Convergence and Stability Results for CR –iterative Procedure using Contractive-like Operators

Madhu Aggarwal*, Renu Chugh and Sanjay Kumar
Department of Mathematics, Maharshi Dayanand University, Rohtak-124001(INDIA)

ABSTRACT

The aim of this paper is to prove weak and strong convergence as well as weak stability results of CR-iterative procedures using contractive-like operators. The results obtained generalize several existing results. An example is also given, using computer programming in C++, to show that CR-iterative procedure converges faster than SP and Noor iterative procedures.

Key Words and Phrases

Iterative procedure, contractive like operators, fixed point, weak and strong convergence, weak stability.

1. INTRODUCTION AND PRELIMINARIES

Let (E, d) be a complete metric space and $T: E \rightarrow E$ be a mapping. Suppose that $F = \{p \in E : Tp = p\}$ is the set of fixed points of T. Now, we fix $x_0 = x \in E$ as a starting point of the iterative procedures under consideration and take $\{a_n\},\{b_n\},\{c_n\}$ as sequences in $[0,1]$.

The Picard, Mann [20], Ishikawa [13] and Noor [6] iterative procedures are defined by the sequence $\{x_n\}$:

$$x_{n+1} = Tx_n, \quad n = 0,1,2,... \tag{1.1.1}$$

$$x_{n+1} = (1-a_n)x_n + a_nTx_n, \quad n = 0,1,2,... \tag{1.1.2}$$

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTx_n, \quad n = 0,1,2,... \end{cases} \tag{1.1.3}$$

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \quad n = 0,1,2,... \end{cases} \tag{1.1.4}$$

In 2007, Agarwal et. al.[11] introduced S- iterative procedure as follows:

$$\begin{cases} x_{n+1} = (1-a_n)Tx_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTx_n, \quad n = 0,1,2,... \end{cases} \tag{1.1.5}$$

The iterative procedure (1.1.4) is independent of (1.1.3) (and hence of (1.1.2)).

Sahu and Petrusel[3] defined the following iterative procedure:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-a_n)x_n + a_nTx_n, \quad n = 0,1,2,... \end{cases} \tag{1.1.6}$$

In 2011, Phuegrattana and Suantai[19] defined the SP- iterative procedureas:

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_nTy_n, \\ y_n = (1-b_n)x_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \quad n = 0,1,2,... \end{cases} \tag{1.1.7}$$

Recently, chugh et. al.[9] introduced the following new three step iterative procedure and called it as CR-iteration:

$$\begin{cases} x_{n+1} = (1-a_n)y_n + a_nTy_n, \\ y_n = (1-b_n)Tx_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \quad n = 0,1,2,... \end{cases} \tag{1.1.8}$$

(1.1.6)

Where $\{a_n\},\{b_n\}$ and $\{c_n\}$are in $[0,1]$.

Remarks

(i) If $a_n = 1$ for all $n \in N$, then Mann iteration (1.1.2) reduces to Picard iteration (1.1.1).

(ii) If $b_n = 0$ for all $n \in N$, then Ishikawa iteration (1.1.3) reduces to Mann iteration (1.1.2).

(iii) If $c_n = 0$ for all $n \in N$, then Noor iteration (1.1.4) reduces to Ishikawa iteration (1.1.3).

(iv) If $a_n = 0$ for all $n \in N$, then iteration (1.1.6) reduces to Picard iteration (1.1.1).

(v) If $b_n = c_n = 0$ for all $n \in N$, then SP iteration (1.1.4) reduces to Mann iteration (1.1.2).

(vi) If $a_n = 0$ for all $n \in N$, then CR-iteration (1.1.8) reduces to S- iteration (1.1.5).

(vii) If $a_n = 0$ and $b_n = 1$ for all $n \in N$, then CR-iteration (1.1.8) reduces to iteration (1.1.6).

Banach[12] used the Picard iteration (1.1.1) to approximate fixed point of operator T satisfying the following contraction or Banach contraction condition :

$$d(Tx,Ty) \le \alpha d(x,y) \tag{1.1.9}$$

for all $x, y \in E$, where $\alpha \in [0,1)$.

The contraction condition was extended by several authors. In 1968, Kannan [10] extended it by using the following contraction condition: there exists $\beta \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le \beta\left[d(x,Tx) + d(y,Ty)\right] \tag{1.1.10}$$

for all $x, y \in E$

Chatterjea [14] defined the following contractive condition: there exists $\gamma \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx,Ty) \le \gamma\left[d(x,Ty) + d(y,Tx)\right] \qquad (1.1.11)$$

for all $x, y \in E$

Zamfirescu[16] defined the following class of operators by combining above three contraction conditions:

Theorem 1.1[16]Let (E, d) be a complete metric space and $T: E \to E$ be a mapping for which there exists real numbers α, β, γ satisfying $\alpha \in [0,1)$, $\beta, \gamma \in \left(0, \frac{1}{2}\right)$ such that for each $x, y \in E,$ at least one of the following conditions hold:

$$\begin{cases} (z_1) & d(Tx,Ty) \le \alpha d(x,y) \\ (z_2) & d(Tx,Ty) \le \beta\left[d(x,Tx) + d(y,Ty)\right] \\ (z_3) & d(Tx,Ty) \le \gamma\left[d(x,Ty) + d(y,Tx)\right] \end{cases} \qquad (1.1.12)$$

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, n=0,1,2,… converges to p for any arbitrary but fixed $x_0 \in X$.

Any operator satisfying the contractive condition (1.1.12) is called the Zamfirescu operator.

In 2004, Berinde[17] approximate fixed points of Ishikawa iterative procedure using quasi-contractive operators which is wider than the class of Zamfirescu operators: there exists $\delta \in [0,1)$ such that

$$d(Tx,Ty) \le 2\delta d(x,Tx) + \delta d(x,y) \qquad (1.1.13)$$

for all $x, y \in E$.

The contractive condition (1.1.12) implies (1.1.13) where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}, 0 \le \delta < 1.$

Osilike and Udomene [8] generalized the class of operators defined by Berinde[17] and defined the following contractive condition: there exists $\delta \in [0,1)$ and $L \ge 0$ such that

$$d(Tx,Ty) \le Ld(x,Tx) + \delta d(x,y) \qquad (1.1.14)$$

for all $x, y \in E$.

Recently, Olatinwo[7] proved the convergence of Ishikawa iterative procedure for an operator satisfying contractive condition:

$$d(Tx,Ty) \le \frac{\varphi(d(x,Tx)) + \delta d(x,y)}{1 + Md(x,Tx)} \qquad (1.1.15)$$

for all $x, y \in E$ where $\delta \in [0,1)$, $M \ge 0$ and $\varphi: R^+ \to R^+$ is a monotone increasing function such that $\varphi(0) = 0.$

If we choose *M=0* and $\varphi(t) = Lt$, then we obtain (1.1.14).

Imoru and Olatinwo[2] defined the following class of contractive-like operators: there exists $\delta \in [0,1)$ and a monotone increasing function $\varphi: R^+ \to R^+$ with $\varphi(0) = 0$, such that

$$d(Tx,Ty) \le \varphi(d(x,Tx)) + \delta d(x,y) \qquad (1.1.16)$$

for all $x, y \in E$.

The contractive condition (1.1.15) reduces to the contractive condition (1.1.16) since

$$d(Tx,Ty) \le \frac{\varphi(d(x,Tx)) + \delta d(x,y)}{1 + Md(x,Tx)} \le \varphi(d(x,Tx)) + \delta d(x,y).$$

Since metric is induced by norm, above contractive condition can be written as:

$$\|Tx - Ty\| \le \varphi(\|x - Tx\|) + \delta\|x - y\|. \qquad (1.1.17)$$

The following are some well known concepts and results**.**

Definition.1.2. A sequence $\{x_n\}$ in a normed linear space *E* is said to be

(i) **strongly convergent** if there exists an element $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. The element x is called the strong limit of the sequence $\{x_n\}$.

(ii) **weakly convergent** if there exists an element $x \in X$ such that $\lim_{n\to\infty} f(x_n) = f(x)$ for every bounded linear functional *f* on *X*. The element x is called the weak limit of the sequence $\{x_n\}$.

It should be noted that strong convergence implies weak convergence but converse may not be true.

Definition.1.3. Let *E* be a Banach space with its dimension greater than or equal to 2. The modulus of *E* is the function $\delta_E(\varepsilon): (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \le 1, \|y\| \le 1 \|x - y\| \ge r\right\}.$$

A Banach space *E* is **uniformly convex** if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Definition.1.4. Let *C* be a nonempty subset of a Banach space *E*. A mapping $T: C \to E$ is said to be **demiclosed** at $y \in E$, if for each sequence $\{x_n\}$ in *C* and each $x \in E$, the condition x_n converges weakly to *x* and Tx_n converges strongly to *y* imply that $x \in C$ and $Tx = y$.

Definition.1.5.[18] Let *E* be a normed linear space and $\{x_n\} \subset E$ be any given sequence. Then $\{y_n\} \subset E$ is said to be approximate sequence of $\{x_n\}$ if for any $k \in N$, there exists $\eta = \eta(k)$ such that

$$\|x_n - y_n\| \le \eta \quad \text{for all } n \ge k.$$

Definition.1.6.[15] Two sequences $\{x_n\}$ and $\{y_n\}$ are said to be equivalent if $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Harder and Hicks[1] gave the following definition of stability of iterative procedure:

Definition.1.7.[1] Let E be a normed linear space and T: $E \to E$ be a map. Let $\{x_n\}$ be an iterative procedure defined by $x_0 \in E$ and $x_{n+1} = f(T, x_n), n \ge 0.$ Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\} \subset E$ be any arbitrary sequence and set $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$ for n=0,1,2,….. We say that the iteration procedure is T-stable or stable if

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ implies that } \lim_{n\to\infty} y_n = p.$$

Berinde[18] defined the concept of weak stability by choosing $\{y_n\}$ to be an approximate sequence of $\{x_n\}$. With the help of some examples, he proved the importance of choosing approximate sequence in place of arbitrary sequence and showed that every stable iteration is also weakly stable but reverse may not be true .

Definition.1.8.[18] Let E be a normed linear space and T: $E \to E$ be a map. Let $\{x_n\}$ be an iterative procedure defined by $x_0 \in E$ and $x_{n+1} = f(T, x_n), n \ge 0.$ Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any approximate sequence $\{y_n\} \subset E$ of $\{x_n\}$

$$\lim_{n\to\infty} \|y_{n+1} - f(T, y_n)\| = 0 \text{ implies } \lim_{n\to\infty} y_n = p,$$

then we say that the iteration procedure is weakly T- stable or weakly stable with respect to T.

In 2010, Timis [4] introduced the concept of weak w^2-stability with respect to T by using equivalent sequence in place of approximate sequence as follows:

Definition.1.9.[4] Let E be a normed linear space and T: $E \to E$ be a map. Let $\{x_n\}$ be an iterative procedure defined by $x_0 \in E$ and $x_{n+1} = f(T, x_n), n \ge 0.$ Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any equivalent sequence $\{y_n\} \subset E$ of $\{x_n\}$

$$\lim_{n\to\infty} \|y_{n+1} - f(T, y_n)\| = 0 \text{ implies } \lim_{n\to\infty} y_n = p,$$

then we say that the iteration procedure is weak w^2-stable with respect to T.

He remarked that any equivalent sequence is an approximate sequence but converse may not be true. Thus this concept generalizes the concept of weak T-stability and T-stability.

Lemma 1.10.[5] Let E be a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n \in N$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that

$\lim_{n\to\infty} \sup \|x_n\| \le r$, $\lim_{n\to\infty} \sup \|y_n\| \le r$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ holds for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0.$

The aim of this paper is to establish weak and strong convergence as well as weak stability results for contractive-like operators satisfying CR-iterative scheme. Moreover, with the help of computer programs in C++, an example is given to show that CR-iteration converges faster than Noor and SP iterative scheme for contractive like operators.

2. RESULTS ON CONVERGENCE AND STABILITY

Lemma 2.1. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T: C \to C$ be a contractive like operator satisfying (1.1.17). Let $\{x_n\}$ be defined by the iterative procedure (1.1.8) where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are in $[0,1]$ for all $n \in N$. If $F \ne \phi$, then

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$ and $\{x_n\}$ is bounded.

(ii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$

Proof. Let $p \in F$. Then

$$\|x_{n+1} - p\| \le (1-a_n)\|y_n - p\| + a_n\|Ty_n - p\|$$
$$\le (1-a_n)\|y_n - p\| + a_n\delta\|y_n - p\| \le \|y_n - p\|. \quad (2.1.1)$$

Now, $$\|y_n - p\| \le (1-b_n)\|Tx_n - p\| + b_n\|Tz_n - p\|$$
$$\le (1-b_n)\delta\|x_n - p\| + b_n\delta\|z_n - p\|, \quad (2.1.2)$$

and also, we get

$$\|z_n - p\| \le (1-c_n)\|x_n - p\| + c_n\|Tx_n - p\|$$
$$\le (1-c_n)\|x_n - p\| + c_n\delta\|x_n - p\| \le \|x_n - p\|. \quad (2.1.3)$$

Using (2.1.3) in (2.1.2), we obtain

$$\|y_n - p\| \le (1-b_n)\delta\|x_n - p\| + b_n\delta\|x_n - p\| = \|x_n - p\|. \quad (2.1.4)$$

From (2.1.4) and (2.1.1), we have

$$\|x_{n+1} - p\| \le (1-a_n)\|x_n - p\| + a_n\|x_n - p\| = \|x_n - p\|. \quad (2.1.5)$$

Hence $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F$, say it q,

that is, $$\lim_{n\to\infty} \|x_n - p\| = q. \quad (2.1.6)$$

Thus, $\{x_n\}$is bounded.

Now, using (2.1.6) in (2.1.3) and (2.1.4), we get

$$\lim_{n\to\infty} \sup \|y_n - p\| \le q. \quad (2.1.7)$$

and $$\lim_{n\to\infty} \sup \|z_n - p\| \le q. \quad (2.1.8)$$

Now, $\|Tx_n - p\| \le \delta\|x_n - p\| < \|x_n - p\|$.

Gives by (2.1.6)

$$\lim_{n\to\infty} \sup \|Tx_n - p\| \le q. \quad (2.1.9)$$

Next, $\|Ty_n - p\| \le \delta\|y_n - p\| < \|y_n - p\|$.

From (2.1.7)

$$\lim_{n\to\infty} \sup\|Ty_n - p\| \le q . \qquad (2.1.10)$$

Next, $\|Tz_n - p\| \le \delta\|z_n - p\| < \|z_n - p\|$.

By (2.1.8)

$$\lim_{n\to\infty} \sup\|Tz_n - p\| \le q . \qquad (2.1.11)$$

Moreover,

$q = \lim_{n\to\infty} \|x_{n+1} - p\| = \|(1-a_n)(y_n - p) + a_n(Ty_n - p)\|$,

then by Lemma 1.10**,** we get

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0 . \qquad (2.1.12)$$

Now,

$$\|x_{n+1} - p\| = \|(1-a_n)y_n + a_nTy_n - p\|$$
$$\le \|y_n - p\| + a_n\|y_n - Ty_n\| ,$$

gives by using (2.1.6) and (2.1.12), that

$$q \le \lim_{n\to\infty} \inf\|y_n - p\| .$$

By (2.1.7), we get

$$\lim_{n\to\infty} \|y_n - p\| = q . \qquad (2.1.13)$$

Thus,

$q = \lim_{n\to\infty}\|y_n - p\| = \lim_{n\to\infty}\|(1-b_n)(Tx_n - p) + b_n(Tz_n - p)\|$, give s by Lemma 1.10 that

$$\lim_{n\to\infty} \|Tx_n - Tz_n\| = 0 . \qquad (2.1.14)$$

Again

$$\|y_n - p\| \le \|Tx_n - p\| + b_n\|Tz_n - Tx_n\| .$$

Using (2.1.14) and (2.1.13), we have

$$q \le \lim_{n\to\infty} \inf\|Tx_n - p\| .$$

By (2.1.9),

$$\lim_{n\to\infty} \|Tx_n - p\| = q . \qquad (2.1.15)$$

From (2.1.8) and (2.1.14), we get

$$\|Tx_n - p\| \le \|Tx_n - Tz_n\| + \|Tz_n - p\|$$
$$\le \|Tx_n - Tz_n\| + \delta\|z_n - p\|$$
$$< \|Tx_n - Tz_n\| + \|z_n - p\| .$$

Using (2.1.14) and (2.1.15), we get

$q \le \lim_{n\to\infty} \inf\|z_n - p\|$.

Now, (2.1.8) gives $\lim_{n\to\infty} \|z_n - p\| = q$.

Thus,

$q = \lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty}\|(1-c_n)(x_n - p) + c_n(Tx_n - p)\|$

gives by Lemma 1.10 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Theorem 2.2. Let E be a uniformly convex Banach space and let C, T and $\{x_n\}$ be taken as in Lemma 2.1. If $F \ne \phi$ and I-T is demiclosed at zero, then $\{x_n\}$ converges weakly to a point of F.

Proof. Let $q \in F$. By lemma 2.1, $\{x_n\}$ is bounded. Also E is a uniformly convex Banach space, so E is reflexive and thus every bounded sequence in E has a weakly convergent subsequence. So, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to $q \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$. Thus $\omega_w(x_n) \ne \phi$. By Lemma 2.1,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (2.2.1)$$

Also I-T is demiclosed with respect to zero, therefore $Tq=q$ and so $q \in F$. Thus $\omega_w(x_n) \subset F$.

Now, for any $q \in \omega_w(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.2)$$

Using (2.2.1) and (2.2.2), we get

$$Tx_{n_i} = (Tx_{n_i} - x_{n_i}) + x_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.3)$$

From (1.1.8), (2.2.2) and (2.2.3), we have

$$z_{n_i} = (1-c_{n_i})x_{n_i} + c_{n_i}Tx_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.4)$$

Now by (2.1.14) and (2.2.3), we get

$$Tz_{n_i} = (Tz_{n_i} - Tx_{n_i}) + Tx_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.5)$$

Again from (1.1.8), (2.2.3) and (2.2.5), we obtain

$$y_{n_i} = (1-b_{n_i})Tx_{n_i} + b_{n_i}Tz_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.6)$$

Also from (2.1.12) and (2.2.6), we have

$$Ty_{n_i} = (Ty_{n_i} - y_{n_i}) + y_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.7)$$

It follows form (2.2.6) and (2.2.7) that

$$x_{n_{i+1}} = (1-a_{n_i})y_{n_i} + a_{n_i}Ty_{n_i} \text{ converges weakly to } q \text{ as } i \to \infty. \qquad (2.2.8)$$

Continuing in this way, by mathematical induction, we can prove that for any $m \ge 0$

$$x_{n_{i+m}} \text{ converges weakly to } q \text{ as } i \to \infty.$$

Again, by applying mathematical induction, we can obtain

$$\bigcup_{m=0}^{\infty}\{x_{n_{i+m}}\}_{i=1}^{\infty} \text{ converges weakly to } q \text{ as } i \to \infty.$$

Since $\{x_n\}_{n=n_1}^{\infty} = \bigcup_{m=0}^{\infty}\{x_{n_{i+m}}\}_{i=1}^{\infty}$, therefore x_n converges weakly to q as $i \to \infty$.

Hence, $\{x_n\}$ converges weakly to a point of F.

Theorem 2.3. Let C be a nonempty closed convex subset of an arbitrary Banach space E and $T: C \to C$ be a contractive-like operator satisfying (1.1.17). Let $\{x_n\}$ be defined by the iterative procedure (1.1.8) and $x_0 \in C$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1] with $\{a_n\}$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then $\{x_n\}$ converges strongly to a point of F.

Proof. Let $p \in F$. Then

$$\| x_{n+1} - p \| \le (1-\alpha_n) \| y_n - p \| + \alpha_n \| Ty_n - p \|$$
$$\le (1-a_n)(1-b_n) \| Tx_n - p \| + (1-a_n)b_n \| Tz_n - p \|$$
$$+a_n \|Ty_n - p\|$$
$$\le \delta (1-a_n)(1-b_n) \| x_n - p \| + \delta(1-a_n)b_n \| z_n - p \|$$
$$+\delta a_n \|y_n - p\|$$
$$\le \delta(1-a_n)(1-b_n) \| x_n - p \| + \delta(1-a_n)b_n(1-c_n) \| x_n - p \|$$
$$+\delta(1-a_n)b_n c_n \|Tx_n - p\| + \delta a_n (1-b_n)\|Tx_n - p\|$$
$$+\delta a_n b_n \|Tz_n - p\|$$
$$\le \delta(1-a_n)(1-b_n) \| x_n - p \| + \delta(1-a_n)b_n(1-c_n) \| x_n - p \|$$
$$+\delta^2(1-a_n)b_n c_n \|x_n - p\| + \delta^2 a_n (1-b_n) \| x_n - p \|$$
$$+\delta^2 a_n b_n (1-c_n) \| x_n - p \| + \delta^3 a_n b_n c_n \| x_n - p \|$$
$$= \delta\{1 - a_n(1-\delta) - (1-\delta)(1-a_n)b_n c_n$$
$$-\delta(1-\delta)a_n b_n c_n\} \| x_n - p \|$$
$$\le \delta [1-\alpha_n(1-\delta)] \| x_n - p \|$$
$$\le \delta^n \prod_{k=0}^{n} [1-a_k(1-\delta)] \| x_0 - p \|$$
$$\le \delta^n e^{-(1-\delta)\sum_{k=0}^{\infty} a_k} \|x_0 - p\| . \qquad (2.3.1)$$

Since $0 \le \delta < 1$, $a_k \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, so $\delta^n e^{-(1-\delta)\sum_{k=0}^{\infty} a_k} \to 0$ as $n \to \infty$. Hence, it follows from (2.3.1) that $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$. Therefore $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

Theorem 2.4. Let E be a normed linear space and $T: E \to E$ be a contractive-like operator satisfying (1.1.17). Let $\{x_n\}$ be defined by iterative procedure (1.1.8) and $x_0 \in E$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in [0,1]. If the sequence $\{x_n\}$ converges to a fixed point p of T, then the CR-iterative procedure (1.1.8) is weak w^2–stable.

Proof. Consider $\{y_n\} \subset E$ to be an equivalent sequence of $\{x_n\}$. Define $\varepsilon_n = \|y_{n+1} - (1-a_n)s_n - a_n Ts_n\|$, where $s_n = (1-b_n)Ty_n + b_n Tt_n$ and $t_n = (1-c_n)y_n + c_n Ty_n$, $n \ge 0$. (2.4.1)

Suppose that $\lim_{n\to\infty} \|y_{n+1} - (1-a_n)s_n - a_n Ts_n\| = 0$. (2.4.2)

To prove that CR-iterative procedure is weak w^2–stable, we show that $\lim_{n\to\infty} y_n = p$.

From (1.1.8), we have

$$\|y_{n+1} - p\| \le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - p\|$$
$$\le \|y_{n+1} - (1-a_n)s_n - a_n Ts_n\| + \|(1-a_n)s_n + a_n Ts_n - x_{n+1}\|$$
$$+\|x_{n+1} - p\|$$
$$\le \varepsilon_n + \|x_{n+1} - p\| + \|(1-a_n)s_n + a_n Ts_n - (1-a_n)y_n - a_n Ty_n\|$$
$$\le \varepsilon_n + \|x_{n+1} - p\| + (1-a_n)\|s_n - y_n\| + a_n\|Ts_n - Ty_n\|$$
$$\le \varepsilon_n + \|x_{n+1} - p\| + (1-a_n)\|s_n - y_n\| + a_n\delta\|s_n - y_n\|$$
$$+a_n\phi(\|y_n - Ty_n\|)$$
$$\le \varepsilon_n + \|x_{n+1} - p\| + (1-a_n+a_n\delta)\|s_n - y_n\| + a_n\phi(\|y_n - Ty_n\|)$$
$$< \varepsilon_n + \|x_{n+1} - p\| + \|s_n - y_n\| + a_n\phi(\|y_n - Ty_n\|). \qquad (2.4.3)$$

Now, using (1.1.8) and (2.4.1), we obtain

$$\|s_n - y_n\|$$
$$\le (1-b_n)\|Ty_n - Tx_n\| + b_n\|Tz_n - Tt_n\|$$
$$\le (1-b_n)\delta\|y_n - x_n\| + b_n\phi(\|y_n - Ty_n\|) + b_n\delta\|z_n - t_n\|$$
$$+b_n\phi(\|z_n - Tz_n\|)$$
$$\le (1-b_n)\delta\|y_n - x_n\| + b_n\phi(\|y_n - Ty_n\|)$$
$$+b_n\delta\left[(1-c_n)\|x_n - y_n\| + c_n\|Tx_n - Ty_n\|\right] + b_n\phi(\|z_n - Tz_n\|)$$
$$\le (1-b_n)\delta\|y_n - x_n\| + b_n\phi(\|y_n - Ty_n\|)$$
$$+b_n\delta\left[(1-c_n)\|x_n - y_n\| + c_n\delta\|x_n - y_n\| + c_n\phi(\|y_n - Ty_n\|)\right]$$
$$+b_n\phi(\|z_n - Tz_n\|)$$
$$< (1-b_n)\delta\|y_n - x_n\| + b_n\phi(\|y_n - Ty_n\|)$$
$$+b_n\delta\left[\|x_n - y_n\| + c_n\phi(\|y_n - Ty_n\|)\right] + b_n\phi(\|z_n - Tz_n\|)$$
$$< \|y_n - x_n\| + b_n(1+\delta c_n)\phi(\|y_n - Ty_n\|) + b_n\phi(\|z_n - Tz_n\|) . \qquad (2.4.4)$$

Also,

$$\|y_n - Ty_n\| \le \|y_n - p\| + \|Ty_n - p\|$$
$$\le \|y_n - p\| + \delta\|y_n - p\|$$
$$= (1+\delta)\|y_n - p\|$$

$\leq (1+\delta)\left[(1-b_n)\|Tx_n - p\| + b_n\|Tz_n - p\|\right]$

$\leq (1+\delta)\left[(1-b_n)\delta\|x_n - p\| + \delta b_n\|z_n - p\|\right]$

$\leq (1+\delta)$

$\left[(1-b_n)\delta\|x_n - p\| + \delta b_n\left((1-c_n)\|x_n - p\| + c_n\|Tx_n - p\|\right)\right]$

$\leq (1+\delta)$

$\left[(1-b_n)\delta\|x_n - p\| + \delta b_n\left((1-\gamma_n)\|x_n - p\| + c_n\delta\|x_n - p\|\right)\right]$

$= \delta(1+\delta)\left[(1-b_n) + b_n(1-c_n) + b_nc_n\delta\right]\|x_n - p\|$

$= \delta(1+\delta)\left[1 - b_nc_n(1-\delta)\right]\|x_n - p\|,$ (2.4.5)

and

$\|z_n - Tz_n\| \leq \|z_n - p\| + \|Tz_n - p\|$

$\leq \|z_n - p\| + \delta\|z_n - p\|$

$= (1+\delta)\|z_n - p\|$

$\leq (1+\delta)\left[(1-c_n)\|x_n - p\| + c_n\|Tx_n - p\|\right]$

$\leq (1+\delta)\left[(1-c_n)\|x_n - p\| + \delta c_n\|x_n - p\|\right]$

$= (1+\delta)\left[1 - c_n + \delta c_n\right]\|x_n - p\|$

$< (1+\delta)\|x_n - p\|.$ (2.4.6)

Using (2.4.5) and (2.4.6) in (2.4.4), we get

$\|s_n - y_n\|$

$< \|y_n - x_n\| + b_n(1+\delta c_n)\phi\left(\delta(1+\delta)\left[1 - b_nc_n(1-\delta)\right]\|x_n - p\|\right)$
$+ b_n\phi\left((1+\delta)\|x_n - p\|\right).$ (2.4.7)

Substituting (2.4.5), (2.4.7) in (2.4.3) and rearranging the terms, we get

$\|y_{n+1} - p\|$

$< \varepsilon_n + \|x_{n+1} - p\| + \|y_n - x_n\|$

$+ b_n(1+\delta c_n)\phi\left(\delta(1+\delta)\left[1 - b_nc_n(1-\delta)\right]\|x_n - p\|\right)$
$+ b_n\phi\left((1+\delta)\|x_n - p\|\right)$

$+ a_n\phi\left(\delta(1+\delta)\left[1 - b_nc_n(1-\delta)\right]\|x_n - p\|\right).$

Since $\{x_n\}$ and $\{y_n\}$ are equivalent sequences, so $\lim_{n\to\infty}\|y_n - x_n\| = 0.$

Also $\{x_n\}$ converges to p, thus $\lim_{n\to\infty}\|x_n - p\| = 0$.

Hence $\lim_{n\to\infty}\|y_{n+1} - p\| = 0.$

Therefore, the CR-iterative procedure (1.1.6) is weak w^2 – stable.

REMARKS:

(i) Theorem 2.4 is generalizations of the Theorem 2.1 of Chugh et. al. [9].

(ii) If $a_n = 0$ for all $n \in N$ in CR-iteration (1.1.8), then Theorem 2.2, 2.3 and 2.4 reduce to convergence and stability results for S- iteration (1.1.5), as well as, if we take $a_n = 0$ and $b_n = 1$ for all $n \in N$ in CR-iteration (1.1.8), then Theorem 2.2, 2.3 and 2.4 reduce to convergence and stability results for iteration (1.1.6) using contractive like operators.

In the following example, the rate of convergence of CR, SP and Noor iterative procedures is compared with the help of computer programs in C++.

Example.2.5. Let E=[0,1] and $T: E \to E$ is defined as $Tx = e^{(1-x)^2} - 1$. Then T is a contractive like operator.

By using computer programs in C++ the comparison of the rate of convergence of CR, SP and Noor iterative procedures to a fixed point of T is shown in the following table, with $x_0 = 0.9$ and $a_n = 0.7, b_n = c_n = 0.6$ for all iterative procedures.

No. of Iterations	CR Iterative Procedure		SP Iterative Procedure		Noor Iterative Procedure	
n	x_{n+1}	Tx_n	x_{n+1}	Tx_n	x_{n+1}	Tx_n
0	0.410292	0.0100502	0.36258	0.01005	0.392037	0.0100502
1	0.412634	0.415885	0.428264	0.501259	0.430968	0.447182
2	0.412363	0.411989	0.407623	0.386638	0.397512	0.382367
3	0.412394	0.412438	0.413858	0.42036	0.42573	0.437624
4	0.412391	0.412386	0.411943	0.409961	0.401513	0.39067
5	0.412391	0.412392	0.412528	0.413136	0.422015	0.430732
6	**0.412391**	**0.412391**	0.412349	0.412163	0.404442	0.396635
7	**0.412391**	**0.412391**	0.412404	0.412461	0.419356	0.425737
-----	-----------	-----------	-----------	----------	----------	-------------
11			0.412391	0.412392	0.416057	0.419466
12			**0.41239**	**0.41239**	0.409301	0.406338

13			**0.41239**	**0.41239**	0.415055	0.417543

30					0.412213	0.412044
31					0.412544	0.412687
32					0.412261	0.412138
-----	-----------	-----------	-----------	-----------	-----------	-----------
73					0.412391	0.412392
74					**0.412391**	**0.412391**
75					**0.412391**	**0.412391**

3. CONCLUSION

In view of above table, CR iterative procedure converges in 7 iterations, SP scheme in 13 iterations and Noor scheme in 75 iterations. Thus CR converges faster than SP and Noor iterative procedures. The decreasing rate of convergence of iterative procedures is as follows: CR, SP and Noor iterative procedure.

4. REFERENCES

[1] A. M. Harder and T. L. Hicks, Stability Results for Fixed Point Iteration Procedures, Math. Japonica, vol. 33, no. 5, (1988), 693-706.

[2] C. O. Imoru and M. O. Olantinwo, On the stability of Picard and Mann iteration processes, Carp. J. Math., vol. 19, no. 2, (2003), 155-160.

[3] D. R. Sahu and A. Petrusel, Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces, NonlinearAnalysis: Theory,Methods &Applications, vol. 74, no. 17, (2011), 6012–6023.

[4] I. Timis, On the weak stability of Picard iteration for some contractive type mappings, Annals of the University of Craiova Mathematics and Computer Science Series, vol. 37, no. 2, (2010), 106-114.

[5] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc., vol. 43, (1991), 153-159.

[6] M. A. Noor, New approximation schemes for general variational inequalities, Journal of Mathematical Analysis and Applications, vol. 251 no. 1, (2000), 217-229.

[7] M. O. Olatinwo, Convergence and stability results for some iterative schemes, Acta Universitatis Apulensis, vol. 26, (2011), 225-236.

[8] M. O. Osilike and A. Udomene, Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings, Indian Journal of Pure and Applied Mathematics, vol. 30, no. 12, (1999), 1229–1234.

[9] R. Chugh, V. Kumar and S. Kumar, Strong convergence of a new three step iterative scheme in Banach spaces, American Journal of computational mathematics, vol. 2, (2012), 345-357.

[10] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., vol. 60, (1968), 71-76.

[11] R. P. Agarwal, D. O'Regan, and D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal., vol. 8, no. 1, (2007), 61-79.

[12] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrals, Fund. Math., vol. 3, (1922), 133-181.

[13] S. Ishikawa, Fixed points by a new iteration method, Proc Am Math Soc., vol. 44, (1974), 147–150.

[14] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci., vol. 25, (1972), 727-730.

[15] T. Cardinali and P. Rubbioni, A generalization of the Caristi fixed point theorem in metric spaces, Fixed Point Theory, vol. 11, no. 1, (2010), 3-10.

[16] T. Zamifirescu, Fixed point theorems in metric spaces, Arch. Math., vol. 23, (1972), 292-298.

[17] V. Berinde, On the convergence of the Ishikawa iteration in the class of quasi contractive operators, Acta Mathematica Universitatis Comenianae, vol. 73, no. 1, (2004), 119–126.

[18] V. Berinde, Iterative Approximation of fixed points, Springer Verlag, Lectures Notes in Mathematics, 2007.

[19] W. Phuengrattana and S. Suantai, On the rate of convergence of Mann Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval, Journal of Computational and Applied Mathematics, vol. 235, (2011), 3006- 3014.

[20] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., vol. 4, (1953), 506- 510.