leads to the Pollaczek-Khinchin formula for the ruin probability:

$$\psi(u) = 1 - P(L \le u) = 1 - \frac{\theta}{1+\theta}\sum_{n=0}^{\infty}\left(\frac{1}{1+\theta}\right)^n F_{L_1}^{*n}(u), \tag{11}$$

Where $F_{L_1}^{*n}(u)$ denotes the n^{th} convolution of the distribution function F_{L_1}. One can use it to derive explicit solutions for a variety of claim amount distributions; particularly those Laplace transform is a rational function. These cases will be discussed in this section. Unfortunately, heavy-tailed distributions like e.g. the log-normal or Pareto one are not included. In such a case various approximations can be applied or one can calculate the ruin probability directly via the Pollaczek-Khinchin formula using Monte Carlo simulations. A briefly present a collection of basic exact results on the ruin probability in infinite time [4]and[8]. The ruin probability $\psi(u)$ is always considered as a function of the initial capital u.

2.1 No initial capital

When $u = 0$ it is easy to obtain the exact formula:

$$\psi(u) = \frac{1}{1+\theta}.$$

Notice that the formula depends only on θ, regardless of the claim frequency rate λ and claim size distribution. The ruin probability is clearly inversely proportional to the relative safety loading.

2.2 Exponential claim amounts

One of the historically first results on the ruin probability is the explicit formula for exponential claims with the parameter, namely

$$\psi(u) = \frac{1}{1+\theta}\exp\left(-\frac{\theta\beta u}{1+\theta}\right). \tag{11}$$

From the **Table 2**, we present the ruin probability values for exponential claims with $\beta = 6.3789.10^{-9}$ and the relative safety loading $\theta = 30\%$ with respect to the initial capital u. Observe that the ruin probability decreases as the capital grows. When $u = 1$ million USD the ruin probability amounts to 18%, whereas $u = 5$ million USD reduces the probability to almost zero.

Table 2: The ruin probability for exponential claims with $\beta = 6.3789.10^{-9}$ and $\theta = 0.3$ (u in USD million)

u	0	1	2	3	4	5
$\psi(u)$	0.769 231	0.176 503	0.040 499	0.009 293	0.002 132	0.000 489

2.3 Gamma claim amounts

Grandell and Segerdahl (1971) showed that for the gamma claim amount distribution with mean 1 and $\alpha \le 1$ the exact value of the ruin probability can be computed via the formula:

$$\psi(u) = \frac{\theta\left(1-R/\alpha\right)\exp(-Ru)}{1+(1+\theta)R-(!+\theta)\left(1-R/\alpha\right)} + \frac{\alpha\theta\sin(\alpha\pi)}{\pi}.I, \tag{12}$$

Where,

$$I = \int_0^\infty \frac{x^\alpha \exp\{-(x+1)\alpha u\}}{\left[x^\alpha\{1+\alpha(1+\theta)(x+1)\}-\cos(\alpha\pi)\right]^2+\sin^2(\alpha\pi)}dx. \tag{13}$$

The integral “I” has to be calculated numerically. Also notice that the assumption on the mean is not restrictive since for claims X with arbitrary mean μ we have that $\psi_X(u) = \psi_{X/\mu}(u/\mu)$. As the gamma distribution is closed under scale changes we obtain that $\psi_{G(\alpha,\beta)}(u) = \psi_{G(\alpha,\alpha)}(\beta u/\alpha)$. This correspondence enables us to calculate the exact ruin probability via equation (10) for gamma claims with arbitrary mean. **Table 3** shows the ruin probability values for gamma claims with $\alpha = 0.9185$, $\beta = 6.1662.10^{-9}$ and the relative safety loading $\theta = 30\%$ with respect to the initial capital u. Naturally, the ruin probability decreases as the capital grows. Moreover, the probability takes similar values as in the exponential case but a closer look reveals that the values in the exponential case are always slightly larger. When $u = 1$ million USD the difference is about 1%. It suggests that a choice of the fitted distribution function may have a an impact on actuarial decisions.

Table 3: The ruin probability for gamma claims with $\alpha = 0.9185, \beta = 6.1662.10^{-9}$ and $\theta = 0.3$ (u in USD million)

u	0	1	2	3	4	5
$\psi(u)$	0.769 229	0.174 729	0.039 857	0.009 092	0.002 074	0.000 473

2.4 Mixture of Two Exponentials Claim Amounts

For the claim size distribution being a mixture of two exponentials with the parameters β_1, β_2 and weights $a, 1-a$, one may obtain an explicit formula by using the Laplace transform inversion (Panjer and Willmot , 1992).

$$\psi(u) = \frac{1}{(1+\theta)(r_2-r_1)}\{(\rho-r_1)\exp(-r_1u)+(r_2-\rho)\exp(-r_2u)\}, \tag{14}$$

Where,

$$r_1 = \frac{\rho+\theta(\beta_1+\beta_2)-\left[\{\rho+\theta(\beta_1+\beta_2)\}^2-4\beta_1\beta_2\theta(1+\theta)\right]^{1/2}}{2(1+\theta)},$$

$$r_2 \frac{\rho+\theta(\beta_1+\beta_2)+\left[\{\rho+\theta(\beta_1+\beta_2)\}^2-4\beta_1\beta_2\theta(1+\theta)\right]^{1/2}}{2(1+\theta)},$$

and

$$p = \frac{a\beta_1^{-1}}{a\beta_1^{-1}+(1-a)\beta_2^{-1}}, \qquad \rho = \beta_1(1-p)+\beta_2 p.$$

Table 4 shows the ruin probability values for mixture of two exponentials claims

with $\beta_1 = 3.5900.10^{-9}, \beta_2 = 7.5088.10^{-9}$, $a = 0.0584$ and the relative safety loading $\theta = 30\%$ with respect to the initial capital u. As before, the ruin probability decreases as the capital grows. Moreover, the increase in the ruin probability values with respect to previous cases is dramatic. When $u = 1$ million USD the difference between the mixture of two exponentials and exponential cases reaches 240%. As the same underlying data set was used in all cases to estimate the parameters of the distributions, it supports the thesis that a choice of the fitted distribution function and checking the goodness of fit is of paramount importance.

Table 4. The ruin probability for mixture of two exponentials claims

u	0	1	5	10	20	50
$\psi(u)$	0.769231	0.587919	0.359660	0.194858	0.057197	0.001447

Finally, note that it is possible to derive explicit formulae for mixture of n ($n \geq 3$) exponentials (Wikstad, 1971, Panjer and Willmot, 1992). They are not presented here since the complexity of formulae grows as n increases and such mixtures are rather of little practical importance due to increasing number of parameters.

3. CONCLUSIONS

In this paper, the infinite-time ruin probabilities in a discrete-time stochastic economic environment in under the assumption that the insurance risk-the total net loss within one time period is unmitigated-repeatedly-varying or suddenly-varying tailed at different cases has been successfully derived and also compared with their results.

4. REFERENCES

[1] Resnick, S. I. 1987. *Extreme Values*, Regular Variation, and Point Processes. Springer, New York.

[2] Rogozin, B. A. 1999. On the constant in the definition of subexponential distributions. *Teor*. Veroyat. Primen.

[3] Paulsen, J. 2002. On Cramér-like asymptotics for risk processes with stochastic return on investments.

[4] Nyrhinen, H. 2001. Finite and infinite time ruin probabilities in a stochastic economic environment. *Stoch*. Process

[5] Norberg, R. 1999. Ruin problems with assets and liabilities of diffusion type.*Stoch*. Process.

[6] Konstantinides, D., Tang, Q. and Tsitsiashvili, G. 2002. Estimates for the ruin probability in the classical risk model with constant interest force in the presence of heavy tails. *Insurance Math*. Econom.

[7] Klüppelberg, C. and Mikosch, T. 1997. Large deviations of heavy-tailed random sums with applications in insurance and finance.

[8] Klüppelberg, C. 1989. Subexponential distributions and characterizations of related classes. *Prob*. Theory Relat

[9] Kalashnikov, V. and Norberg, R. (2002). Power tailed ruin probabilities in the presence of risky investments. *Stoch*. Process

[10] Kalashnikov, V. and Konstantinides, D. (2000). Ruin under interest force and subexponential claims: a simple treatment

[11] Geluk, J. L. and de Haan, L. (1987). *Regular Variation*, Extensions and Tauberian Theorems (CWI Tract **40**). CWI, Amsterdam

[12] De Haan, L. (1970). *On Regular Variation and its Application to the Weak Convergence of Sample Extremes* (Math. Centre Tracts **32**)

[13] Chover, J., Ney, P. and Wainger, S. (1973a). Functions of probability measures

[14] Sato, K.-I. (1999). L´evy Processes and Infinitely Divisible Distributions. Cambridge Univ. Press.

[15] Schmidli, H. (2001). Distribution of the first ladder height of a stationary risk process perturbed by α-stable L´evy motion. Insurance Math. Econom

[16] Kluppelberg ¨ , C., Kyprianou, A. E. and Maller, R. A. (2003). Ruin probabilities and overshoots for general L´evy insurance risk processes

[17] Zolotarev, V. M. (1964). The first passage time of a level and behavior at infinity for a class of processes with independent increments.