Beta Combination Graphs

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ABSTRACT

Let G(V,E) be a graph with p vertices and q edges.A graph G(p,q) is said to be a Beta combination graph if there exist a bijection f: V(G) → {1,2 …., p } such that the induced function B_f: E(G)→N, N is a natural number, given by B_f (uv)= $\frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, every edges uv ∈ G and are all distinct and the function f is called the Beta combination labeling.In this paper, we proved the Petersen graph , Complete graph K_n (n≤ 8),Ladder L_n (n≥2), fan f_n (n≥ 2), wheel W_n(n≥ 3), path P_n , cycle C_n(n≥3),friendship graph F_n (n≥1),complete bipartite graph $K_{n,n}$ (n≥ 2), Tree T_n , triangle snake , n-bistar graph $B_{n,n}$ and Star graph $K_{1,n}$ (n>1) are the Beta combination graphs.Also we proved Complete graph K_n (n>8) is not a Beta combination graph.

General Terms
Mathematical subject classification(2010) 05C78.

Keywords
Beta combination graph and Beta combination labeling.

1. INTRODUCTION

The research in graph enumeration and graph labeling started way back in 1857 by Arthur Cayley.Graph labeling and enumeration finds the application in chemical graph theory, social networking and computer networking and channel assignment problem.Abundant literature exists as of today concerning the structure of graphs admitting a variety of functions assigning real numbers to their elements so that certain given conditions are satisfied.

Throughout this paper, by a graph we mean a finite, undirected, simple graph. The vertex set and the edge set of a graph G are denoted by V(G) and E(G) respectively. Let G(p,q) be a graph with p = | V(G) | vertices and q = | E(G) | edges. A detailed survey of graph labeling can be found in [4].Combinations play a major role in combinatorial problems.The new labeling introduced in this paper is a logical-mathematical attempt. We used the following definitions in the subsequent sections.

Definition 1.1

A graph G(p,q) is said to be a Beta combination graph if there exist a bijection f: V(G) → {1,2 …., p } such that the induced function B_f: E(G) → N, N is a natural number, given by B_f (uv) = $\frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, every edges uv ∈ G and are all distinct and the function f is called the Beta combination labeling.

Definition 1.2.[6]

The Ladder L_n (n≥2) is the product graph P_2 x P_n and contains 2n nodes and 3n-2 edges.

Definition 1.3.[6]

The Fan f_n (n≥2) is obtained by joining all nodes of P_n to a further node called the center and contains n+1 nodes and 2n-1 edges.

Definition 1.4.[6]

The n-bistar graph $B_{n,n}$ is the graph obtained from two copies of $K_{1,n}$ by joining the vertices of maximum degree by an edge.

Definition 1.5 .[6]

The wheel W_n (n≥ 3) is obtained by joining all nodes of cycle C_n to a further node called the center, and contains (n+1) nodes and 2n edges.

Definition 1.6.[6]

The friendship graph F_n(n ≥ 1) consists of n triangles with a common vertex.

Definition 1.7.[6]

A triangular snake is obtained from a path $v_1,v_2,\ldots,v_n$ by joining v_i and v_{i+1} to a new vertex w_i for i=1,2,….,n-1.

In this paper, we proved the Petersen graph , Complete graph K_n (n≤8),Ladder L_n (n≥2), fan f_n (n≥ 2),wheel W_n(n≥ 3), path P_n , cycle C_n(n≥ 3),friendship graph F_n (n≥1),complete bipartite graph $K_{n,n}$ (n≥ 2) ,Tree T_n , triangle snake , n-bistar graph $B_{n,n}$ and Star graph $K_{1,n}$ (n>1) are the Beta combination graphs.Also we proved Complete graph K_n (n>8) is not a Beta combination graph.

2. MAIN RESULTS

Theorem 2.1

The Petersen graph is a Beta combination graph.

Proof:

Let G be the Petersen graph with 10 vertices $u_1, u_2, \ldots u_5, v_1, v_2, \ldots v_5$. Let $u_1, u_2, \ldots u_5$ be the outer vertices and let $v_1, v_2, \ldots v_5$ be the vertices of the star. Define f: V(G) $\rightarrow$ {1,2, …., 10 } by $f(u_i) = 2i-1$ if $1 \le i \le 5$ and $f(v_i) = 2i$ if $1 \le i \le 5$. And f induces that $B_f: E(G) \rightarrow N$ by

$B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of G and are all distinct.

Theorem:2.2

Every n-Bistar $B_{n,n}$ admits a beta combination labeling.

Proof:

The graph n-Bistar $B_{n,n}$ is the graph obtained from two copies of $K_{1,n}$ by joining the vertices of maximum degree by an edge. Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the vertices of first copy of $K_{1,n}$ and let $v_1, v_2, \ldots, v_n, v_{n+1}$ be the vertices of second copy of $K_{1,n}$ respectively of $B_{n,n}$. Let u_{n+1} and v_{n+1} are adjacent in $B_{n,n}$. Define $f: V(B_{n,n}) \rightarrow \{1,2,\ldots,2n+2\}$ by $f(u_i)=2i-1$ if $1 \le i \le n+1$ and $f(v_i)=2i$ if $1 \le i \le n+1$. And f induces $B_f: E(B_{n,n}) \rightarrow N$ by $B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of $B_{n,n}$ and are all distinct.

Theorem:2.3

All Ladders L_n ($n \geq 2$) are the Beta combination graph.

Proof:

Let the Ladder L_n be the graph with 2n vertices $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_{2n}$. Let u_i is adjacent to u_{i+1} if $1 \le i \le 2n-1$ and u_i is adjacent to u_{2n+1-i} if $1 \le i \le n-1$. Let $e_1, e_2, \ldots, e_{3n-2}$ be the 3n-2 edges of the ladder L_n such that $e_j = u_j u_{j+1}$ if $1 \le i \le 2n-1$ and $e_{2n-1+j} = u_j u_{2n+1-j}$ if $1 \le j \le n-1$.

Define $f: V(L_n) \rightarrow \{1,2,\ldots,2n\}$ by $f(u_i)=2i-1$ if $1 \le i \le n$ and $f(u_{2n+1-i})=2i$ if $1 \le i \le n$. And f induces that $B_f: E(L_n) \rightarrow N$ by

$B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of L_n and are all distinct.

Example:2.4

The beta combination labeling of the Petersen graph and $B_{5,5}$ are shown in the Fig-1 and Fig-2 respectively.

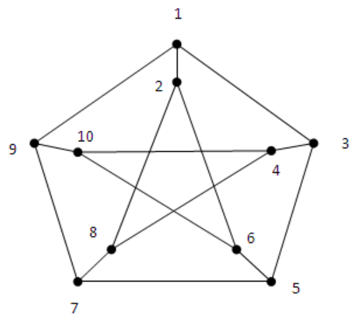

Fig-1

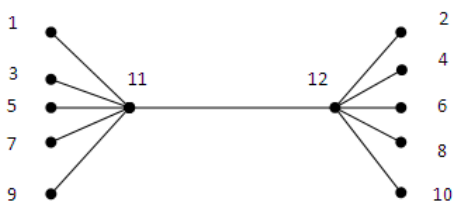

Fig-2

Theorem 2.5

Every wheel $W_n(n \geq 3)$ admits a beta combination labeling.

Proof:

Let $W_n = C_n + K_1$. Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n and u_{n+1} be the vertex of K_1.

Case (i) [$n \neq (2m+1)C_m-2$, $m \geq 2$ and $n \neq (2m+1)C_m-1$, $m \geq 2$]

Define $f: V(W_n) \rightarrow \{1,2,\ldots,n+1\}$ by $f(u_i) = i$ if $1 \leq i \leq n+1$.

Case(ii) [$n = (2m+1)C_m-2$, $m \geq 2$]

Define $f: V(W_n) \rightarrow \{1,2,\ldots,n+1\}$ by $f(u_{m-1}) = m$, $f(u_m) = m-1$ and $f(u_i) = i$ if $1 \leq i \leq n+1$, $i \neq m-1, m$.

Case(iii) [$n=(2m+1)C_m-1$, $m \geq 2$]

Define $f: V(W_n) \rightarrow \{1,2,\ldots,n+1\}$ by $f(u_1) = 2$, $f(u_2) = 1$ and $f(u_i) = i$ if $3 \leq i \leq n+1$.

In all above cases f induces that $B_f: E(W_n) \rightarrow N$ by

$B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of W_n and are all distinct.

Theorem:2.6

Every Complete bipartite graph $K_{n,n}$ admits beta combination labeling.

Proof:

Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be the vertices of $K_{n,n}$. Let $K_{n,n} = G_1 + G_2$ such that $u_1, u_2, \ldots, u_n$ be the vertices of G_1 and $v_1, v_2, \ldots, v_n$ be the vertices of G_2. Define $f: V(K_{n,n}) \rightarrow \{1,2,\ldots,2n$ by $f(u_i)=2i-1$ if $1 \leq i \leq n$ and $f(v_i)=2i$ if $1 \leq i \leq n$. And f induces that $B_f: E(K_{n,n}) \rightarrow N$ by $B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of $K_{n,n}$ and are all distinct.

Theorem 2.7

Every star $K_{1,n}$ ($n \geq 2$) is a beta combination graph.

Proof:

Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the vertices of $K_{1,n}$. Then $K_{1,n} = G_1 + G_2$ such that u_1 be the vertex of G_1 and $u_2, \ldots, u_n, u_{n+1}$ be the vertices of G_2. Define $f: V(K_{1,n}) \rightarrow \{1,2,\ldots,n+1\}$ by $f(u_i) = i$ if $1 \leq i \leq n+1$. And f induces that $B_f: E(K_{1,n}) \rightarrow N$ by $B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of $K_{1,n}$ and are all distinct.

Example 2.8

The beta combination labeling of F_4 and a triangle snake are shown in the Fig-3 and Fig-4 respectively.

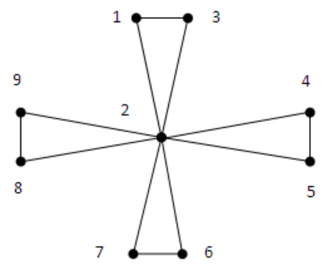

Fig-3

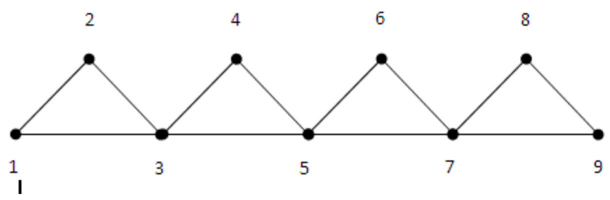

Fig-4

Theorem:2.9

Every triangle snake is a beta combination graph.

Proof:

Let G be a triangle snake with vertices $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, v_{n+1}\}$. Then $\{u_iv_i, u_iv_{i+1}, v_iv_{i+1} / 1 \leq i \leq n\}$ be the edges of G. Define $f: V(G) \rightarrow \{1,2,\ldots,2n+1\}$ by $f(u_i)=2i$ if $1 \leq i \leq n$ and $f(v_i)=2i-1$ if $1 \leq i \leq n+1$. And f induces that $B_f: E(G) \rightarrow N$ by $B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of G and are all distinct.

Theorem:2.10

Friendship graph F_n($n \geq 1$) is a beta combination graph.

Proof:

Let $u_1, u_2, \ldots, u_{2n+1}$ be the vertices of F_n. Also F_n has n number of triangles with common vertex u_{2n+1}. Then F_n has n number of pairs such as $(u_1,u_2),(u_3,u_4),\ldots,(u_{2n-1},u_{2n})$. Let $e_1, e_2, \ldots, e_{2n}$ be the 3n number of edges of F_n such that $e_i = u_iu_{i+1}$, $i = 2m-1$ if $1 \leq m \leq n$ and $e_{n+i} = u_iu_{2n+1}$ if $1 \leq i \leq 2n$.

[see fig 3]. Define $f: V(F_n) \rightarrow \{1,2,\ldots,2n+1\}$ by $f(u_1)=1$, $f(u_i) = i+1$ if $2 \leq i \leq 2n$ and $f(u_{2n+1}) = 2$. And f induces $B_f: E(F_n) \rightarrow N$ by $B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv of F_n are all distinct.

Theorem :2.11

Every path P_n is a Beta Combination graph.

Proof:

Let $u_1, u_2, \ldots, u_n$ be the vertices of path P_n and let $e_1, e_2, \ldots, e_{n-1}$ be the n-1 edges of P_n such that $e_i = u_iu_{i+1}$ if $1 \leq i \leq n-1$. A bijection $f: V(P_n) \rightarrow \{1,2,3,\ldots,n\}$ is defined by $f(u_i) = i$, if $1 \leq i \leq n$. And f induces that $B_f: E(P_n) \rightarrow N$ by $B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$ for every edges uv in P_n and are all distinct.

Theorem :2.12

Every Cycle C_n ($n \geq 3$) is a Beta Combination graph.

Proof:

Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n such that u_1 is adjacent to u_n and u_i is adjacent to u_{i+1} if $1 \leq i \leq n-1$. Then $e_i = u_iu_{i+1}$ if $1 \leq i \leq n-1$ and $e_n = u_1u_n$ be the n edges of C_n.

.**Case(i)**[$n \neq (2m+1)C_m - 1, m \geq 2$]

Define a bijection $f: V(C_n) \rightarrow \{1,2,3,\ldots,n\}$ is defined by $f(u_i) = i$, if $1 \leq i \leq n$.

Case(ii)[$n = (2m+1)C_m - 1, m \geq 2$]

Define a bijection $f: V(C_n) \rightarrow \{1,2,3,\ldots,n\}$ is defined by

$f(u_1)=2, f(u_2)=1$ and $f(u_i)= i$ if $3\leq i\leq n$.

In all above cases f induces that $B_f:E(C_n)\rightarrow\{1,2,3,.....\}$ by

$B_f(uv) = \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$ for every edges uv in C_n and are all distinct.

Theorem:2.13

Every Complete graph $K_n(n\leq 8)$ is a Beta Combination graph.

Proof:

Let $u_1,u_2,......u_n$ be the n vertices of complete graph K_n. Define $f:V(K_n)\rightarrow\{1,2,3,.....,n\}$ by $f(u_i)=i$ if $1\leq i\leq n$.And f induces that $B_f:E(K_n)\rightarrow N$ by $B_f(uv)= \frac{[f(u)+f(v)]!}{f(u)!f(v)!}$,for every edges uv in K_n and are all distinct.

Theorem 2.14

Complete graph K_n is not a Beta Combination graph for n>8.

Proof:

For n>8, we get $10C_1=5C_2=10$.Therefore K_n does not admit a beta combination labeling as induced edge function is not injective for n>8[see Fig-5].

Example 2.15

An example of a non beta combination graph K_9 is displayed in Fig-5.

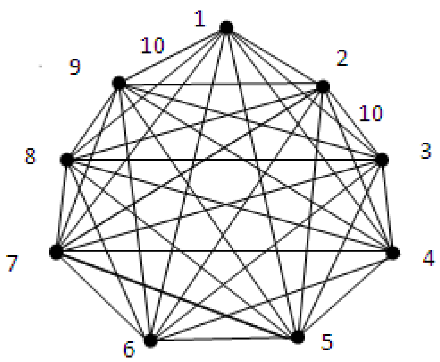

Fig-5

Theorem:2.16

Every fan graph $f_n(n\geq 2)$ is a beta combination graph.

Proof:

Let $u_1,u_2,......u_n,u_{n+1}$ be the n+1 vertices of f_n.Then $f_n=K_1+P_n$.Let $K_1= u_{n+1}$ and u_{n+1} is adjacent to every vertices $u_1,u_2,......u_n$ of path P_n .

Case(i) [$n\neq (2m+1)C_m-2$, $m\geq 2$]

Define $f:V(f_n)\rightarrow\{1,2,3,.....,n+1\}$ by $f(u_i)= i$, $1\leq i\leq n+1$.

Case(ii) [$n= (2m+1)C_m-2$, $m\geq 2$]

Define $f:V(f_n)\rightarrow\{1,2,3,.....,n+1\}$ by $f(u_{m-1})= m, f(u_m)= m-1$, $f(u_i)= i$ if $1\leq i\leq n+1$ and $i\neq m-1$, m .

In all above cases f induces that $B_f:E(f_n)\rightarrow N$ by $B_f(uv)=\frac{[f(u)+f(v)]!}{f(u)!f(v)!}$, for every edges uv in f_n and are all distinct.

Theorem 2.17

Every tree is a beta combination graph.

Proof:

Let u_1 denote the root vertex of the tree graph and labeling this vertex u_1 as '1'. If the vertex u_1 has 'i' sub trees with rooted vertices as $u_2,u_3,\ldots u_i$ then labeling each vertex as 2,3,4,…i. similarly each of the above vertices has one or more sub trees with labeling i+1,i+2,… continuing this process we get beta combination tree graph.[see Fig-6].

Example 2.18

The beta combination labeling of a tree T_6 is shown in the Fig-6 .

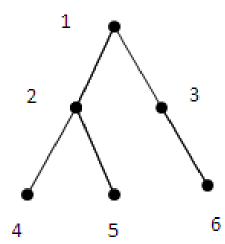

Fig-6

3. ACKNOWLEDGMENTS

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4. CONCLUSION

We have planned to find more beta combination graphs

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