

# New Techniques for Daubechies Wavelets and Multiwavelets Implementation using Quantum Computing

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## ABSTRACT

In this paper, new techniques to implement the Daubechies wavelets and multiwavelets are presented using quantum computing synthesis structures. Also, a new quantum implementation of inverse Daubechies multiwavelet transform is proposed. The permutation matrices, particular unitary matrices, play a pivotal role. The particular set of permutation matrices arising in quantum wavelet and multiwavelet transforms is considered, and efficient quantum circuits that implement them are developed. This allows the design of efficient and complete quantum circuits for the quantum wavelet and multiwavelet transforms.

## General Terms

Design, Theory, Algorithms

## Keywords

Quantum Circuits, Quantum Computing, Wavelet Transforms, Multiwavelet Transforms

## 1. INTRODUCTION

As it happens in classical signal analysis, it is natural to expect that quantum wavelet transform will find important future applications [1] for the treatment of quantum databases and quantum data compression. Therefore, it is important to develop quantum circuits for implementing wavelet and multiwavelet transforms. Wavelets have been around since the late 1980s, and have found many applications in signal processing, numerical analysis, operator theory, and other fields [2]. The wavelet transform is a tool that cuts up data or functions or operators into different frequency components, and then studies each component with a resolution matched to its scale. The wavelet transform of a signal evolving in time depends on two variables: scale (or frequency) and time; wavelets provide a tool for time-frequency localization [3]. One generalization is multiwavelets, which have been around since the early 1990s. Multiwavelets have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal. They also have some disadvantages: the discrete multiwavelet transform requires preprocessing and post processing steps. Also, the theory becomes more complicated [2].

In this paper, efficient and complete quantum circuits are derived for the quantum Daubechies  $D^{(4)}$  wavelet and multiwavelet transform representations. Quantum algorithms are describable in terms of unitary transformations.

Fortunately, there is an important class of computations, the unitary transforms, such as the Fourier transform, Walsh-Hadamard transform and wavelet transforms, that are describable in terms of unitary operators [4]. The Fourier and Walsh-Hadamard transforms have been used most extensively by the quantum computing algorithms [5] – [11]. However, the wavelet transforms are every bit as useful as the Fourier transform; therefore it is considered, here, how to achieve a quantum wavelet transform.

The process of finding a quantum circuit that implements the unitary operator of the wavelet transform is to factor the wavelet operator into the direct sum, direct product and dot product of smaller unitary operators. These operators correspond to 1-qubit and 2-qubit quantum gates. The permutation matrices play a pivotal role in the factorization of the unitary operators that arise in the wavelet transforms. The main issue in deriving feasible and efficient quantum circuits for the quantum wavelet transforms considered in this paper, is the design of efficient quantum circuits for certain permutation matrices. The permutation matrices, due to their specific structure, represent a very special subclass of unitary matrices [4]. Therefore, the exploitation of this specific structure represents the key to achieve an efficient quantum implementation of permutation matrices. In this paper, two representative wavelet kernels are considered, the Daubechies  $D^{(4)}$  wavelets and multiwavelets. Two new decompositions which lead to gate-level circuits for their implementations are developed.

The rest of the article is organized as follows. Two fundamental quantum wavelet pyramidal and packet algorithms are introduced in Section 2, as well as quantum circuits for the perfect shuffle permutation matrices which arise in quantum wavelet transforms are discussed in this section, too. New quantum implementations of the most popular discrete wavelet transform, namely, the 4-coefficient Daubechies wavelet and multiwavelet transforms are developed in Section 3 and Section 4, respectively. While Section 5 deals with quantum implementation of the inverse Daubechies multiwavelet transform. Finally, the conclusion section summarizes this work.

## 2. WAVELET PYRAMIDAL AND PACKET ALGORITHMS

Given a wavelet kernel, its corresponding wavelet transform is usually performed according to a packet algorithm (PAA) or a pyramidal algorithm (PYA). The first step in devising

quantum counterparts of these algorithms is the development of suitable factorizations. Consider the Daubechies fourth-order wavelet kernel of dimension  $2^i$ , denoted as  $D_{2^i}^{(4)}$ . The factorizations of PAA and PYA for a  $2^n$ -dimensional vector are given as [4]

$$PAA = \left( I_{2^{n-2}} \otimes D_4^{(4)} \right) \left( I_{2^{n-3}} \otimes \Pi_8 \right) \cdots \left( I_{2^{n-i}} \otimes D_{2^i}^{(4)} \right) \times \left( I_{2^{n-i-1}} \otimes \Pi_{2^{i+1}} \right) \cdots \left( I_2 \otimes D_{2^{n-1}}^{(4)} \right) \Pi_{2^n} D_{2^n}^{(4)} \quad (1)$$

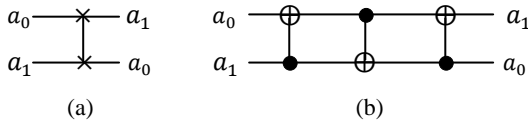
$$PYA = \left( D_4^{(4)} \oplus I_{2^{n-4}} \right) \left( \Pi_8 \oplus I_{2^{n-8}} \right) \cdots \left( D_{2^i}^{(4)} \oplus I_{2^{n-2i}} \right) \left( \Pi_{2^{i+1}} \oplus I_{2^{n-2i+1}} \right) \cdots \Pi_{2^n} D_{2^n}^{(4)} \quad (2)$$

These factorizations allow analysis of the feasibility and efficiency of quantum implementations of the packet and pyramid algorithms.

A set of efficient and practically realizable circuits for implementation of Qubit Permutation Matrices can be built by using the qubit swap gate,  $\Pi_4$ , where

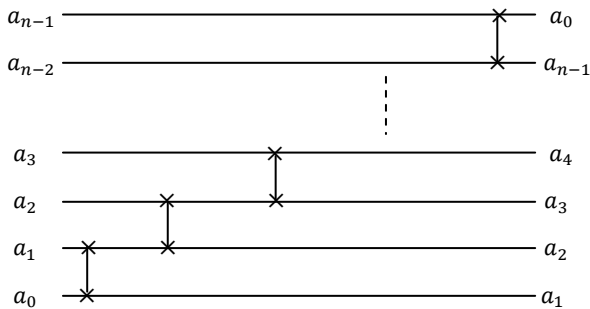
$$\Pi_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The  $\Pi_4$  gate, shown in Figure 1a, can be implemented with three EXOR (or Controlled-NOT) gates as shown in Figure 1b.



**Fig 1: The  $\Pi_4$  gate (a) and its implementation by using three EXOR (Controlled-NOT) gates (b).**

A circuit for implementation of  $\Pi_{2^n}$  by using  $\Pi_4$  gates is shown in Figure 2. This circuit is based on a simple idea of successive swapping of the neighboring qubits.



**Fig 2: Implementation of perfect shuffle permutation matrix,  $\Pi_{2^n}$ .**

The most popular discrete wavelet transform, namely, the Daubechies fourth-order wavelet kernel of dimension  $2^n$  is given in a matrix form as [4]

$$D_{2^n}^{(4)} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & & & & & & & \\ c_3 & -c_2 & c_1 & -c_0 & & & & & & & \\ & & c_0 & c_1 & c_2 & c_3 & & & & & \\ & & c_3 & -c_2 & c_1 & -c_0 & & & & & \\ & & & \vdots & & & \ddots & & & & \\ & & & & & & & c_0 & c_1 & c_2 & c_3 \\ & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\ & & & & & & & & & & \vdots \\ c_2 & c_3 & & & & & & & & & c_0 & c_1 \\ c_1 & -c_0 & & & & & & & & & c_3 & -c_2 \end{bmatrix} \quad (3)$$

Where  $c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$ ,  $c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$ ,  $c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$ , and  $c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$ .

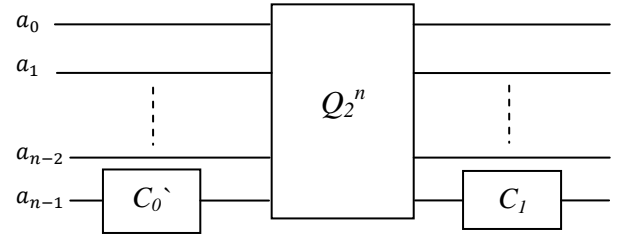
A factorization of  $D_{2^n}^{(4)}$  is proposed in [4] as

$$D_{2^n}^{(4)} = \left( I_2^{n-1} \otimes C_1 \right) Q_2^n \left( I_2^{n-1} \otimes C_0 \right) \quad (4)$$

Where  $C_0 = 2 \begin{bmatrix} c_2 & c_3 \\ c_3 & -c_2 \end{bmatrix}$  and  $Q_2^n$  is the *downshift permutation matrix* given by

$$Q_2^n = \begin{bmatrix} 0 & 1 & & & & & \\ 0 & 0 & 1 & \cdots & & & \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 & \\ 1 & 0 & 0 & & 0 & 0 & \end{bmatrix} \quad (5)$$

Figure 3 shows a block-level implementation of Equation (4). Clearly, the main issue for a practical quantum gate-level implementation of Equation (4) is the quantum implementation of matrix  $Q_2^n$ .



**Fig 3: A block-level circuit for implementation of  $D_{2^n}^{(4)}$  [4].**

### 3. QUANTUM IMPLEMENTATION OF DAUBECHIES $D^{(4)}$ WAVELET TRANSFORM

A new circuit for implementation of permutation matrix  $Q_2^n$  is developed based on its description as a *quantum arithmetic operator*. Such a quantum arithmetic description of  $Q_2^n$  is given as:

$$Q_2^n: |a_{n-1} a_{n-2} \cdots a_1 a_0\rangle \rightarrow |b_{n-1} b_{n-2} \cdots b_1 b_0\rangle \quad (6)$$

$$\text{Where } b_i = a_i - 1 \pmod{2^n}. \quad (7)$$



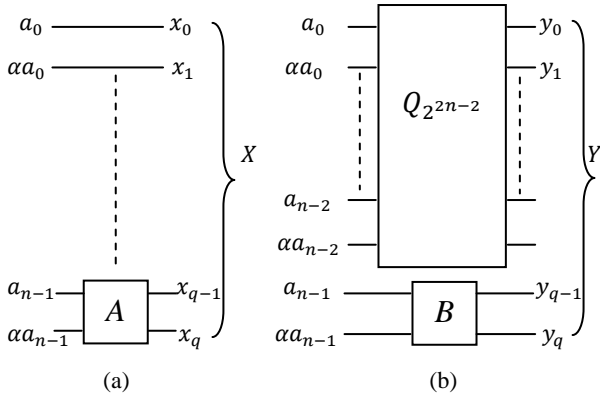
$$Y = \begin{bmatrix} & H_2 & H_3 & & & \\ & G_2 & G_3 & & & \\ & & & H_2 & H_3 & \dots \\ & & & G_2 & G_3 & \\ & & & \vdots & \vdots & \ddots \\ H_2 & H_3 & & & & \dots \\ G_2 & G_3 & & & & \dots \end{bmatrix} \quad (16)$$

The matrix  $Y$  can be described as:

$$Y = Q_{2^{2n-2}} \otimes B \quad (17)$$

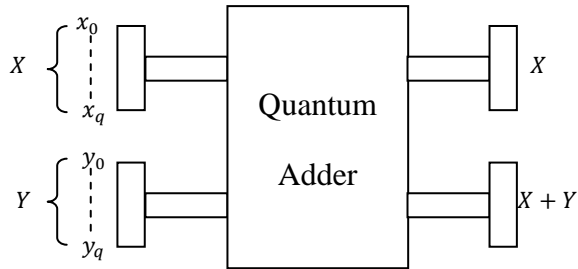
$$\text{Where, } B = \begin{bmatrix} H_2 & H_3 \\ G_2 & G_3 \end{bmatrix} \quad (18)$$

$Q_{2^{2n-2}}$  is the downshift permutation matrix as described in the previous Section. The  $H_2$ ,  $H_3$ ,  $G_2$ , and  $G_3$  are as given in Equations (10) and (11), respectively. Figure 5 shows the quantum realizations of the matrices  $X$  and  $Y$  as specified by Equations (14) and (17), respectively. The circuit of the block  $Q_{2^{2n-2}}$  in Figure 5b is as shown previously in Figure 4.



**Fig 5: The realization of: (a) Equation 14, (b) Equation 17; where  $q = 2n - 1$ .**

Now, to obtain a single level of the  $D^{(4)}$  multiwavelet transform, the  $X$  and  $Y$  outputs of Figure 5a&b are applied as inputs to a quantum adder circuit as shown in Figure 6.



**Fig 6: The realization of the Daubechies  $D^{(4)}$  multiwavelet transform.**

The addition of two registers  $|X\rangle$  and  $|Y\rangle$  can be written as  $|X, Y\rangle \rightarrow |X, X + Y\rangle$ , where the result of addition is written into one of the input registers. To prevent overflows, the second register (initially loaded in state  $|Y\rangle$ ) should be of size  $q + 1$  if both  $X$  and  $Y$  are encoded on  $q$  qubits. In addition, a temporary register of size  $q$  is required, initially in state  $|0\rangle$ , to which the carries of the addition are written (the last carry is the most significant bit of the result and is written in the last qubit of the second register) [12].

## 5. QUANTUM IMPLEMENTATION OF INVERSE DAUBECHIES $D^{(4)}$ MULTI WAVELET TRANSFORM

A new quantum implementation of inverse Daubechies  $D^{(4)}$  multiwavelet transform is proposed here, as follows.

The reconstruction matrix  $R$ , Equation (19), which is the transformation matrix  $T$  transposed, can be decomposed as the summation of two matrices  $Z$  and  $W$ .

$$R = \begin{bmatrix} H_0 & G_0 & & & & H_2 & G_2 \\ H_1 & G_1 & & & & H_3 & G_3 \\ H_2 & G_2 & H_0 & G_0 & \dots & & \\ H_3 & G_3 & H_1 & G_1 & \dots & & \\ & & H_2 & G_2 & & & \\ & & H_3 & G_3 & & & \\ \vdots & & & & \ddots & & \vdots \\ & & & & & H_0 & G_0 \\ & & & & & \dots & H_1 & G_1 \end{bmatrix} \quad (19)$$

The matrix  $Z$  is shown below:

$$Z = \begin{bmatrix} H_0 & G_0 & & & & \\ H_1 & G_1 & & & & \\ & & H_0 & G_0 & \dots & \\ & & H_1 & G_1 & & \\ & & \vdots & \vdots & \ddots & \vdots \\ & & & & \dots & H_0 & G_0 \\ & & & & & H_1 & G_1 \end{bmatrix} \quad (20)$$

The matrix  $Z$  can be described as:

$$Z = I_{2^{2n-2}} \otimes C \quad (21)$$

$$\text{Where, } C = \begin{bmatrix} H_0 & G_0 \\ H_1 & G_1 \end{bmatrix} \quad (22)$$

The  $H_0$ ,  $H_1$ ,  $G_0$ , and  $G_1$  are as given in Equations (10) and (11), respectively. The matrix  $W$  is:

$$W = \begin{bmatrix} & & & & & H_2 & G_2 \\ & & & & & H_3 & G_3 \\ H_2 & G_2 & \dots & & & & \\ H_3 & G_3 & & & & & \\ \vdots & & \ddots & & & & \vdots \\ & & & \dots & H_2 & G_2 & \\ & & & & H_3 & G_3 & \end{bmatrix} \quad (23)$$

The matrix  $W$  can be described as:

$$W = Q_{2^{2n-2}}^t \otimes D \quad (24)$$

Where,

$$Q_{2^{2n-2}}^t = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (25)$$

$$\text{and } D = \begin{bmatrix} H_2 & G_2 \\ H_3 & G_3 \end{bmatrix}. \quad (26)$$

The  $H_2$ ,  $H_3$ ,  $G_2$ , and  $G_3$  are as given in Equations (10) and (11), respectively. The realization of Equation (21) is similar to that of Equation (14). Also, the realization of Equation (24) is similar to that of Equation (17), but the  $Q_{2^{2n-2}}^t$  (i.e. the transpose of the downshift permutation matrix) is used instead

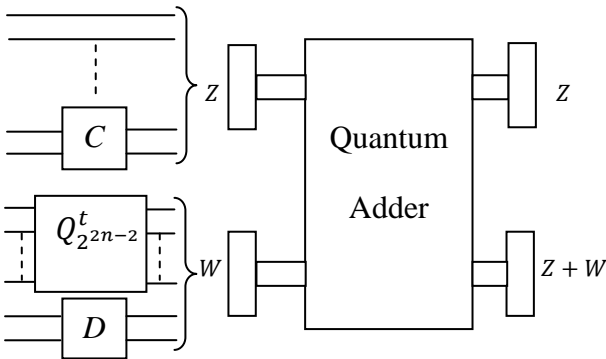
of  $Q_{2^{2n-2}}$ . Figure 7 shows the entire quantum implementation of the inverse  $D^{(4)}$  multiwavelet transform.

The quantum arithmetic description of  $Q_{2^n}^t$  is given as:

$$Q_{2^n}^t : |a_{n-1} a_{n-2} \dots a_1 a_0\rangle \rightarrow |b_{n-1} b_{n-2} \dots b_1 b_0\rangle$$

Where  $b_i = a_i + 1 \pmod{2^n}$ .

This description of  $Q_{2^n}^t$  allows its quantum implementation. Hence, it is required to construct a quantum circuit for  $|m\rangle \rightarrow |m+1 \pmod{2^n}\rangle$ . In binary representation, this mapping can be specified in terms of the following operations:



**Fig 7: Implementation of the inverse Daubechies  $D^{(4)}$  multiwavelet transform.**

$|a_{n-1} a_{n-2} \dots a_1 a_0\rangle \rightarrow |b_{n-1} b_{n-2} \dots b_1 b_0\rangle$ , with

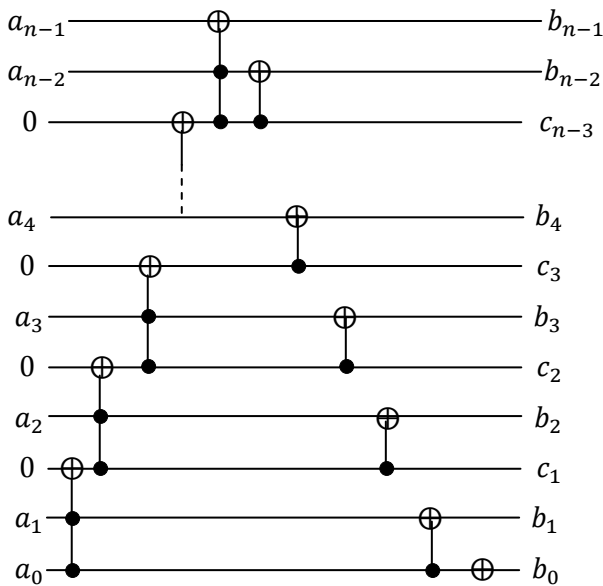
$$b_0 = a_0 \oplus 1$$

$$b_1 = a_1 \oplus a_0$$

$$b_2 = a_2 \oplus c_1, \text{ where } c_1 = a_1 a_0$$

$$b_i = a_i \oplus c_{i-1}, \text{ where } c_{i-1} = a_{i-1} c_{i-2} \text{ for } 3 \leq i \leq n-1$$

Calculating the  $c_i$ 's and then the  $b_i$ 's, the circuit in Figure 8 is obtained.



**Fig 8: A new circuit for implementation of permutation  $Q_{2^n}^t$  matrix.**

## 6. CONCLUSIONS

In this paper, fast algorithms for quantum Daubechies  $D^{(4)}$  wavelet, multiwavelet and inverse multiwavelet transforms are developed. Three complete circuits for these three types of Daubechies  $D^{(4)}$  transforms are described. Permutation matrices  $\Pi_{2^n}$ ,  $Q_{2^n}$ , and  $Q_{2^n}^t$  play a pivotal role in the development of the three types of transform. In fact, not only they arise explicitly in the packet and pyramid algorithms but also they play a key role in factorization of wavelet kernels and in decomposition of multiwavelet and inverse multiwavelet transforms.

In this paper, the Daubechies quantum wavelet and multiwavelet transforms are implemented. In a similar manner, the other types of wavelet and multiwavelet transformations can be achieved (a promising task for the future).

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