

Solving Differential Equations of Second Order using Quadratic Legendre Multi-wavelets (QLMW) with Operational Matrix of Integration

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ABSTRACT

In this paper is suggested an efficient method to solve differential equations. Using quadratic Legendre multi-wavelets approximation method, differential equations are converted into the system of algebraic equations with the help of operational matrix of integration and its product. Some illustrative examples are included to show the efficiency and applicability of the method.

AMS Subject Classifications: 65T60; 65L10; 65M70.

Keywords

Quadratic Legendre wavelets; Quadratic Legendre multi-wavelets; Operational matrix of integration; Differential equations.

1. INTRODUCTION

Wavelet is compactly supported [1] square integrable function in time domain and frequency domain. Firstly, wavelet was introduced by N. Ricker [2] in Seismology to provide a time dimension to seismic analysis. Also, it was repeated by E. A. Robinson [3, 4]. But finally wavelet was defined by J. Morlet et al [5, 6]. After that A. Grossman [7], a French theoretical physicist recognized the importance of wavelet transform and developed an inversion formula with the help of admissibility condition. Wavelet is new developing area which is being applied mathematics, medical science, technology and more widely in signal analysis. There is a reason of more success of wavelet and their transform to the comparison of Fourier analysis because Fourier analysis does not contain the local information of non-stationary signals but wavelets examine signals simultaneously in both time and frequency. In recent years, wavelet techniques are used to find out the numerical solution of linear and non-linear problems [10,13]. Current applications of wavelet contain climate analysis, financial denoising, denoising of astronomical images, fast solution of differential equations, computer graphic and so on.

The basic approach initiates from the integral of basis vector $\chi(t)$ which is defined by

$$\int_0^x \chi(t) dt \approx P \chi(x), \tag{1}$$

where, $\chi(t) = [\varphi_1, \varphi_2, \dots, \varphi_n]^T$ in which the elements $\varphi_i(t)$; $i = 1, 2, 3, \dots, n$ are orthonormal basis function on certain interval $[0, 1]$, and the matrix P can be uniquely illustrated based on the particular orthonormal function.

In section 2, quadratic Legendre multi-wavelet (QLMW) is introduced. Approximation of any function within interval $[0, 1]$, using QLMW, followed by error analysis, is included in section 3. In section 4, the operational matrix of integration for QLMW is calculated. The product of operational matrix is produced in section 5. In section 6, these computations are used to derive a method for solving initial value problem. Finally, the advantages of present method with the help of two examples are shown in section 7.

2. THE QUADRATIC LEGENDRE MULTI-WAVELET (QLMW)

Let ψ be a function of $L_2(\mathfrak{S})$, called mother wavelet, satisfying the following properties [7, 8]:

$$\int_{\mathfrak{S}} \psi(t) dt = 0 \quad \text{and} \quad \int_{\mathfrak{S}} |\psi(t)|^2 dt = 1.$$

This is also called unit energy function.

A family of such type of functions constructed from translation and dilation of a single function ψ , wavelets [7, 8] can be defined as:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathfrak{S}; a \neq 0, \tag{2}$$

where a and b are dilation and translation parameters respectively.

By discretization [1, 7] of these parameters $a = 2^{-k}$ and $b = n2^{-k}$, one gets

$$\psi_{k,n}(t) = 2^{k/2} \psi(2^k t - n), \quad k, n \in \mathbb{C} \tag{3}$$

Let ϕ be a function of $L_2(\odot)$ space. It is said to be scaling function for V_0 , if it satisfies the following condition

$$\int_{-\infty}^{\infty} \phi(t) dt = 1 \text{ and } V_0 = \text{span}_k \{\phi(t-k)\}. \quad (4)$$

2.1. Definition: The nested sequence $\{V_j\}_{j=-\infty}^{\infty}$ of the subspaces of $L_2(\odot)$ with scaling function ϕ is called multi-resolution analysis (MRA) if it satisfies the following conditions:

- (i) $\lim_{j \rightarrow +\infty} V_j = L_2(\odot)$ and $\lim_{j \rightarrow -\infty} V_j = \{0\}$
- (ii) $\phi(t) \in V_0 \Leftrightarrow \phi(2^j t) \in V_j$
- (iii) $\{\phi(t-n)\}_{n=-\infty}^{\infty}$ is a Riesz basis of V_0 .

For any orthogonal MRA with a multi-scaling function ϕ , there exist [12] a multi-wavelet function Ψ and a multi-scaling function ϕ orthogonal to each other, given as below:

$$\psi(t) = \sqrt{2} \sum_{n=-\infty}^{\infty} b_n \phi(2t-n) \quad (5)$$

and $\{\psi_{j,n}\}_{j,n=-\infty}^{\infty}$ forms an orthonormal basis for $L_2(\odot)$ under certain condition [11].

For constructing the quadratic Legendre multi-wavelet, when the single scaling function $\phi(t)$ is replaced with vector of scaling function $\Phi(t)$ yields:

$$\Phi(t) = [\phi_0(t), \phi_1(t), \phi_2(t)]^T,$$

where

$$\begin{aligned} \phi_0(t) &= 1, \quad \phi_1(t) = \sqrt{3}(2t-1) \text{ and} \\ \phi_2(t) &= \sqrt{5}(6t^2-6t+1), \quad 0 \leq t \leq 1. \end{aligned} \quad (6)$$

Since $\Psi(t) = [\psi_0(t), \psi_1(t), \psi_2(t)]^T$ are the corresponding mother wavelet functions [11], using equation (2) one gets the quadratic Legendre wavelet (QLW) as given

below:

$$\begin{aligned} \psi_0(t) &= \begin{cases} -\frac{1}{3}(120t^2 - 72t + 7), & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{3}(120t^2 - 168t + 55), & \frac{1}{2} \leq t \leq 1, \end{cases} \\ \psi_1(t) &= \begin{cases} \sqrt{3}(30t^2 - 14t + 1), & 0 \leq t \leq \frac{1}{2} \\ \sqrt{3}(30t^2 - 46t + 17), & \frac{1}{2} \leq t \leq 1, \end{cases} \end{aligned} \quad (7)$$

$$\psi_2(t) = \begin{cases} -\frac{\sqrt{5}}{3}(48t^2 - 18t + 1), & 0 \leq t \leq \frac{1}{2} \\ \frac{\sqrt{5}}{3}(48t^2 - 78t + 31), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

Translating and dilating the quadratic Legendre wavelet $\Psi(t)$, we get

$$\psi_{k,n}^j(t) = 2^{j/2} \psi^j(2^k t - n), \quad k, n, j \in \mathbb{C} \quad (8)$$

The family $\{\psi_{k,n}^j(t)\}$ forms an orthonormal basis for $L_2(\odot)$ and subfamily is orthonormal in $L_2[0, 1]$ for $n=0, 1, 2, \dots, 2^k-1; k=0, 1, 2, \dots$ and $j=0, 1, 2$. If $n=0, 1; k=0, 1$ and $j=0, 1, 2$; the scaling functions and subfamily members are plotted in the figures 1-12. Here, variable t and family members $\{\phi_0, \phi_1, \phi_2, \psi_{00}^0, \psi_{00}^1, \psi_{00}^2, \psi_{10}^0, \psi_{10}^1, \psi_{10}^2, \psi_{11}^0, \psi_{11}^1, \psi_{11}^2\}$ are shown along horizontal and vertical axes respectively.

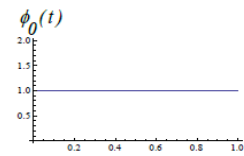


Fig.1

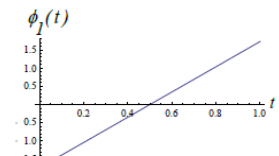


Fig.2

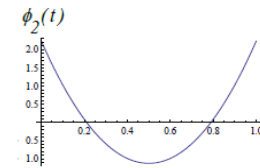


Fig.3

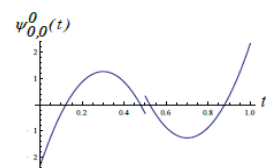


Fig.4

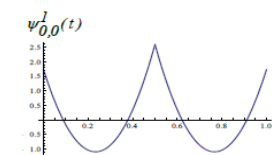


Fig.5

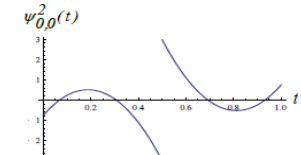


Fig.6

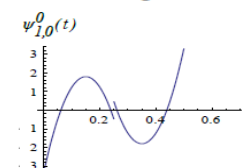


Fig.7

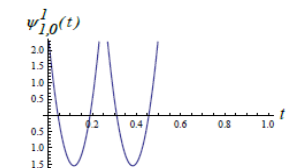


Fig.8

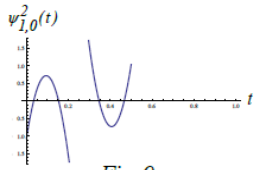


Fig. 9

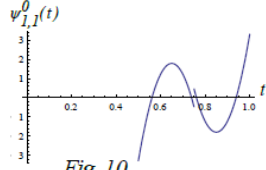


Fig. 10

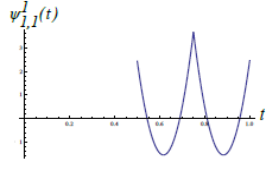


Fig. 11

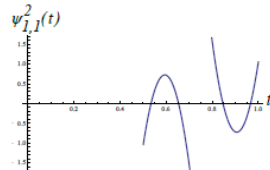


Fig. 12

3. FUNCTION APPROXIMATIONS

The function $f(x)$, using quadratic Legendre multi-wavelets in the interval $[0, 1]$, is approximated as given below:

$$f(x) = \sum_{i=0}^2 c_i \phi_i + \sum_{k=0}^{\infty} \sum_{j=0}^2 \sum_{n=0}^{2^k-1} c_{k,n}^j \psi_{k,n}^j(x), \quad (9)$$

where,

$$c_i = \langle f(x), \phi_i(x) \rangle \text{ and } c_{k,n}^j = \langle g(x), \psi_{k,n}^j(x) \rangle.$$

After truncating the infinite series, equation (9) takes the following form:

$$f(x) \approx \sum_{i=0}^2 c_i \phi_i + \sum_{k=0}^M \sum_{j=0}^2 \sum_{n=0}^{2^k-1} c_{k,n}^j \psi_{k,n}^j(x) = C^T \Theta(x), \quad (10)$$

where,

$$C = [c_0, c_1, c_2, c_{00}^0, \dots, c_{M(2^M-1)}^0, c_{00}^1, \dots, \dots, c_{M(2^M-1)}^1, c_{00}^2, \dots, c_{M(2^M-1)}^2]^T \quad (11)$$

and

$$\Theta(x) = [\phi_0, \phi_1, \phi_2, \psi_{00}^0, \dots, \psi_{M(2^M-1)}^0, \psi_{00}^1, \dots, \psi_{M(2^M-1)}^1, \psi_{00}^2, \dots, \psi_{M(2^M-1)}^2]^T. \quad (12)$$

Theorem 3.1 Let the function $f : [0, 1] \rightarrow \mathbb{R}$ and $f \in C^3[0, 1]$. Then $C^T \Theta$ approximates f with mean error bound as follows:

$$\|f - C^T \Theta\| \leq \frac{1}{3! 2^{3k}} \sup_{x \in [0,1]} |f'''(x)|,$$

where $\|\cdot\|$ denotes the norm in $L_2(\mathbb{R})$ space.

Proof. The proof of this theorem can be seen in [2].

It can be seen from this theorem; the factor $\frac{1}{3! 2^{3k}}$ shows best

approximation of function because if k increases then error decreases rapidly.

4. OPERATIONAL MATRIX OF INTEGRATION OF QLMW

With the help of equation (12), equation (1) takes the form:

$$\int_0^x \Theta(t) dt \approx P_{12 \times 12} \Theta(x),$$

where, $P_{12 \times 12}$ an operational matrix of integration of QLMW [11] is given below:

$$P_{12 \times 12} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{13}}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{13}}{30} & 0 & \frac{3}{16\sqrt{3}} & 0 & 0 & \frac{3\sqrt{3}}{256\sqrt{3}} & \frac{3\sqrt{3}}{256\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3}{16\sqrt{3}} & 0 & \frac{5}{48\sqrt{3}} & 0 & -\frac{\sqrt{2}}{64} & \frac{\sqrt{2}}{64} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-5}{48\sqrt{3}} & 0 & \frac{1}{6\sqrt{13}} & \frac{3\sqrt{6}}{256} & \frac{3\sqrt{6}}{256} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6\sqrt{13}} & 0 & -\frac{\sqrt{2}}{32\sqrt{3}} & -\frac{\sqrt{2}}{32\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3\sqrt{2}}{256\sqrt{3}} & \frac{\sqrt{2}}{64} & \frac{-3\sqrt{6}}{32\sqrt{3}} & \frac{\sqrt{2}}{32\sqrt{3}} & 0 & 0 & \frac{5}{96\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{-3\sqrt{2}}{256\sqrt{3}} & -\frac{\sqrt{2}}{64} & \frac{-3\sqrt{6}}{32\sqrt{3}} & -\frac{\sqrt{2}}{32\sqrt{3}} & 0 & 0 & 0 & \frac{5}{96\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-5}{96\sqrt{3}} & 0 & 0 & 0 & \frac{1}{12\sqrt{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-5}{96\sqrt{3}} & 0 & 0 & 0 & \frac{1}{12\sqrt{13}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{12\sqrt{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{12\sqrt{13}} & 0 & 0 \end{bmatrix}$$

In general

$$P = \begin{bmatrix} A_{(3.2^n) \times (3.2^n)} & B_{(3.2^n) \times (3.2^n)} \\ -B_{(3.2^n) \times (3.2^n)} & C_{(3.2^n) \times (3.2^n)} \end{bmatrix}, \quad (13)$$

where order of P , given by [11], is $(3.2^{n+1}) \times (3.2^{n+1})$.

5. THE PRODUCT OF QLMW BASIS VECTOR AND IT'S TRANSPOSE

In this section the product of operational integration matrix of QLMW is obtained, which is useful in solving non homogeneous differential and integral equations. From equation (12) we have

$$\Theta(x) \Theta^T(x) = \sum_{j=0}^2 \phi_j I_j + \sum_{j=0}^2 \sum_{i=0}^2 \sum_{k=0}^{2^M-1} \psi_{i,k}^j I_{i,k}^j, \quad (14)$$

where,

$$I_j = \begin{bmatrix} A_j & B_j \\ [B_j]^T & C_j \end{bmatrix}_{(3.2^{n+1}) \times (3.2^{n+1})},$$

$$I_{i,k}^j = \begin{bmatrix} A_{i,k}^j & B_{i,k}^j \\ [B_{i,k}^j]^T & C_{i,k}^j \end{bmatrix}_{(3.2^{n+1}) \times (3.2^{n+1})},$$

$$A_j = \begin{bmatrix} (a')_j & (a'')_j \\ [(a'')_j]^T & (a''')_j \end{bmatrix}_{(3.2^n) \times (3.2^n)}, \quad (15)$$

$$A_{i,k}^j = \begin{bmatrix} (a')_{i,k}^j & (a'')_{i,k}^j \\ [(a'')_{i,k}^j]^T & (a''')_{i,k}^j \end{bmatrix}_{(3.2^n) \times (3.2^n)},$$

$$C_j = \begin{bmatrix} (c')_j & (c'')_j \\ [(c'')_j]^T & (c''')_j \end{bmatrix}_{(3.2^n) \times (3.2^n)},$$

$$C_{i,k}^j = \begin{bmatrix} (c')_{i,k}^j & (c'')_{i,k}^j \\ [(c'')_{i,k}^j]^T & (c''')_{i,k}^j \end{bmatrix}_{(3.2^n) \times (3.2^n)}.$$

The order of the elements of the matrices $A_j, B_j, C_j, A_{i,k}^j, B_{i,k}^j, C_{i,k}^j$ is $(3.2^{n-1}) \times (3.2^{n-1})$. For

$i = 0, 1; j = 0, 1, 2; k = 0, 1$ and $M = I$, the matrices I_j and $I_{i,k}^j$ are of order 12×12 . Also, the matrices I_0, A_0 are identities and $A_{1,1}^1, A_{1,1}^2$ are null matrices. Thus, the entries of matrices $A_j, A_{i,k}^j$ are given below:

$$\begin{aligned}
 (a')_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}, & (a'')_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{9\sqrt{3}}{8\sqrt{5}} & 0 & 0 \end{bmatrix} \\
 (a''')_1 &= \begin{bmatrix} 0 & \frac{17}{24} & 0 \\ \frac{17}{24} & 0 & \frac{2}{3\sqrt{5}} \\ 0 & \frac{2}{3\sqrt{5}} & 0 \end{bmatrix}, & (a')_2 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{2}{\sqrt{5}} & 0 \\ 1 & 0 & \frac{2\sqrt{5}}{7} \end{bmatrix} \\
 (a'')_2 &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{9\sqrt{3}}{8\sqrt{5}} & 0 & 0 \\ 0 & \frac{3\sqrt{3}}{7} & 0 \end{bmatrix}, & (a''')_2 &= \begin{bmatrix} \frac{58}{63\sqrt{5}} & 0 & \frac{173}{504} \\ 0 & -\frac{4}{7\sqrt{5}} & 0 \\ \frac{173}{504} & 0 & \frac{-26\sqrt{5}}{63} \end{bmatrix} \\
 (a')_{0,0}^0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{9\sqrt{3}}{8\sqrt{5}} \\ 0 & \frac{9\sqrt{3}}{8\sqrt{5}} & 0 \end{bmatrix}, & (a'')_{0,0}^0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{17}{24} & 0 \\ \frac{58}{63\sqrt{5}} & 0 & \frac{173}{504} \end{bmatrix} \\
 (a''')_{0,0}^0 &= \begin{bmatrix} 0 & \frac{2}{21\sqrt{3}} & 0 \\ \frac{2}{21\sqrt{3}} & 0 & \frac{-31}{168\sqrt{15}} \\ 0 & \frac{-31}{168\sqrt{15}} & 0 \end{bmatrix}, & (a')_{0,0}^1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3\sqrt{3}}{7} \end{bmatrix} \\
 (a'')_{0,0}^1 &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{17}{24} & 0 & \frac{2}{3\sqrt{5}} \\ 0 & -\frac{4}{7\sqrt{5}} & 0 \end{bmatrix}, & (a''')_{0,0}^1 &= \begin{bmatrix} \frac{2}{21\sqrt{3}} & 0 & 0 \\ 0 & \frac{3}{7\sqrt{3}} & 0 \\ \frac{-31}{168\sqrt{15}} & 0 & \frac{52}{21\sqrt{3}} \end{bmatrix} \\
 (a')_{0,0}^2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & (a'')_{0,0}^2 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{2}{3\sqrt{5}} & 0 \\ \frac{173}{504} & 0 & \frac{-23\sqrt{5}}{63} \end{bmatrix} \\
 (a''')_{0,0}^2 &= \begin{bmatrix} 0 & \frac{-31}{168\sqrt{15}} & \frac{3}{21\sqrt{3}} \\ \frac{-31}{168\sqrt{15}} & 0 & \frac{52}{21\sqrt{3}} \\ 0 & \frac{52}{21\sqrt{3}} & 0 \end{bmatrix}
 \end{aligned}$$

Remaining twelve matrices are the generalization of the following six matrices $X_1, X_2, X_3, X_4, X_5, X_6$:

$$\begin{aligned}
 X_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{9\sqrt{3}}{64\sqrt{10}} \\ 0 & \frac{9\sqrt{3}}{64\sqrt{10}} & \frac{(-1)^{l+1}27}{16\sqrt{2}} \end{bmatrix}, & X_2 &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{(-1)^{l+1}3\sqrt{3}}{16\sqrt{2}} & \frac{27}{64\sqrt{2}} & \frac{(-1)^{l+1}3\sqrt{3}}{8\sqrt{10}} \\ \frac{24}{16\sqrt{10}} & \frac{(-1)^{l+1}63\sqrt{3}}{64\sqrt{10}} & \frac{27}{64\sqrt{2}} \end{bmatrix} \\
 X_3 &= \begin{bmatrix} \frac{(-1)^{l+1}3}{4\sqrt{2}} & \frac{3\sqrt{3}}{16\sqrt{2}} & \frac{(-1)^l3}{16\sqrt{10}} \\ \frac{3\sqrt{3}}{16\sqrt{2}} & \frac{(-1)^l27}{64\sqrt{2}} & \frac{-57\sqrt{3}}{64\sqrt{2}} \\ \frac{(-1)^l3}{16\sqrt{10}} & \frac{-57\sqrt{3}}{64\sqrt{2}} & \frac{(-1)^l3}{4\sqrt{2}} \end{bmatrix}, & X_4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3\sqrt{3}}{112\sqrt{2}} \end{bmatrix} \\
 X_5 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{(-1)^{l+1}\sqrt{15}}{28\sqrt{2}} & \frac{9\sqrt{5}}{112\sqrt{2}} & \frac{(-1)^{l+1}\sqrt{3}}{14\sqrt{2}} \end{bmatrix}, & X_6 &= \begin{bmatrix} \frac{5}{7\sqrt{6}} & \frac{(-1)^{l+1}}{28\sqrt{2}} & \frac{\sqrt{10}}{7\sqrt{3}} \\ \frac{(-1)^{l+1}}{28\sqrt{2}} & \frac{15}{112\sqrt{2}} & \frac{45\sqrt{3}}{14\sqrt{2}} \\ \frac{\sqrt{10}}{7\sqrt{3}} & \frac{-3\sqrt{5}}{14\sqrt{2}} & \frac{2\sqrt{2}}{7\sqrt{3}} \end{bmatrix}
 \end{aligned}$$

such that

$$\begin{aligned}
 X_1 &= \begin{cases} (a'')_{l,0}^0, & l=0 \\ (a'')_{l,0}^1, & l=1 \end{cases}, & X_2 &= \begin{cases} (a'')_{l,0}^0, & l=0 \\ (a'')_{l,0}^1, & l=1 \end{cases}, \\
 X_3 &= \begin{cases} (a'')_{l,0}^0, & l=0 \\ (a'')_{l,0}^1, & l=1 \end{cases}, & X_4 &= \begin{cases} (a'')_{l,0}^2, & l=0 \\ (a'')_{l,0}^1, & l=1 \end{cases}, \\
 X_5 &= \begin{cases} (a'')_{l,0}^2, & l=0 \\ (a'')_{l,0}^1, & l=1 \end{cases}, & X_6 &= \begin{cases} (a'')_{l,0}^2, & l=0 \\ (a'')_{l,0}^1, & l=1 \end{cases}
 \end{aligned}$$

Thus, the entries of C_j and $C_{i,k}^j$ are achieved. Also, it can be seen that C_0 is obtained as identity matrix, $(c')_{1,0}^1, (c'')_{1,1}^1, (c'')_{1,0}^0, (c')_{1,1}^2$ as null matrices and the following as non-zero matrices:

$$\begin{aligned}
 (c')_1 &= \begin{bmatrix} \frac{-\sqrt{3}}{2} & 0 & \frac{17}{48} \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{17}{48} & 0 & \frac{-\sqrt{3}}{2} \end{bmatrix}, & (c'')_1 &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{17}{48} & 0 & 0 \\ 0 & \frac{1}{3\sqrt{5}} & 0 \end{bmatrix} \\
 (c'')_{1,1} &= \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{3\sqrt{5}} \\ 0 & \frac{-\sqrt{3}}{2} & 0 \\ \frac{1}{3\sqrt{5}} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}, & (c')_{1,0}^0 &= \begin{bmatrix} 0 & 0 & \frac{2\sqrt{2}}{21\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{2\sqrt{2}}{21\sqrt{3}} & 0 & 0 \end{bmatrix} \\
 (c'')_{1,0}^0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-31}{84\sqrt{30}} & 0 \end{bmatrix}, & (c'')_{1,0}^1 &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{2\sqrt{3}}{21\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 (c')_{1,0}^2 &= \begin{bmatrix} \frac{2\sqrt{2}}{21\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3\sqrt{6}}{7} \end{bmatrix}, & (c')_{1,1}^0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2\sqrt{2}}{21\sqrt{3}} & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 (c'')_{i,l}^0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-31}{84\sqrt{30}} \\ 0 & 0 & 0 \end{bmatrix}, & (c''')_{i,l}^0 &= \begin{bmatrix} \frac{3\sqrt{6}}{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{52\sqrt{2}}{21\sqrt{3}} \end{bmatrix}, \\
 (c''')_{i,l}^2 &= \begin{bmatrix} 0 & 0 & \frac{52\sqrt{2}}{21\sqrt{3}} \\ 0 & 0 & 0 \\ \frac{52\sqrt{2}}{21\sqrt{3}} & 0 & 0 \end{bmatrix}, & (c'')_{i,o}^2 &= \begin{bmatrix} 0 & \frac{-31}{84\sqrt{30}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [(c'')_{i,l}^2]^T, \\
 (c'')_{i,l}^1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{52\sqrt{2}}{21\sqrt{3}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = (c''')_{i,o}^2, & (c'')_{i,l}^1 &= \begin{bmatrix} 0 & 0 & \frac{-31}{84\sqrt{30}} \\ 0 & 0 & 0 \\ \frac{-31}{84\sqrt{30}} & 0 & 0 \end{bmatrix} = (c''')_{i,o}^1.
 \end{aligned}$$

Similarly, corresponding to the entries of C_j and $C_{i,k}^j$ the twelve matrices are the generalization of the following six matrices $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$:

$$\begin{aligned}
 Z_1 &= (\sqrt{15})^l \begin{bmatrix} \frac{29}{126\sqrt{5}} & 0 & \frac{17\sqrt{5}}{32\sqrt{3}} \left(\frac{-1}{15}\right)^1 \\ 0 & \frac{29}{126\sqrt{5}} & 0 \\ \frac{17\sqrt{5}}{32\sqrt{3}} \left(\frac{-1}{15}\right)^1 & 0 & \frac{-1}{7\sqrt{5}} \end{bmatrix}, \\
 Z_2 &= (\sqrt{15})^l \begin{bmatrix} 0 & \frac{173}{2016} & 0 \\ \frac{17\sqrt{5}}{32\sqrt{3}} \left(\frac{1}{15}\right)^1 & 0 & \frac{173}{2016} \\ 0 & \frac{1}{2\sqrt{3}} \left(\frac{1}{15}\right)^{1+1} & 0 \end{bmatrix}, \\
 Z_3 &= (\sqrt{15})^l \begin{bmatrix} \frac{-1}{7\sqrt{5}} & 0 & \frac{1}{2\sqrt{3}} \left(\frac{-1}{15}\right)^1 \\ 0 & \frac{-13}{126\sqrt{5}} (5)^1 & 0 \\ \frac{1}{2\sqrt{3}} \left(\frac{-1}{15}\right)^1 & 0 & \frac{-13}{126\sqrt{5}} (5)^1 \end{bmatrix}, \\
 Z_4 &= \left(\frac{1}{2\sqrt{5}}\right)^l \begin{bmatrix} \frac{5}{189} \left(\frac{-547}{5}\right)^1 & 0 & \frac{17(5^1)}{24\sqrt{3}} \\ 0 & \frac{5}{189} \left(\frac{-547}{5}\right)^1 & 0 \\ \frac{17(5^1)}{24\sqrt{3}} & 0 & \frac{11(-19)^1}{21(11)} \end{bmatrix}, \\
 Z_5 &= \begin{bmatrix} 0 & \frac{-173\sqrt{5}}{1512} \left(\frac{2}{\sqrt{5}}\right)^1 & 0 \\ \frac{17}{24\sqrt{3}} \left(\frac{-2}{\sqrt{5}}\right)^{-1} & 0 & \frac{-173\sqrt{5}}{1512} \left(\frac{2}{\sqrt{5}}\right)^1 \\ 0 & \frac{17}{24\sqrt{3}} \left(\frac{-2}{\sqrt{5}}\right)^{-1} & 0 \end{bmatrix},
 \end{aligned}$$

$$Z_6 = (\sqrt{5})^l \begin{bmatrix} \frac{-11(-19)^1}{21(110)} & 0 & \frac{2}{3\sqrt{15}} \left(\frac{-1}{2}\right)^1 \\ 0 & \frac{193}{189} \left(\frac{41}{386}\right)^1 & 0 \\ \frac{2}{3\sqrt{15}} \left(\frac{-1}{2}\right)^1 & 0 & \frac{-193}{189} \left(\frac{41}{386}\right)^1 \end{bmatrix},$$

such that

$$\begin{aligned}
 Z_1 &= \begin{cases} (c')_2, l=0 \\ (c'')_{0,o}^1, l=1 \end{cases} & Z_2 &= \begin{cases} (c'')_2, l=0 \\ (c'')_{0,o}^1, l=1 \end{cases} \\
 Z_3 &= \begin{cases} (c''')_2, l=0 \\ (c''')_{0,o}^1, l=1 \end{cases} & Z_4 &= \begin{cases} (c'')_{0,o}^0, l=0 \\ (c'')_{0,o}^2, l=1 \end{cases} \\
 Z_5 &= \begin{cases} (c'')_{0,o}^0, l=0 \\ (c'')_{0,o}^2, l=1 \end{cases} & Z_6 &= \begin{cases} (c''')_{0,o}^0, l=0 \\ (c''')_{0,o}^2, l=1 \end{cases}
 \end{aligned}$$

And in equation (15) the matrix B_0 is the null matrix while the other matrices $B_j, B_{i,k}^j; i, k = 0, 1; j = 0, 1, 2$ are given below:

$$\begin{aligned}
 B_l &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{9\sqrt{3}}{64\sqrt{10}} & \frac{9\sqrt{3}}{64\sqrt{10}} & 0 & 0 & 0 & 0 \\ \frac{-3\sqrt{3}}{16\sqrt{2}} & \frac{3\sqrt{3}}{16\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{27}{64\sqrt{2}} & \frac{27}{64\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{64\sqrt{2}}{-3\sqrt{3}} & \frac{64\sqrt{2}}{3\sqrt{3}} & 0 & 0 & 0 & 0 \\ \frac{8\sqrt{10}}{8\sqrt{10}} & \frac{8\sqrt{10}}{8\sqrt{10}} & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 Y_l &= (\sqrt{15})^l \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{9\sqrt{3}}{64\sqrt{10}} & \frac{9\sqrt{3}}{64\sqrt{10}} & 0 & 0 & 0 & 0 \\ -\frac{27}{64\sqrt{2}} \left(\frac{7}{15}\right)^1 & \frac{27}{64\sqrt{2}} \left(\frac{7}{15}\right)^1 & \frac{3\sqrt{3}}{112\sqrt{2}} & \frac{3\sqrt{3}}{112\sqrt{2}} & 0 & 0 \\ \frac{27}{16\sqrt{10}} \left(\frac{1}{9}\right)^1 & \frac{27}{16\sqrt{10}} \left(\frac{1}{9}\right)^1 & \frac{-\sqrt{15}}{28\sqrt{2}} & \frac{\sqrt{15}}{28\sqrt{2}} & 0 & 0 \\ -\frac{63\sqrt{3}}{64\sqrt{10}} \left(\frac{-1}{7}\right)^1 & \frac{63\sqrt{3}}{64\sqrt{10}} \left(\frac{-1}{7}\right)^1 & \frac{9\sqrt{5}}{112\sqrt{2}} & \frac{9\sqrt{5}}{112\sqrt{2}} & 0 & 0 \\ \frac{27}{64\sqrt{2}} \left(\frac{-19}{45}\right)^1 & \frac{27}{64\sqrt{2}} \left(\frac{-19}{45}\right)^1 & \frac{\sqrt{3}}{14\sqrt{2}} & \frac{\sqrt{3}}{14\sqrt{2}} & 0 & 0 \end{bmatrix}, \\
 Y_2 &= \left(\frac{2}{\sqrt{5}}\right)^l \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3\sqrt{3}}{16\sqrt{2}} & \frac{3\sqrt{3}}{16\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{27}{16\sqrt{10}} \left(\frac{5}{8}\right)^1 & \frac{27}{16\sqrt{10}} \left(\frac{5}{8}\right)^1 & \frac{-\sqrt{15}}{28\sqrt{2}} & \frac{\sqrt{15}}{28\sqrt{2}} & 0 & 0 \\ \frac{-3}{4\sqrt{2}} \left(\frac{-1}{8}\right)^1 & \frac{3}{4\sqrt{2}} \left(\frac{-1}{8}\right)^1 & \frac{5}{7\sqrt{6}} \left(\frac{1}{2}\right)^1 & \frac{5}{7\sqrt{6}} \left(\frac{1}{2}\right)^1 & 0 & 0 \\ \frac{3\sqrt{3}}{16\sqrt{2}} \left(\frac{-19}{8}\right)^1 & \frac{3\sqrt{3}}{16\sqrt{2}} \left(\frac{-19}{8}\right)^1 & \frac{-15}{28\sqrt{2}} \left(\frac{1}{2}\right)^1 & \frac{15}{28\sqrt{2}} \left(\frac{1}{2}\right)^1 & 0 & 0 \\ \frac{3(10)^1}{16\sqrt{10}} & \frac{-3(10)^1}{16\sqrt{10}} & \frac{\sqrt{10}}{7\sqrt{3}} & \frac{\sqrt{10}}{7\sqrt{3}} & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$Y_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{17}{48} & 0 & \frac{(-1)^1\sqrt{3}}{2} & 0 & \frac{1}{3\sqrt{5}} & 0 \\ \frac{(-1)^1 17}{32\sqrt{3}} & 0 & \frac{-1}{7\sqrt{5}} & 0 & \frac{(-1)^1}{2\sqrt{3}} & 0 \\ \frac{17}{24\sqrt{3}} & 0 & \frac{(-1)^{1+1} 11}{21} & 0 & \frac{2}{3\sqrt{15}} & 0 \\ \frac{(-1)^{1+1} 17}{96} & 0 & \frac{-\sqrt{3}}{7} & 0 & \frac{(-1)^{1+1}}{6\sqrt{5}} & 0 \\ \frac{(-1)^1 17\sqrt{5}}{48\sqrt{3}} & 0 & \frac{(-1)^1 19}{42\sqrt{5}} & 0 & \frac{-1}{3\sqrt{5}} & 0 \end{bmatrix}$$

$$Y_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{5}} & 0 \\ 0 & 0 & \frac{1}{3\sqrt{5}} & 0 & \frac{(-1)^1\sqrt{3}}{2} & 0 \\ \frac{173}{2016} & 0 & \frac{(-1)^1}{2\sqrt{3}} & 0 & \frac{-13\sqrt{5}}{126} & 0 \\ \frac{(-1)^1 173\sqrt{5}}{1512} & 0 & \frac{2}{3\sqrt{15}} & 0 & \frac{(-1)^{1+1} 193}{189} & 0 \\ \frac{173\sqrt{5}}{672\sqrt{2}} & 0 & \frac{(-1)^{1+1}}{6\sqrt{5}} & 0 & \frac{-65}{42\sqrt{3}} & 0 \\ \frac{(-1)^1 173}{756} & 0 & \frac{1}{3\sqrt{3}} & 0 & \frac{(-1)^{1+1} 41\sqrt{5}}{378} & 0 \end{bmatrix}$$

$$Y_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{(-1)^1\sqrt{3}}{2} & 0 & \frac{17}{48} & 0 & 0 & 0 \\ \frac{29}{26\sqrt{5}} & 0 & \frac{(-1)^1 17\sqrt{5}}{32\sqrt{3}} & 0 & \frac{173}{2016} & 0 \\ \frac{(-1)^{1+1} 5}{189} & 0 & \frac{17}{24\sqrt{3}} & 0 & \frac{(-1)^1 173\sqrt{5}}{1512} & 0 \\ \frac{29}{42\sqrt{3}} & 0 & \frac{(-1)^{1+1} 17}{96} & 0 & \frac{173\sqrt{5}}{673\sqrt{3}} & 0 \\ \frac{(-1)^1 547}{378\sqrt{5}} & 0 & \frac{-17\sqrt{5}}{48\sqrt{3}} & 0 & \frac{(-1)^1 173}{756} & 0 \end{bmatrix}$$

When $l = 0$ then $Y_3 = B_{1,0}^1$, $Y_4 = B_{1,l}^1$ and $Y_5 = B_{1,0}^0$ and if $l = 1$ then Y_3, Y_4, Y_5 are related to $B_{1,l}^0, B_{1,l}^2, B_{1,0}^1$ respectively with cyclic operation of columns $c_{s+1} = c_s, s = 1, 2, 3, 4, 5$ and

$$Y_l = \begin{cases} B_2, & l = 0 \\ B_{0,0}^1, & l = 1 \end{cases} \quad Y_2 = \begin{cases} B_{0,0}^0, & l = 0 \\ B_{0,0}^2, & l = 1 \end{cases}$$

With the help of equation (11) and the matrices I_j and $I_{i,k}^j$, the equation (34) of [9] takes the form:

$$\Theta(x)\Theta^T(x)C \approx \tilde{C}\Theta(x). \quad (16)$$

6. METHOD OF SOLUTION OF INITIAL VALUE PROBLEMS

In this section is suggested an approximate solution of second order differential equation with initial conditions using QLMW and operational matrix of integration.

Consider the equation

$$a(x)y'' + b(x)y' + c(x)y(x) = f(x) \quad (17)$$

with $y(0) = \alpha$ and $y'(0) = \beta$,

where, a, b, c and f are function of x or constants.

First, approximating a, b, c and f by using equation (10) we get

$$a(x) = A^T \Theta(x), \quad b(x) = B^T \Theta(x), \quad c(x) = C^T \Theta(x),$$

$$f(x) = F^T \Theta(x), \quad y'(0) = E_0^T \Theta(x) \text{ and } y(0) = D_0^T \Theta(x). \quad (18)$$

Now

$$y''(x) = Y^T \Theta(x)$$

and

$$y'(x) = \int_0^x y''(x) dx + y'(0) \approx Y^T P \Theta(x) + E_0^T \Theta(x) \quad (19)$$

$$y(x) = \int_0^x y'(x) dx + y(0)$$

$$\approx Y^T P^2 \Theta(x) + E_0^T P \Theta(x) + D_0^T \Theta(x)$$

By using equations (18) and (19), the equation (17) yields

$$\begin{aligned} \Theta^T(x)F &= A^T \Theta(x)\Theta^T(x)Y + B^T \Theta(x)(\Theta^T(x)P^T Y \\ &\quad + \Theta^T(x)E_0) + C^T \Theta(x)(\Theta^T(x)[P^T]^2 Y \\ &\quad + \Theta^T(x)PE_0^T + \Theta^T(x)D_0) \\ &= A^T \Theta(x)\Theta^T(x)Y + B^T \Theta(x)\Theta^T(x)P^T Y \\ &\quad + B^T \Theta(x)\Theta^T(x)E_0 + C^T \Theta(x)\Theta^T(x)[P^T]^2 Y \\ &\quad + C^T \Theta(x)\Theta^T(x)PE_0^T + C^T \Theta(x)\Theta^T(x)D_0. \end{aligned} \quad (20)$$

Now using the product operation matrix \tilde{C} of equation (16) we obtained

$$\tilde{A}Y + \tilde{B}P^T Y + \tilde{B}E_0 + \tilde{C}P^2 Y + \tilde{C}PE_0^T + \tilde{C}D_0 = F$$

$$\text{or } \tilde{A}Y + \tilde{B}P^T Y + \tilde{C}P^2 Y = F - \tilde{B}E_0 - \tilde{C}PE_0^T - \tilde{C}D_0.$$

After finding Y the approximated solution of differential equation can be obtained.

7. EXAMPLES

The following two examples have been solved by proposed method in this paper.

Example 1.1 Consider

$$xy'' + 2y' - 6x = 0 \quad (21)$$

with $y(0) = 0, y'(0) = 0$.

Putting,

$$y''(x) = Y^T \Theta(x), \quad (22) \text{ where}$$

$$Y = [y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}]^T \quad (23)$$

and

$$\Theta(x) = [\phi_0, \phi_1, \phi_2, \psi_{00}^0, \psi_{00}^1, \psi_{00}^2, \psi_{10}^0, \psi_{10}^1, \psi_{10}^2, \psi_{11}^0, \psi_{11}^1, \psi_{11}^2]^T. \quad (24)$$

Using initial condition, we can get

$$y'(x) = Y^T P \Theta(x), \quad y(x) = Y^T P^2 \Theta(x) \quad (25)$$

and

$$x = E^T \Theta(x), \quad 2 = F^T \Theta(x), \quad 6 = G^T \Theta(x). \quad (26)$$

Now inserting equations (22)-(26) into equation (21) we obtain

$$\Theta^T (\tilde{E} + \tilde{F} P^T) Y = \Theta^T \tilde{G} E.$$

Which is hold for each x in defined interval, therefore we get

$$(\tilde{E} + \tilde{F} P^T) Y = \tilde{G} E,$$

so yields

$$Y = [2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T. \quad (27)$$

By virtue of equation (25) we get exact solution

$$y(x) = x^2. \quad (28)$$

Example 1.2 Consider

$$y''(x) = 1,$$

$$\text{with } y(0) = 0, y'(0) = 0. \quad (29)$$

Taking,

$$y''(x) = Y^T \Theta(x), \quad (30)$$

where

$$Y = [y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}]^T \quad (31)$$

and

$$\Theta(x) = [\phi_0, \phi_1, \phi_2, \psi_{00}^0, \psi_{00}^1, \psi_{00}^2, \psi_{10}^0, \psi_{10}^1, \psi_{10}^2, \psi_{11}^0, \psi_{11}^1, \psi_{11}^2]^T. \quad (32)$$

From initial condition, we have

$$y'(x) = Y^T P \Theta(x), \quad y(x) = Y^T P^2 \Theta(x) \quad (33)$$

and

$$I = F^T \Theta(x). \quad (34)$$

Using equations (28)-(33) into equation (28) we get

$$\Theta^T Y = \Theta^T F$$

which is satisfy for each x in finite interval, therefore we get

$$Y = F$$

and hence

$$Y = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T. \quad (35)$$

Using equation (33) and we get exact solution

$$y(x) = \frac{x^2}{2}. \quad (36)$$

8. CONCLUSION

The aim of the present work is to suggest an efficient method for solving non homogeneous differential equation with IVP by reducing an integral equation into a set of algebraic equations with the help of operational matrix. It is also shown the QLMW provides an exact solution. This work shall facilitate in solving

real-world problems, such as related to the mathematical physics and digital electronics etc.

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