Bounds for the Order of Symmetry Group of Automorphism of Compact Riemann Surface

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ABSTRACT
In this paper the authors have considered the molecule C_6H_{12} (the cyclohexane) and then constructed the group of symmetries of C_6H_{12} which is a group of order 4. Then they proved that the bounds of the order of symmetry group of Automorphisms of compact Riemann surfaces on which the symmetry group of C_6H_{12} acts as a group of Automorphism is 4(g-1) where g ($\geq$ 2) , the genus of the corresponding Riemann surface and the corresponding minimum genu g=2 and associated Fuchsian group has signature Δ(2,2,2,2,2)
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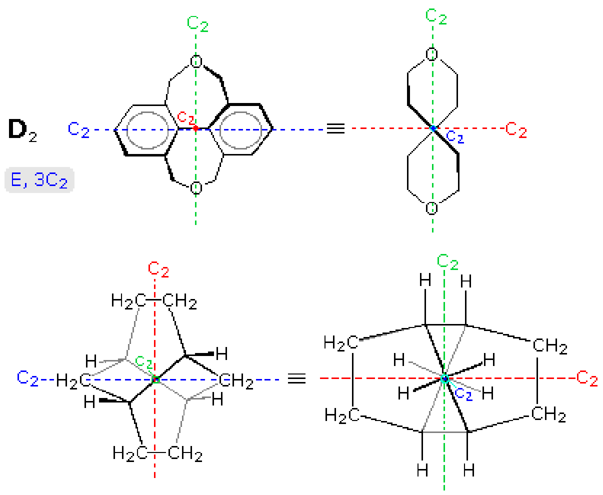
GeneralTerms
(i) Group of Symmetries-It is formed by the symmetry element , point ,line or planes corresponding to the symmetry operations which are rotations ,reflections and inversions on an objects.
(ii) Molecular Symmetry Groups(point groups) – Molecules are assigned to point groups based on the number and orientation of their symmetry elements.
(iii)Diagrammatical Representation of the symmetry elements of the moleculeC_6H_{12} -



(iv)Compact Riemann Surface –A Compact Riemann Surface is a complex manifold which has an analytic structure .
(v) Hurwitz Formula – If g be the genus of a complete curve C ,$|G|$ be the degree of a finite group G which acts on C then

2g-2=$|G|$(R –2) as 2g-2 >0 then R > 2, R = $\sum_{i=1}^{k}\left(1-\frac{1}{r_i}\right)$
(vi) Smooth Homoimorphism –A homomorphism from a Fuchsian group to any group is called a smooth homomorphism if the kernel of the homomorphism is a surface group.

Key words:
point group, symmetry, Fuchsian group, smooth quotient, Riemann surface, Automorphism group, Genus

1. INTRODUCTION:

Every finite group can be realized as a group of transformations of algebraic curve of genus g$\geq$ 2 [3] . It was proved that an algebraic curve could be conceived of as a compact Riemann surface and its birational transformations can be realized as biholomorphic self transformation, usually called automorphisms, of such surface .Schwarz proved that if S is a compact Riemann surface of genus g (the maximum number of non intersecting closed curves in S) , then Aut(S) is (i) infinite for g = 0,1 and (ii) finite for g $\geq$ 2. Hurwitz[10] showed that for a compact Riemann surface S of genus g there exists an upper bound on the order of Aut(S) when it is finite (g$\geq$ 2) and the order of Aut(S) cannot exceed 84 (g – 1) and is attained for g = 3 , and also infinitely many values of g , but is not attained for g = 2 , 4 , 5 , 6 .The study of Automorphism of compact Riemann surfaces was initiated by A.M.Macbeath[11] . He formulated two broad problems on this topic as follows:

(A) Find all large groups of automorphisms including Hurwitz groups. [the groups attained the order 84 (g – 1)].

(B) Determine bounds for different classes of finite groups.

In this paper the authors considered a molecule C_6H_{12} and then construct its group of symmetries. If the group of symmetries is denoted by G(C_6H_{12}) then G(C_6H_{12}) = { E, $C_2(x), C_2(y), C_2(z)$} where E is the identity and C_2 are mutually perpendicular axes such that $C_2 = \frac{2\pi}{2} = \pi$

Clearly $[C_2(x)]^2$ = E , $[C_2(y)]^2$ = E , $[C_2(z)]^2$ = E , also $C_2(x)C_2(y) = C_2(z) = C_2(y)C_2(x)$ i.e every element of G(C_6H_{12}) has order 2 except the identity E .Further it is an

abelian group . For simplicity the authors represent the elements as E= 1, $C_2(x)$ = a, $C_2(y)$= b,
$C_2(z)$ = ab then they have the presentation
G(C_6H_{12}) = $< a, b \mid a^2 = 1 = b^2 ; ab = ba >$ ----(1.1)
Which is nothing but the D_2 group of order 4.

As already mentioned that every finite group can be realized as the group of automorphisms of some compact Riemann surface of genus g(≥ 2) . This inspires the authors to take this molecular group of symmetries G(C_6H_{12}) and to determine the associated Riemann surface on which G(C_6H_{12}) is acting as a group of automorphisms . The concept of Fuchsian group is directly related with the automorphism groups of compact Riemann surface.

A Fuchsian group Γ is an infinite group having a compact orbit space D/ Γ is called co-compact and has presentation of the form
$< y_1, y_2, \ldots, y_k ; \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_\gamma, \beta_\gamma ; \quad y_1^{m_1} = y_2^{m_2} = \cdots = y_k^{m_k} = \prod_{i=1}^{k} y_i \prod_{j=1}^{\gamma} (\alpha_j^{-1}\beta_j^{-1}\alpha_j\beta_j) = 1 >$ (1.2) Where $y_1, y_2, \ldots, y_k$ are of finite order generators and $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_\gamma, \beta_\gamma$ *are* infinite order generators of Γ.
Also the measure of Γ is given by
$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i}) > 0$..(1.3)

Such a Fuchsian group is denoted by Γ=$\Delta(\gamma: m_1, m_2, \ldots\ldots, m_k)$ called signature of Γ . If Γ has no finite order element then Γ= $\Delta(\gamma: -)$ called a surface group. A.M.Macbeath [11] proved that if $\Gamma_1$is a subgroup of Γ of finite index
then Γ_1 is also a Fuchsian group and $[\Gamma : \Gamma_1] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)}$…(1.4)

A homomorphism ϕ from a Fuchsian group Γ to a finite group G is called smooth if the kernel of $\phi$is a surface group of Γ . The factor
group Γ/Ker $\phi \cong \phi(\Gamma)$ is called smooth quotient. [9]

A finite group G is representable as an automorphism group of compact Riemann surface of genus g if and only if there is a smooth epimorphism ϕ from a Fuchsian group Γ to G such that Ker ϕ is a surface group of genus g. [11]

In an another paper of the same authors it is proved that there is a smooth epimorphism $\phi : \Gamma \to$ G(C_6H_{12}) as follows:

Theorem:
There is a smooth epimorphism $\phi: \Gamma \to$ G(C_6H_{12}) where Γ and G(C_6H_{12}) are given by (1.2) and (1.1) resepectively , if and only if
(1)If k = 0 i.e. Γ = $\Delta(\gamma: -)$, a surface group then $\gamma \geq 0$
(2) If $k \neq 0$ then m_i =2 and k ≥ 2 .
Moreover, (i)$\gamma \geq 0$ if k ≥ 5, (ii)$\gamma \geq 1$ if k = 2, 3, 4
Following this result the authors shall prove the main result as given below:

Main Result: Let S be a compact Riemann surface of genus g admitting G(C_6H_{12}) as an automorphism group . Then there is a Fuchsian group Γ such that G(C_6H_{12})≅ Γ/K where K is a surface group of genus g. Now from (1.4) we have

$$\frac{2(g-1)}{o[G(C_6H_{12}]} = 2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right)$$

$$\Rightarrow \frac{2(g-1)}{4} = 2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right)$$

⇒ g = 1 + 4($\gamma - 1$) + k, k being any integer (≥2) Therefore, Minimum value of g is 2 and the corresponding Fuchsian group is Γ = $\Delta(2,2,2,2,2)$, which follows the result of Maclachlan [12]. Again the upper bound of Aut(S) is obtained for minimum value of g .In this case we have

$$\frac{2(g-1)}{o[G(C_6H_{12}]} \geq -2 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$\Rightarrow o[G(C_6H_{12})] \leq \frac{2(g-1)}{\frac{1}{2}} = 4(g-1)$$

Hence the maximum bound of the automorphism group of compact Riemann surface on which the symmetry group of the molecule C_6H_{12} acts as a group of automorphism is 4(g-1) which is attained for g=2 . Now one can summarize the result follows:

Theorem:
An upper bound for the order of the symmetry -group of the cyclohaxane molecule C_6H_{12} acting on a compact Riemann surface of genus g is 4(g-1) where the minimum genus is 2 and the corresponding Fuchsian group has the signature $\Delta(2,2,2,2,2)$. This completes the objective of this paper.

2. CONCLUSION

The discussion of this paper has come to the conclusion that the molecular symmetry group is the most appropriate group to calculate the bounds for the order of Automorphism of Compact Riemann Surface, which is an appealing topic for Algebric Topology as well as the study of molecular symmetry.

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