$\overrightarrow{P_5}$-Factorization of Complete Bipartite Symmetric Digraphs

U S Rajput
Department of mathematics and astronomy,
Lucknow University, Lucknow

Bal Govind Shukla
Department of Mathematics and Astronomy,
Lucknow University, Lucknow

Abstract

In path factorization, H.Wang [1] gives the necessary and sufficient conditions for the existence of P_k-factorization of a complete bipartite graph for $k,$ an even integer. Further, Beiling Du [2] extended the work of H.Wang, and studied the P_{2k}-factorization of complete bipartite multigraph. For odd value of k the work on factorization was done by a number of researchers. P_3-factorization of complete bipartite graph was studied by K.Ushio [3]. P_5-factorization of complete bipartite graph was studied by J.Wang [4]. In the present paper, we study $\overrightarrow{P_5}$-factorization of complete bipartite symmetric digraphs and show that the necessary and sufficient conditions for the existence of $\overrightarrow{P_5}$-factorization of complete bipartite symmetric digraphs are:

(1)$3m \geq 2n,$

(2) $3n \geq 2m,$

(3)$m + n \equiv 0(mod\ 5)$ and

(4)$5mn/2(m + n)$ is an integer.

Mathematics Subject Classification

68R10, 05C70, 05C38.

Key words

Complete bipartite Graph, Factorization of Graph,
Spanning Graph.

1.INTRODUCTION

A $\overrightarrow{P_5}$-factorization of $K^*_{m,n}$ is sum of arc-disjoint $\overrightarrow{P_5}$-factors, where $K^*_{m,n}$ be the complete bipartite symmetric digraph with two partite sets having m and n vertices. A spanning sub graph $\vec{F}$ of $K^*_{m,n}$ is called a path factor if each component of $\vec{F}$ is a path of order at least two. In particular, a spanning sub graph $\vec{F}$ of $K^*_{m,n}$ is called a $\overrightarrow{P_5}$ −factor if each component of $\vec{F}$ is isomorphic to $\overrightarrow{P_5}$. If $K^*_{m,n}$ is expressed as an arc disjoint sum of $\overrightarrow{P_5}$ −factors, then this sum is called $\overrightarrow{P_5}$ −factorization of $K^*_{m,n}$.

Here, we take path of order 5. A $\overrightarrow{P_5}$ is the directed path on 5 vertices. A $\overrightarrow{P_5}$ −factorization of $K^*_{m,n}$ gives rise to a P_5 -factorization of 2K m, n i.e. 2K m, n has a P_5 - factorization if and only if ;

(1)$3m \geq 2n,$

(2) $3n \geq 2m,$

(3)$m + n \equiv 0(mod\ 5)$ and

(4)$5mn/2(m + n)$ is an integer.

2. Mathematical Analysis:

The necessary and sufficient conditions for the existence of $\overrightarrow{P_5}$ −factorization of complete bipartite symmetric digraph are given below in theorem 1.

Theorem 1: Let m, n be the positive integers then $K^*_{m,n}$ has a $\overrightarrow{P_5}$ −factorization iff:

(1)$3m \geq 2n,$

(2) $3n \geq 2m,$

(3)$m + n \equiv 0(mod\ 5)$ and

(4)$5mn/2(m + n)$ is an integer.

Proof of necessity

Proof: Let r be the number of $\overrightarrow{P_5}$ factor in the factorization, and t be the number of copies of $\overrightarrow{P_5}$ in a factor, which can be computed by using

$$r = \frac{5mn}{2(m+n)} \qquad \ldots (1)$$

and

$$t = \frac{m+n}{5} \qquad \ldots (2)$$

respectively.

Obviously, r and t will be integers. Thus conditions 3 and 4 in theorem 1 are necessary. Let a and b be the number of copies of $\overrightarrow{P_5}$ with its end points in Y and X, respectively in a particular $\overrightarrow{P_5}$ −factor. Then by simple arithmetic we can obtain,

$3a + 2b = m$

and

$2a + 3b = n.$

From this, we can compute a and b, which are as follows:

$$a = \frac{(3m-2n)}{5} \qquad \dots (3)$$

$$b = \frac{(3n-2m)}{5} \qquad \dots (4)$$

Since, by definition a and b are positive integers, therefore equations (1) and (2) imply,

$$\frac{3m-2n}{5} \geq 0$$

and

$$\frac{3n-2m}{5} \geq 0.$$

This implies $3m \geq 2n$ and $3n \geq 2m$, therefore conditions 1 and 2 in theorem 1 are necessary.

This proves the necessity of theorem 1.

Proof of sufficiency:

Further, we need the following number theoretic result (lemma 1) to prove the sufficiency of theorem 1.The proof of

following lemma 1 can be found in any good text related to elementary number theory.

Lemma 1: Let g, p and q be any positive integers. If gcd (p, q) = 1, then

gcd ($p.q$, $p+g.q$) = gcd (p, g).

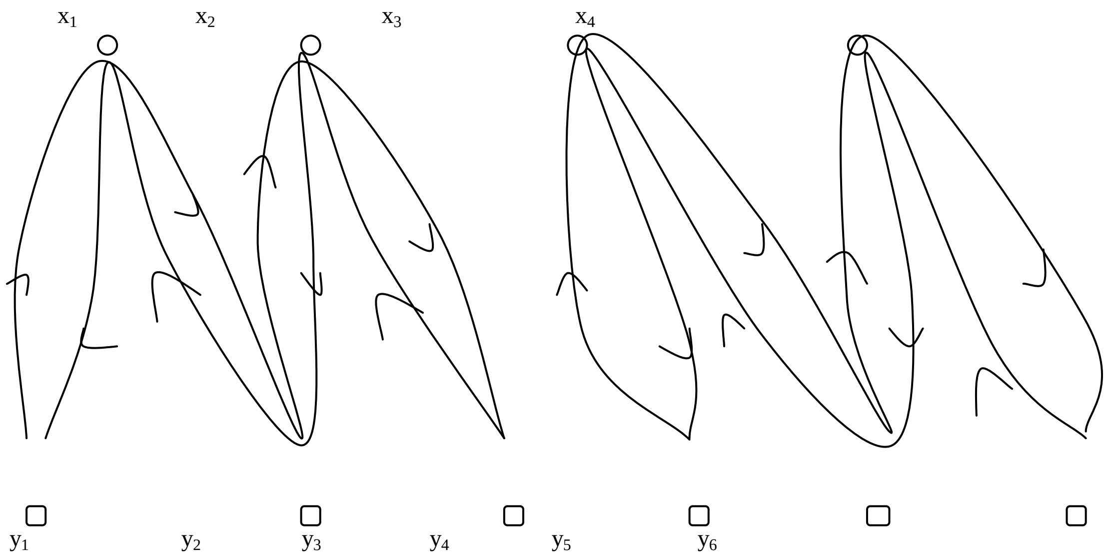
The following lemma will also be used in the proof.

emma 2: If $K^*_{m,n}$ has $\overrightarrow{P_5}$ −factorization then $K^*_{sm,sn}$ has $\overrightarrow{P_5}$ −factorization for every positive integer s.

Proof: Let $K_{s,s}$ is 1-factorable [6], and {H_1, H_2, … H_s} be a 1-factorization of it. For each i with $\{1 \leq i \leq s\}$, replace every edge of H_i with a $K^*_{m,n}$ to get a spanning sub graph G_i of $K^*_{sm,sn}$ such that the $G_i's\{1 \leq i \leq s\}$ are pair wise edge disjoint and there sum is $K^*_{sm,sn}$. Since $K^*_{m,n}$ is $\overrightarrow{P_5}$ −factorable, therefore G_i is also $\overrightarrow{P_5}$ −factorable, and hence, $K^*_{sm,sn}$ is also $\overrightarrow{P_5}$ -factorable.

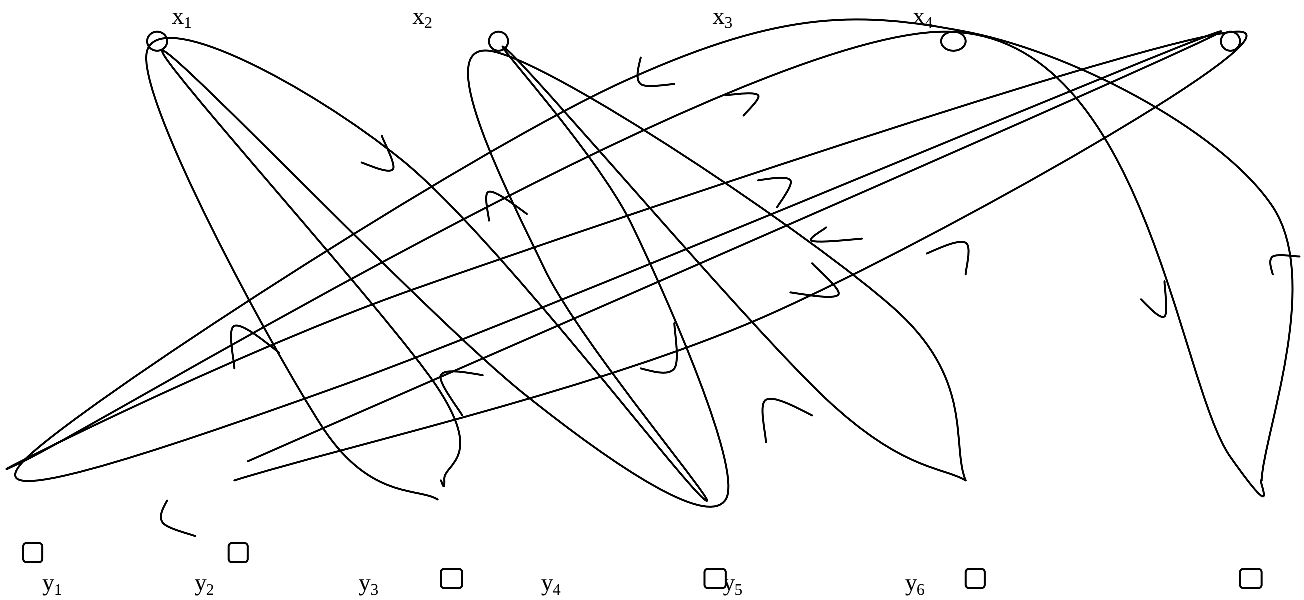
Now to prove the sufficiency of theorem 1, there are three cases to consider

Case I ($3m = 2n$): In this case $K^*_{m,n}$ has a $\overrightarrow{P_5}$-factorization. For instance $K^*_{4,6}$ has $\overrightarrow{P_5}$ −factorization as follows (fig. 1 to fig. 3)



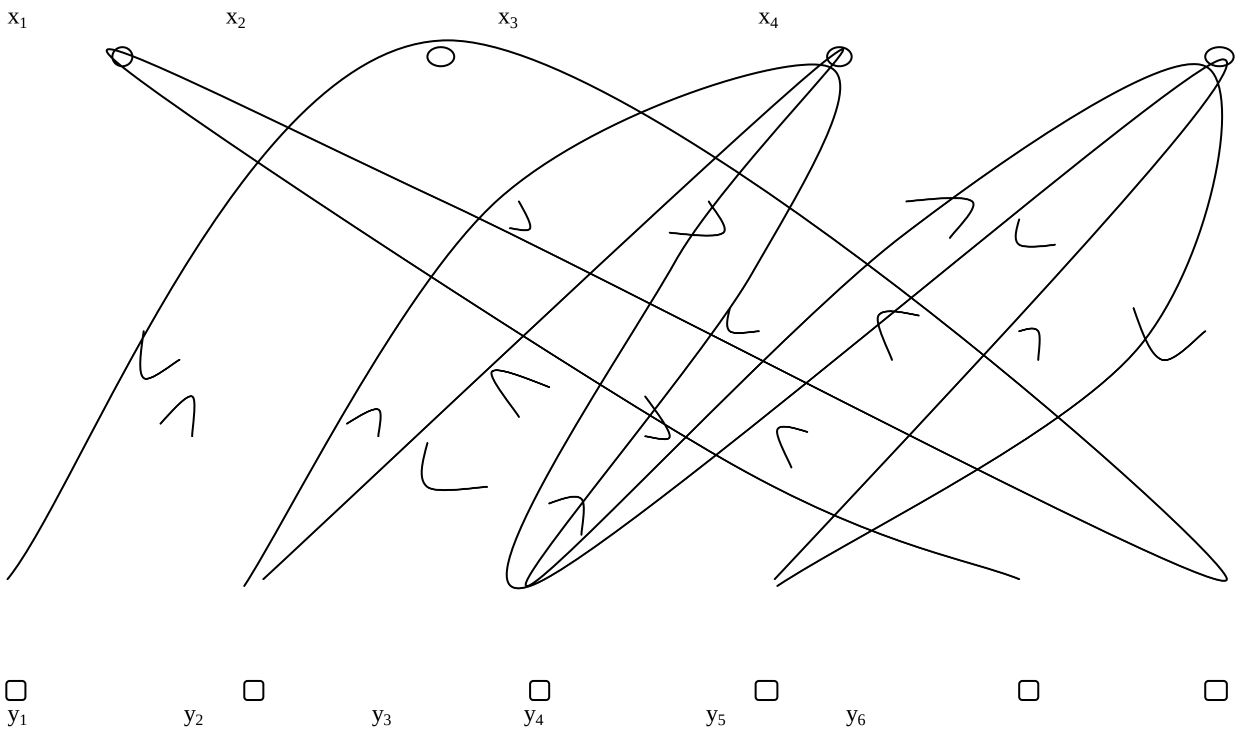
$y_1\, x_1\, y_2\, x_2\, y_3$, $y_4\, x_3\, y_5\, x_4\, y_6$; $y_1\, x_1\, y_2\, x_2\, y_3$, $y_4\, x_3\, y_5\, x_4\, y_6$:

fig (1).



$y_3\ x_1\ y_4\ x_2\ y_5$, $y_6\ x_3\ y_1\ x_4\ y_2$; $y_5\ x_2\ y_4\ x_1\ y_3$, $y_2\ x_4\ y_1\ x_3\ y_6$:

Fig (2)



$y_5\ x_1\ y_6\ x_2\ y_1$, $y_2\ x_3\ y_3\ x_4\ y_4$; $y_1\ x_2\ y_6\ x_1\ y_5$, $y_4\ x_4\ y_3\ x_3 y_2$.

Fig(3).

Case II 3n = 2*m*. Obviously $\square^*_{\square,\square}$ has $\overrightarrow{\square_5}$ −factorization.

Case III ($3\square > 2\square$ and $3\square > 2\square$): Let $\square = \frac{3\square-2\square}{5}$, $\square = \frac{3\square-2\square}{5}$, $\square = \frac{\square+\square}{5}$ and $\square = \frac{5\square\square}{2(\square+\square)}$. Where $\square$, $\square$, $\square$ and $\square$ will be integers and $0 < \square < \square$, $0 < \square < \square$.

As mentioned previously

$$\square = 3\square + 2\square$$

and

$$\square = 2\square + 3\square.$$

Hence

$$\square = 3(\square + \square) + \frac{\square\square}{2(a+\square)}.$$

Since $\square$ is a positive integer, therefore,

$$\frac{\square\square}{2(\square + \square)}$$

must be a positive integer.

Let

$$\square = \frac{\square\square}{2(\square + \square)}$$

here,

$\square$= The number of copies of $\overrightarrow{\square_5}$ in any factor,

$\square$ = The number of $\overrightarrow{\square_5}$-factor in the factorization,

$\square$ = The number of copies of $\overrightarrow{\square_5}$ with its endpoints in Y in a particular $\overrightarrow{\square_5}$- factor (type M),

$\square$ = The number of copies of $\overrightarrow{\square_5}$ with its endpoints in X in a particular $\overrightarrow{\square_5}$- factor (type W),

$\square$ = The total number of copies of $\overrightarrow{\square_5}$ in the whole factorization.

Let $\gcd(2\square, 3\square) = \square$ and therefore $2\square = \square\square$ and $3\square = \square\square$ for some $\square, \square$.

Here $\gcd(\square, \square) = 1$. Consequently

$$\square = \frac{\square\square\square}{2(3\square + 2\square)}$$

These equalities imply the following equalities:

$$\square = \frac{2(3\square + 2\square)\square}{\square\square},$$

$$\square = \frac{2(\square + \square)(3\square + 2\square)\square}{\square\square},$$

$$\square = \frac{(9\square + 4\square)(3\square + 2\square)\square}{3\square\square},$$

$$\square = \frac{(\square + \square)(9\square + 4\square)\square}{\square\square},$$

$$\square = \frac{\square(3\square + 2\square)\square}{\square\square}$$

and

$$\square = \frac{2\square(3\square + 2\square)\square}{3\square\square}.$$

Now we establish the following lemma:

Lemma 3:

Case (1):

If $\gcd(\square, 4) = 1$ and $\gcd(\square, 9) = 1$, the then there exists a positive integer $\square$ such that

$$\square = 6(\square + \square)(3\square + 2\square)\square,$$

$$\square = (9\square + 4\square)(3\square + 2\square)\square,$$

$$\square = 3(\square + \square)(9\square + 4\square)\square,$$

$$\square = 3\square(3\square + 2\square)\square \text{ and}$$

$$\square = 2\square(3\square + 2\square)\square.$$

Case (2):

If $\gcd(\square, 4) = 1$ and $\gcd(\square, 9) = 3$, let $\square = 3\square_1$.

Then there exists a positive integer $\square$ such that

$$\square = 6(\square + 3\square_1)(\square + 2\square_1)\square,$$

$$\square = 3(3\square + 4\square_1)(\square + 2\square_1)\square,$$

$$\square = 3(\square + 3\square_1)(3\square + 4\square_1)\square,$$

$$\square = 3\square(\square + 2\square_1)\square \text{ and}$$

$$\square = 6\square_1(\square + 2\square_1)\square.$$

Case (3):

If $\gcd(\square, 4) = 1$ and $\gcd(\square, 9) = 9$, let $\square = 9\square_2$. Then there exists a positive integer $\square$ such that

$$\square = 2(\square + 9\square_2)(\square + 6\square_2)\square,$$

$$\square = 3(\square + 4\square_2)(\square + 6\square_2)\square,$$

$$\square = 18(\square + 9\square_2)(\square + 6\square_2)\square,$$

$$\square = \square(\square + 6\square_2)\square \text{ and}$$

$$\square = 6\square_2(\square + 6\square_2)\square.$$

Case (4):

If $\gcd(\square, 4) = 2$ and $\gcd(\square, 9) = 1$, let $\square = 2\square_1$. Then there exists a positive integer $\square$ such that

$$\square = 6(2\square_1 + \square)(3\square_1 + \square)\square,$$

$$\square = 2(9\square_1 + 2\square)(3\square_1 + \square)\square,$$

$$\square = 3(2\square_1 + \square)(9\square_1 + 2\square)\square,$$

$$\square = 6\square_1(3\square_1 + \square)\square \text{ and}$$

$$\square = 2\square(3\square_1 + \square)\square.$$

Case (5):

If $\gcd(\square, 4) = 2$ and $\gcd(\square, 9) = 3$, let $\square = 2\square_1, \square = 3\square_1$. Then there exists a positive integer $\square$ such that

$$\square = 6(2\square_1 + 3\square_1)(\square_1 + \square_1)\square,$$

$$\square = 6(3\square_1 + 2\square_1)(\square_1 + \square_1)\square,$$

$$\square = 18(2\square_1 + 3\square_1)(3\square_1 + 2\square_1)\square,$$

$$\square = 6\square_1(\square_1 + \square_1)\square \text{ and}$$

$$\square = 6\square_1(\square_1 + 2\square_1)\square.$$

Case (6):

If $\gcd(\square, 4) = 2$ and $\gcd(\square, 9) = 9$, let $\square = 2\square_1$, $\square = 9\square_2$.

Then there exists a positive integer $\square$ such that

$$\square = 2(2\square_1 + 9\square_2)(\square_1 + 3\square_2)\square,$$

$$\square = 6(\square_1 + 2\square_2)(\square_1 + 3\square_2)\square,$$

$$\square = 3(2\square_1 + 9\square_2)(\square_1 + 2\square_2)\square,$$

$$\square = 2\square_1(\square_1 + 3\square_2)\square\ \square\square\square$$

$$\square = 6\square_2(\square_1 + 3\square_2)\square.$$

Case (7):

If $\gcd(\square, 4) = 4$ and $\gcd(\square, 9) = 1$, let $\square = 4\square_2$. Then there exists a positive integer $\square$ such that

$$\square = 3(4\square_2 + \square)(6\square_2 + \square)\square,$$

$$\square = 2(9\square_2 + \square)(6\square_2 + \square)\square,$$

$$\square = 12(4\square_2 + \square)(9\square_2 + \square)\square,$$

$$\square = 6\square_2(6\square_2 + \square)\square\ \square\square\square$$

$$\square = \square(6\square_2 + \square)\square.$$

Case (8):

If $\gcd(\square, 4) = 4$ and $\gcd(\square, 9) = 3$, let $\square = 4\square_2, \square = 3\square_1$. Then there exists a positive integer $\square$ such that

$$\square = 3(4\square_2 + 3\square_1)(2\square_2 + \square_1)\square,$$

$$\square = 6(3\square_2 + \square_1)(2\square_2 + \square_1)\square,$$

$$\square = 3(4\square_2 + 3\square_1)(3\square_2 + \square_1)\square,$$

$$\square = 6\square_2(2\square_2 + \square_1)\square\ \square\square\square$$

$$\square = 3\square_1(2\square_2 + \square_1)\square.$$

Case (9):

If $\gcd(\square, 4) = 4$ and $\gcd(\square, 9) = 9$, let $\square = 4\square_2, \square = 9\square_2$. Then there exists a positive integer $\square$ such that

$$\square = (4\square_2 + 9\square_2)(2\square_2 + 3\square_2)\square,$$

$$\square = 6(\square_2 + \square_2)(2\square_2 + 3\square_2)\square,$$

$$\square = 3(4\square_2 + 9\square_2)(\square_2 + \square_2)\square,$$

$$\square = 2\square_2(2\square_2 + 3\square_2)\square\ \square\square\square$$

$$\square = 3\square_2(2\square_2 + 3\square_2)\square.$$

Proof of lemma 3:

For proving lemma 3, here we are giving the proof of case (1) only, proofs of other cases are similar.

Let gcd(p, q) =1, gcd (p, 4) =1 and gcd (q, 9) =1,

then

gcd (9p + 4q, 3) = 1 = gcd(3p + 2q, 3)

and if

gcd(9p, 4) = gcd(3p, 2) =1

then gcd (9p + 4q, 2) = 1 .

Hence,

gcd (9p + 4q, $p.q$) = gcd (3p + 2q, $p.q$) =1 (lemma 1).

Since,

$$\square = \frac{(9\square + 4\square)(3\square + 2\square)\square}{3\square\square}$$

is an integer, we observe that $\frac{\square}{3\square\square}$(call it $\square$) will be an integer.

Hence

$$\square = (9\square + 4\square)(3\square + 2\square)\square,$$

Similarly all values of $\square, \square, \square$ and $\square$ are positive integers in case (1) of lemma 3. The proofs of other equalities in lemma 3 are similar to case (1) of lemma 3.

This is proof of lemma 3.

Now in lemma 4 we will establish the value of $\square$ and $\square$ for $\overrightarrow{\square_5}$ −factorization.

We observe that cases (1) and (9), (2) and (8), (3) and (7), (4) and (6) in lemma 3 are symmetrical, Therefore we give the direct construction of only one case (at $\square = 1$)and for remaining it will be obvious.

Lemma 4:

For any positive integer $\square$ and $\square$ let $\square = 6(\square + \square)(3\square + 2\square)$ and

$$\square = (9\square + 4\square)(3\square + 2\square).$$

Then $□^*_{□,□}$ has $\overrightarrow{□_5}$ −factorization.

Proof:-

Let $a = 3p\ (3p + 2q)$ and $b = 2q\ (3p + 2q)$. Hence $t = a + b = 2(3p + 2q)^2$,

and $□ = □_1 . □_2$. Where $□_1 = 3(□ + □)$and $□_2 = (9□ + 4□)$

Let X and Y be two partite sets of $□^*_{□,□}$ such that,

$X = \{x_{ij};\ 1 \le i \le r_1,\ 1 \le j \le m_0\}$,

and

$Y = \{y_{ij};\ 1 \le i \le r_2,\ 1 \le j \le n_0\}$,

where

$m_0 = m/r_1 = 2(3p + 2q)$ and

$n_0 = n/r_2 = (3p+2q)$.

In $\overrightarrow{□_5}$ −factor, there are

$a = 3p(3p + 2q)$

type M, $\overrightarrow{□_5}$ −factor and

$b = 2q(3p + 2q)$

type W, $\overrightarrow{□_5}$ −factor ,

where type M denote the $\overrightarrow{□_5}$ −factor with its end point in Y and type W with its end point in X.

Now for each $1 \le i \le 3p$,

let $1 \le j \le (3p + 2q)$, $0 \le v,\ u \le 1$.

$□_□ = \{x_{i+1,\ j+(3p+2q)u},\ y_{3(i-1)+u+v+1,\ j+(i-1)+u}\}$,

Now for each $1 \le i \le q$,

let $1 \le j \le (3p + 2q)$, $1 \le u \le 2$, $0 \le v,w \le 1$,

$E_{3p+i} = \{x_{3p+3(i-1)+u+v,\ j+(3p+2q)w},$

$y_{9p+4(i-1)+2w+u,\ j+3p+2(i-1)+u+v+w-1}\}$

Let, $\vec{F} = \cup_{1 \le i \le 3p+q} □_□$.

Obviously $\vec{F}$ contains

$t = a + b = 2(3p + 2q)^2 = 2(3p + 2q)\ n_0$ vertex disjoint and edge disjoint $\overrightarrow{□_5}$ −component and span $K^*_{m,n}$ then the digraph $\vec{F}$ is a $\overrightarrow{□_5}$ −factor of $□^*_{□,□}$.

Define a bijection σ from XUYonto XUY,

$$\sigma : X \cup Y \xrightarrow[onto]{} X \cup Y$$

such that $\sigma(x_{i,j}) = x_{i+1,j}$ and $\sigma(y_{i,j}) = y_{i+1,j}$ where $i \in (1,2\ldots r_1)$ and $j \in (1,2\ldots r_2)$.

Construct $□_1 . □_2$ $\overrightarrow{□_5}$ −factor

$\vec{F}_{\xi,\eta}$ $\{ 1 \le \xi \le r_1 ,\ 1 \le \eta \le r_2 \}$

such that

$\vec{F}_{\xi,\eta} = \{\sigma^{\xi}(x)\ \sigma^{\eta}(y) : x \in X,\ y \in Y,\ xy \in F\}$.

It is shown that the digraph,

$\vec{F}_{\xi,\eta}$ $\{ 1 \le \xi \le r_1 ,\ 1 \le \eta \le r_2 \}$,

are line disjoint $\overrightarrow{□_5}$ −factor of $□^*_{□,□}$ and its union is also $□^*_{□,□}$.

Thus $\{\{ \vec{F}_{\xi,\eta}$; $1 \le \xi \le r_1,\ 1 \le \eta \le r_2\}$ is a $\overrightarrow{□_5}$ −factorization of $□^*_{□,□}$. This proves the lemma 4.

Proof of Theorem 1:

By applying lemma 2 with lemma 3 to 4, it can be seen that when the parameters m and n satisfy conditions (1) – (4) in theorem 1, the graph $□^*_{□,□}$ has a $\overrightarrow{□_5}$ −factorization. This completes the proof of Theorem 1.

3. CONCLUSION:

Here in this paper, we have obtained the necessary and sufficient conditions for path factorization of complete bipartite symmetric digraph with 5 number of vertices having symmetric disjoint edges.

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