GENERAL DEGREE OF PERIODIC SPLINE FUNCTIONS

1. Manprit Kaur,
Deptt.of Mathematics, Gyan Ganga College of Technology, Jabalpur

2. Arun Kumar,
Deptt. of Mathematics and Computer Science, R.D.University, Jabalpur

ABSTRACT

In this paper we consider spline function of general degree m which has the same area as the function does in each partition of the sub-intervals. The existence and uniqueness in general of spline functions also been studied and obtained the result.

: In this paper we consider spline function of general degree, i.e.,of degree m, m=2,3,.... We first give definitions and notations. We take $0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ a subdivision of the interval [0,1]. The periodic spline function of degree m was defined by Ahlberg, Nilson and Walsh [1] in the following way:

DEFINITION 1 :

A function φ is said to be periodic spline function of degree m if it satisfies the following conditions:

(a) In each sub-interval $[x_{i-1}, x_i]$, i = 1,2,...,n , the function φ coincides with a polynomial of degree at most m i.e., $\varphi(x) \in \pi_m$;

(b) its derivatives upto m-1 order are continuous, i.e., $\varphi \in C^{(m-1)}[0,1]$;

(c) φ holds the boundary conditions; $\varphi^{(j)}(0) = \varphi^{(j)}(1)$, j = 0,1,...,m-1.

H. ter. Morsche [10] had defined the periodic spline function of degree m which is as follows:

DEFINITION 2

By S(m,n) we denote the set of spline functions ϕ, defined on [0,∞) , that have the following properties :

(a) The restrictions of ϕ to an arbitrary sub-interval $[x_{i-1}, x_i]$, i = 1,2,..., $x_i = ih$, belongs to π_m;

(b) $\phi \in C^{m-1}[0,\infty)$.

The set of periodic spline functions of degree m corresponding to the uniform sub-division of the interval [0,1] into n sub-intervals will be denoted by $S_0(m,n)$.

DEFINITION 3:

Truncated Power Function : The truncated power function is x_+^m. It is defined by

$x_+^m =$

[Sorry. Ignored \begin{cases} ... \end{cases}]

where m is a positive real number.

1. INTRODUCTION

Ahlberg, Nilson and Walsh [1] had obtained the result for arbitrary periodic function *f* with period *l* . This states that there exists a unique periodic spline function of odd degree that interpolates f at the nodes x_i. Further added that it may not hold in general for even degree. Even degree case was considered by Subbotin [11]. He established that the result holds good if the nodes are equally spaced with arbitrary periodic function f, if the interpolation points are taken to be the mid-points of the sub-intervals $[x_{i-1}, x_i]$. Meir and Sharma [8,9] considered the above study for cubic case with a view to have more flexibility in the interpolating points for the class of function $C^2[0,1]$. Specifically, if the interpolation points are denoted by $y_1, y_2, ..., y_n$ and if $y_i = x_i - \lambda h$ with $0 \le \lambda \le \frac{1}{3}$ or $\frac{2}{3} \le \lambda \le 1$, then they established the existence and uniqueness of a periodic cubic spline function interpolating arbitrary element of $C^2[0,1]$ at the points y_i. About there convergence they proved the following theorem [9] :

THEOREM A: Let $f(x)\in C^2[0,1]$ be 1-periodic, and let $\varphi(x)\in C^2[0,1]$ be the 1-periodic cubic spline with joints $x_i=i/n$, satisfying $\varphi\left(\frac{i+\lambda-1}{n}\right)=f\left(\frac{i+\lambda-1}{n}\right)$, $i=1,2,...,n$, where $0\le\lambda\le\frac{1}{3}$ or $\frac{2}{3}\le\lambda\le 1$. Then $max_x|\varphi''(x)-f''(x)|\le 15\omega_2\left(\frac{1}{n}\right)$, where ω_2 is the second modulus of continuity of f''. Instead of considering point interpolation, Dikshit [5] studied certain area-matching of the spline function and the function in each sub-interval of the partition. Essentially he obtained the following [1978]

THEOREM B:

Let $f\in C^2[0,1]$ be a 1-periodic locally integrable function with respect to a non-negative measure $d\mu$ satisfying $\mu(x+h)-\mu(x)=K$ (constant).Suppose further that either $\int_0^h \alpha(x)d\mu>0$ or $\int_0^h \alpha(h-x)d\mu>0$,where $\alpha(x)=3x^3-6hx^2+h^3$. Then there exists a unique $s(x)\in S(3,\Delta)$ satisfying the following conditions:

$$\int_{x_{i-1}}^{x_i}\{f(x)-s(x)\}d\mu=0,\quad i=1,2,...,n,\text{and}$$

$s^{(r)}(0)=s^{(r)}(1)$, $r=0,1,2$. It was also shown that the result of Meir and Sharma [8,9] follows from the above theorem as particular case. Since this result does not cover the interpolation at the mid-points of the mesh so a separate result was proved by Dikshit ([5], Th.2, per Remark 2) which allows to consider only mid-points of a mesh as interpolatory condition. For every odd n, above result continues to hold if we assume the conditions $\int_0^h(-4x^3+6h^2x-h^3)d\mu=0$ and $\int_0^h d\mu>0$ in place of $\int_0^h\alpha(x)d\mu>0$ or $\int_0^h\alpha(h-x)d\mu>0$,where $\alpha(x)=3x^3-6hx^2+h^3$. A similar result for area-matching was studied by Kumar and Govil [7]. They obtained the following

:**THEOREM C**

Let f ∈ L[0,1]. Then there exists a unique spline s(x)∈$S(3,\Delta)$ which bounds the same area as the function does, precisely,

$$\int_{x_{i-1}}^{x_i} f(x)dx=\int_{x_{i-1}}^{x_i} s(x)dx,\quad i=1,2,...,n,\text{if } s''(0)=s''(1)=0$$

and $h(s''(1-h)-s''(h))=24S'(0)$. The study of area-matching spline function was further investigated by Das (Thesis 2004[6], Th.3.1(a), (b)) and obtained the following result :

THEOREM D :

(a) Let $f\in G_p$ (by $f\in G_p$ it is meant that f is integrable with respect to measure dg and also it is periodic with period-1) , then there exists a unique 1-periodic spline function $s\in S(3,\Delta)$ such that, for $0\le\alpha\le 1$,

$$\int_{x_{i-1}}^{x_i}s(x)dg+\alpha\int_{x_i}^{x_{i+1}}s(x)dg=\int_{x_{i-1}}^{x_i}f(x)dg+\alpha\int_{x_i}^{x_{i+1}}f(x)dg,$$

i = 1,2,...,n,if $\int_0^h\mu(x)dg=\int_0^h\mu(h-x)dg=\delta_a>0$, where $\mu(x)=3x^3-6hx^2+h^3$, and n is odd when $\alpha=1$. For $\alpha=1$ the results fails to exist if n is even.(b) Let β, $0\le\beta\le 4$ be zero of $\int_0^h\sigma(\beta,x)dg=0$,

where
$\sigma(\beta,x)=-\beta^3(h-x)^3+\beta^2(3x^3-6hx^2+4h^3)-\beta(-3x^3+3hx^2+3h^2x+h^3)+x^3$.
The above theorem is true, in case $\beta\ne 1$, if $\int_0^h\{-\beta^2(h-x)^3+\beta(4x^3-9hx^2+3h^2x+3h^3)-x^3\}dg=\delta_b>0$, and, in case if $\beta=1$, iff n is odd. The results of Meir and Sharma [9] have been generalized by H.ter Morsche [10] for interpolating periodic splines for general degree. This result is contained in the following theorems :

THEOREM E :

Let f be a periodic function with period l. If the linear system

$$\sum_{r=0}^{m}(m!)^{-1}M_{i-1+r}\sum_{j=0}^{r}(-1)^{j}m+2j(r-j+\lambda)^{m}=h^{1-m}\Delta^{m-1}f(x_i),$$

$i=1,2,...,n$, in the unknowns $M_0,M_1,...,M_{n-1}$, where $M_{n+k}=M_k$ for all k, has a unique solution $M=(M_0,M_1,...,M_{n-1})$, then there exists one and only one function $\phi\in S_0(m,n)$ with $\phi(x_i)=f(x_i)$ $(i=1,2,...,n)$.

THEOREM F

: Let there be given a uniform subdivision $0=x_0<x_1<...<x_{n-1}<x_n=1$ of the interval [0,1] together with a periodic function f with period l. Furthermore, let the interpolation points x_i be defined as, $x_i=y_i-\lambda h$, $0<\lambda<1$. Then there exists a uniquely determined periodic spline function $\phi\in S_0(m,n)$ with the interpolation properties

$$\phi(x_i)=f(x_i),\quad i=1,2,...,n,$$

in each of the following cases:
(i) n is odd.
(ii) m is odd and $\lambda\ne\frac{1}{2}$.
(iii) m is even and $\lambda\ne 0$ and $\lambda\ne 1$.

The proof of the results obtained in Morsche[10] make use of the following interesting results:

THEOREM G :

Let,

$P_m(z,\lambda)= \sum_{r=0}^{m} a^r z^r.$

(a) For all z and λ one has

$$P_m(z,\lambda)-zP_m(z,\lambda+1)=(m!)^{-1}(1-z)^{m+1}\lambda^m.$$

(b) For all λ and $z\neq 0$ one has

$$P_m(z,\lambda)=z^m P_m(z^{-1},1-\lambda).$$

(c)

$\sum_{r=0}^{n-1} a_r=1.$

$a_r=$

[Sorry. Ignored \begin{cases} ... \end{cases}]

(d) For k = 0,1,...,n-1 one has $a_k\geq 0$.(e) With respect to $\lambda, P_m(-1,\lambda)$ is a polynomial of degree m and there holds $P_m(-1,\lambda)=(m!)^{-1}2^m E_m(\lambda)$, where E_m is the so called Euler polynomial of degree m .(f) For all λ one has $\frac{d}{d\lambda}P_m(-1,\lambda)=2P_{m-1}(-1,\lambda)$. (g) For $0<\lambda<1$ the polynomial $P_m(z,\lambda)$ has m negative and distinct zeros. For $\lambda=0$ the polynomial $P_m(z,\lambda)$ has m-1 negative and distinct zeros, while in addition $z=0$ is a zero.(h) For even m the only zeros of $P_m(-1,\lambda)$ on [0,1] are $\lambda=0$ and $\lambda=1$. For odd m the only zero of $P_m(-1,\lambda)$ on [0,1] is $\lambda=\frac{1}{2}$. The main object of this paper is to consider spline of general degree i.e., of degree m which has the same area as the function does in each partition of the sub-intervals. We prove the following:**THEOREM** : Let there be given a uniform subdivision $0=x_0<x_1<...<x_{n-1}<x_n=1$ of the interval [0,1] together with a periodic function f with period l. Then there exists uniquely determined periodic spline function $\phi\in S_0(m,n)$ for odd values of n satisfying

$$\int_{x_{i-1}}^{x_i} \phi(x)dx= \int_{x_{i-1}}^{x_i} f(x)dx, \qquad i=1,2,...,n.$$

We need the following lemma, for proof of the theorem:**LEMMA 1** : Let there be given an infinite sequence of real numbers $M_0, M_1,...$. A function $\phi\in S(m,n)$ has the properties: $\phi^{(m-1)}(x_i)=M_i$ (i = 0,1,2,....) if and only if ϕ can be written in the form

$$\phi(x)=p(x)+(m!)^{-1}\sum_{k=0}^{\infty} M_k \Delta^2 (x-x_{k+1})_+^m,$$

where $p(x)\in\pi_{m-2}$ is an arbitrary polynomial.

The above lemma is contained in (cf [10], p.199).

LEMMA 2 : We have

$$\sum_{j=0}^{m+2}(-1)^{m+2-j}{}_{m+2}j e^{(j-1)t}\left(\frac{d}{dt}\right)^{m+1}\left(\frac{e^{(m+3-j)t}}{1+e^t}\right)=0$$

The lemma follows from (cf.[10],pg. 205) on taking $m=m+1$.

The coefficient matrix of the equation (3.8) is

$$A=\begin{bmatrix} a_0 & a_1 & \cdot & ..a_{n-1} \\ a_{n-1} & a_0 & a_1 & ..a_{n-2} \\ \cdot & \cdot & \cdot & \cdot\cdot \quad \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \quad \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot \quad \cdot \\ a_1 & a_2 & \cdot & \cdot\cdot \quad a_0 \end{bmatrix}.$$

The solutions of the equations are unique if A is not singular.

Let Q be the circulant matrix, precisely $Q=C(0,1,...,0)$. We can write

(3.9)$A=a_0Q^0+a_1Q^1+...+a_{n-1}Q^{n-1}.$

In order to obtain eigen values of the matrix of A we consider the polynomial $R_m(s)$. We have Morsche ([10] , cf. p.204) ,

$$R_m(s)(m+1)!=\sum_{j=0}^{m+2}\sum_{j=0}^{r}(-1)^{m-j}{}_{m+2}j(j-r-1)^{m+1}s^r.$$

On changing the order of summation, we obtain

(3.10)$R_m(s)(m+1)!=\sum_{j=0}^{m+2}(-1)^{m-j}{}_{m+2}j\sum_{r=j}^{m+2}(j-r-1)^{m+1}.s^r,$

We substitute

$r-j=l$ and $s=-e^t$.

The above equation becomes

$$R_m(s)(m+1)!=-\sum_{j=0}^{m+2}(-1)^{m-j}{}_{m+2}j\sum_{l=0}^{m+2-j}(l+1)^{m+1}.(-e^t)^{l+j}$$

$$=-\sum_{j=0}^{m+2}(-1)^j{}_{m+2}j\sum_{l=0}^{m+2-j}(l+1)^{m+1}.(-e^t)^{l+j}$$

$$=-\sum_{j=0}^{m+2}(-1)^j({}_{m+2}j(-1)^j e^{(j-1)t}\sum_{l=0}^{m+2-j}(l+1)^{m+1}(-1)^l e^{(l+1)t}$$

$$=-\sum_{j=0}^{m+2}{}_{m+2}j e^{(j-1)t}\sum_{l=0}^{m+2-j}(-1)^l(l+1)^{m+1}e^{(l+1)t}$$

$$=-\sum_{j=0}^{m+2}{}_{m+2}j e^{(j-1)t}\left(\frac{d}{dt}\right)^{m+1}\sum_{l=0}^{m+2-j}(-1)^l e^{(l+1)t}$$

Error!

$$=-\sum_{j=0}^{m+2} m+2je^{(j-1)t}\left(\frac{d}{dt}\right)^{m+1}\left(\frac{e^t}{1+e^t}\right),$$

by Lemma 2 .
Therefore

$$R_m(s)=\frac{(1+e^t)^{m+2}}{(m+1)!}e^{-t}\left(\frac{d}{dt}\right)^{m+1}\left(\frac{e^t}{1+e^t}\right).$$

Now we see zeros of $R_m(s)$. It is direct to see that the zeros of $R_m(s)$ are zeros of

$$e^{-t}\left(\frac{d}{dt}\right)^{m+1}\left(\frac{e^t}{1+e^t}\right)=K_{m+1}(t),$$

say. We have

$$K_1(t)=\frac{1}{(1+e^t)^2}$$

and it has no zero. Further,

$$K_2(t)=-\frac{-1+e^t}{(1+e^t)^3},$$

and this is zero only for $t=0$ i.e., $s=-1$. Next

$$K_3(t)=-\frac{-1+4e^t-e^{2t}}{(1+e^t)^4},$$

it has got only two real zeros, namely, $e^t=2+\sqrt{3}$, $2-\sqrt{3}$ i.e., $s=-(2+\sqrt{3}),-(2-\sqrt{3})$.

Now we show by induction that $K_{m+1}(t)$ has got m real zeros. We have

$$K_{l+2}(t)=K'_{l+1}(t)=\frac{d}{dt}K_{l+1}(t).$$

Suppose, $K_{l+1}(t)$ has got l zeros. By Rolle's theorem between any two real zeros there exists one real zero. Hence, derivative of $K_{l+2}(t)$ has got $l-1$ real zeros between the zeros of $K_{l+1}(t)$.

Further , $K_{l+1}(t)$ becomes zero for $t\to\pm\infty$. Hence $K_{l+2}(t)$ has got one zero left to the zero of $K_{l+1}(t)$ and one zero between the right zero of $K_{l+1}(t)$ and 0. Thus $K_{l+2}(t)$ has got $l+1$ negative zeros.

.

Hence $K_{m+1}(t)$ has got m real zeros.

Now we proceed to locate eigen values of A. It is known that the n eigen values of Q are the roots of unity i.e., $\omega_k=e^{\frac{2\pi ik}{n}}$, k = 0,1,...,n-1. From above equation it follows that the eigen values of A correspond to $R(s)$ for $s=\omega_k$, k = 0,1,...,n-1. One of the root ω_k becomes real i.e., only for even n. Thus A has possibly an eigen value zero for even n. For odd n there is no eigen value zero. Hence, A is invertible for odd n. This proves the theorem.

2. REFERENCES:

[1] Ahlberg , J. H. , Nilson , E. N. and Walsh , J. L., *The theory of splines and their applications*, Academic Press, (New York, 1967).

[2] de Boor, C., *Best approximations properties of splines of odd-degree* , J. Math. Mech. 12,(1968),747-749.

[3] Das, V.B., *Spline interpolation by lower degree polynomials using area matching condition*, Ph.D. Thesis, Rani Durgawati Uni. (2004), Jabalpur.

[4] Demko, S., *Interpolation by Quadratic Splines*, J. Approx. Theory, 23(1978), 392-400.

[5] Dikshit, H.P., *On cubic interpolatory Splines*, J. Approx. Theory, 22(1978), 105-107.

[6] Kumar and Das, *Convergence of Cubic Piecewise Function, Journal of Computat1ional and Applied Mathematics* , 194(2006),388-394.

[7] Kumar and Govil, L.K., *On Deficient Cubic Splines Interpolants*, J. Approx. Theory, 68(1992), 175-182.

[8] Meir, A. and Sharma, A., *A Degree of Approximation*, J. Math. Mech., 15(1966), 759-768.

[9] Meir, A. and Sharma, A., *Convergence of a class of interpolatory splines*, J. Approx. Theory, 1(1968), 243-250.

[10] Morsche H.ter., *On the existence and convergence of interpolating periodic spline functions of arbitrary degree*, Deptt. of mathematics, Tech.Uni. Eindhoven, The Netherlands, 197-214.

[11] Subbotin, Yu. N., *Piecewise Polynomial (spline) Interpolation*, Math. Notes, 1(1967), 41-45