*International Journal of Computer Applications (0975 - 8887)*
*Volume 73 - No. 17, July 2013*

# Existence Results for Fractional Order Mixed Type Functional Integro-differential Equations with Impulses

S. Dhanalakshmi
Department of Mathematics
Kongunadu Arts and Science college
Coimbatore-641 029
Tamil Nadu, India.

R.Murugesu
Department of Mathematics
SRMV College of Arts and Science
Coimbatore-641 020,
Tamil Nadu, India.

## ABSTRACT

In this paper, we prove the existence of mild solutions for the semilinear fractional order functional of Volterra-Fredholm type differential equations with impulses in a Banach space. The results are obtained by using the theory of fractional calculus, the analytic semigroup theory of linear operators and the fixed point techniques.
■ifx

**Keywords:** Impulsive conditions; Fractional differential equations; Fixed point theorems.
**2010 MSC Subject Classification:** 34K37, 34A08, 47H10

## 1. INTRODUCTION

Impulsive differential equations have become more important in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics. Integro differential equations play an important role in many branches of linear and non-linear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, biology, economics, electrostatics. For some general and recent works on the theory of impulsive differential and integrodifferential equations, we refer the reader to (see [1, 2, 4, 5, 7, 9, 10, 13, 14, 20, 29, 30]).
Fractional differential equations have many applications in various fields of engineering and science, for example, vibration, viscoelasticity, control and electromagnetic theory. Many recent works are devoted to physical application of fractional calculus and fractional differential equations. The theory of fractional differential equations and their applications has been extensively studied by several authors (see [12, 22–24]) and references therein.
In recent years, there has been a significant development in impulsive fractional differential equations. For more details, we refer the monographs (see [21, 25, 28]) and the papers (see [3, 8, 11, 26, 27, 32, 33]).
In particular ( [17–19, 31]) discussed some existence results for nonlinear fractional differential equations with impulse. Very recently, in ( [16]), the author discussed about the existence of mild solutions for the fractional order semilinear functional differential equations with impulse.
Motivated by above mentioned works ( [16, 31]), the purpose of this paper, we shall consider the existence of mild solutions for the fractional order semilinear functional Volterra-Fredholm type of differential equations with impulses as follows,

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + f\left(t, x_t, \int_0^t h(t,s,x_s)ds,\right. \\ &\qquad \left.\int_0^T k(t,s,x_s)ds\right), \\ &\qquad t \in J = [0,T],\ t \neq t_k, \\ \Delta x|_{t=t_k} &= I_k(x(t_k^-)),\ k = 1, 2, \cdots, m, \\ x(t) &= \phi \in \Lambda, \end{aligned} \tag{1}$$

where $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{T(t), t \geq 0\}$ on a Banach space X, $f : J \times \Lambda \times X \times X \to X, h : D \times \Lambda \to X,\ k : D \times \Lambda \to X$ are appropriate functions and $D := \{(t,s) \in [0,T] \times [0,T] : s \leq t\}$ , where $\Lambda$ is a phase space defined in preliminaries, $0 = t_0 < t_1 <, \cdots, < t_m < t_{m+1} = T, I_k \in C(X,X)$ $(k = 1,\ 2,\ \cdots,\ m)$ are bounded functions. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-),\ x(t_k^+)$ and $x(t_k^-)$ represent the left and right limits of x(t) at $t = t_k$ respectively. We assume that the histories $x_t : [-\tau, 0] \to X,\ x_t(s) = x(t+s),\ s \in [-\tau, 0]$ belong to an abstract phase space $\Lambda$.
In this paper, we use the analytic semigroup theory of linear operators and fixed point method to prove the existence and uniqueness of mild solution. In Section 2, we present some definition and preliminary facts. In Sections 3, we prove the existence of mild solution to the fractional order mixed type functional integrodifferential equations with impulses.

## 2. PRELIMINARIES

Throughout this work, $(X, ||.||)$ is a Banach space. An operator A is said to be sectorial if there are constants $\omega \in R,\ \theta \in [\pi/2, \pi],\ M > 0$ such that the following two conditions are satisfied:

$$\begin{cases} (1)\rho(A) \subset \sum_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega, \\ \qquad\qquad |arg(\lambda - \omega)| < \theta\}, \\ (2)||R(\lambda, A)||_{L(X)} \leq \frac{M}{|\lambda - \omega|},\ \lambda \in \sum_{\theta,\omega}. \end{cases}$$

Consider the following Cauchy problem for the Caputo fractional derivative evolution equation of order $\alpha(m-1 < \alpha < m,\ m > 0$ is an integer):

$$\begin{cases} D^\alpha x(t) &= Ax(t), \\ x(0) &= x,\ x^{(k)}(0) = 0,\ k = 1,\ 2,\ \cdots,\ m-1 \end{cases} \tag{2.1}$$

where A is a sectorial operator. The solution operators $S_\alpha(t)$ of (2.1) is defined by (see [17])

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda,$$

where $\Gamma$ is a suitable path lying on $\sum_{\theta,\omega}$.

An operator A is said to belong to $C^\alpha(X; M, \omega)$, if problem (2.1) has a solution operator $S_\alpha(t)$ satisfying $||S_\alpha(t)|| \leq Me^{\omega t},\ t \geq\ 0$. Denote $C^\alpha(\omega) := \{C^\alpha(X; M, \omega) : M \geq 1\}$, and $C^\alpha := \{C^\alpha(\omega) : \omega \geq 0\}$.

DEFINITION 1. *(see [31]). A solution operator $S_\alpha(t)$ of (2.1) is called analytic if $S_\alpha(t)$ admits an analytic extension to a sector $\sum_{\theta_0} := \{\lambda \in C\backslash\{0\} : |arg\lambda| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type $(\theta_0, \omega_0)$ if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that $||S_\alpha(t)|| \leq Me^{\omega Ret}$, $\sum_\theta := \{t \in C\backslash\{0\} : |argt| < \theta\}$. Denote $A^\alpha(\theta_0, \omega_0) := \{A \in C^\alpha : A\ generates\ analytic\ solution\ operators\ S_\alpha(t)\ of\ type\ (\theta_0, \omega_0)\}$.*

LEMMA 2. *(see [31]). Let $\alpha \in (0, 2)$, a linear closed densely defined operator A belong to $A^\alpha(\theta_0, \omega_0)$ iff $\lambda^\alpha \in \rho(A)$ for each $\lambda \in \sum_{\theta_0+\frac{\pi}{2}}$, and for any $\theta < \theta_0,\ \omega > \omega_0$, there is a constant $C = C(\theta, \omega)$ such that*

$$\left\|\lambda^{\alpha-1}R(\lambda^\alpha, A)\right\| \leq \frac{C}{|\lambda - \omega|},\ \lambda \in \sum_{\theta+\frac{\pi}{2}}(\omega).$$

*For any $\tau > 0$, we have*

$$\Lambda = \{\Phi : [-\tau, 0] \to\ X such\ that\ \Phi(t)\ is\ bounded\ and\ measurable\}$$

*and equip the space $\Phi$ with the norm*

$$||\Phi||_\Lambda = \sup_{s\in[-\tau,0]}|\Phi(s)|,\ \forall\ \Phi \in \Lambda.$$

*We consider the space*

$$\Lambda_h = \{x : [-\tau, T] \to X\ such\ that\ x_k \in C((t_k, t_{k+1}],\ X)\ and\ there\ exist\ x(t_k^+)\ and\ x(t_k^-)\ with\ x(t_k) = x(t_k^-),\ x_0 = \phi \in \Lambda,\ k = 0, 1, \cdots, m\}$$

*where $x_k$ is the restriction of $x$ to $J_k = (t_k, t_{k+1}],\ k = 0, 1, \cdots, m$. Set $||.||_{\Lambda_h}$ to be a seminorm in $\Lambda_h$ defined by*

$$||x||_{\Lambda_h} = ||\phi||_\Lambda + sup\{|x(s)| : s \in [0, T]\},\ x \in \Lambda_h.$$

DEFINITION 3. *( [6]) Let $f : J \times \Lambda \to X$ be a continuous function, and A is a sectorial operator. A continuous solution x(t) of the integral equation*

$$x(t) = \begin{cases} S_\alpha(t)\phi + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \\ \qquad f(s, x_s)ds,\ \ 0 \leq t \leq T, \\ \phi(t), \quad -\tau \leq t \leq 0, \end{cases}$$

*where*

$$S_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, A)d\lambda,$$
$$T_\alpha(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda^\alpha, A)d\lambda$$

*and $\Gamma$ is suitable path lying on $\sum_{\theta,\omega}$, is said to be a mild solution of the initial value problem*

$$\begin{cases} D^\alpha x(t) &= Ax(t) + f(t, x_t),\ \ t \in J = [0, T], \\ x(t) &= \phi \in \Lambda. \end{cases}$$

LEMMA 4. *(see [31]). If $\alpha \in (0, 1)$ and $A \in A^\alpha(\theta_0, \omega_0)$, then for any $x \in X$ and $t > 0$, we have*

$$||T_\alpha(t)|| \leq Ce^{\omega t}(1 + t^{\alpha-1}),\ \ t > 0,\ \omega > \omega_0.$$

DEFINITION 5. *(see [16]). A function $x \in \Lambda_h$ is a solution of fractional integral equation*

$$x(t) = \begin{cases} S_\alpha(t)\phi + \int_0^t T_\alpha(t-s) \\ \quad f\left(s, x_s, \int_0^s h(s,\theta,x_\theta)d\theta, \int_0^T k(s,\theta,x_\theta)d\theta\right)ds, \\ \qquad\qquad t \in [0, t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] + \int_{t_1}^t T_\alpha(t-s) \\ \quad f\left(s, x_s, \int_0^s h(s,\theta,x_\theta)d\theta, \int_0^T k(s,\theta,x_\theta)d\theta\right)ds, \\ \qquad\qquad t \in (t_1, t_2]; \\ \vdots \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-))] + \int_{t_m}^t T_\alpha(t-s) \\ \quad f\left(s, x_s, \int_0^s h(s,\theta,x_\theta)d\theta, \int_0^T k(s,\theta,x_\theta)d\theta\right)ds, \\ \qquad\qquad t \in [t_m, T]; \\ \phi(t), \qquad\qquad -\tau \leq t \leq 0 \end{cases}$$

*will be called a mild solution of problem (1.1).*

*In this paper, we will employ an axiomatic definition for the phase space $\Lambda$, $\Lambda$ is a linear space of functions mapping $[-\tau, 0]$ into X endowed with a seminorm $||.||_\Lambda$, which satisfies the following conditions:*

*(A1) If $x : [-\tau, T] \to X$ is a continuous on $[0, T]$ and $x_0 \in \Lambda$, then $x_t \in \Lambda$ and $x_t$ is continuous in $t \in [0, T]$.*

*(A2) $||\phi(0)|| \leq K_1||\phi||_\Lambda$ for $\phi \in \Lambda$ and some constant $K_1$.*

*(A3) There exist a measurable and locally bounded functions K(t) and M(t) of $t \geq 0$ such that*

$$||x_t||_\Lambda \leq K(t)\sup_{\theta\in[0,t]}||x(\theta)|| + M(t)||x_0||_\Lambda,$$

*for $t \in [0, T]$ and x as in (A1).*

*In order to prove the existence of mild solution of IVP (1.1), we need following lemma.*

LEMMA 6. *( [15]) Let X be a Banach space, and $U \subset X$ convex with $0 \in U$, let $F : U \to U$ be a completely continuous operator, then either*
*(a)F has a fixed point, or*
*(b)the set $E = \{x \in U : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded.*

## 3. EXISTENCE RESULTS

In this section, we study the existence of mild solutions for the system (1.1)
To establish our results, we introduce the following conditions;

**(H1)** $f : [0, T] \times \Lambda \times X \times X \to X$ is continuous, and these exists $H_f$ such that

$$||f(t, \varphi, \varphi_1, \varphi_2) - f(t, \Psi, \Psi_1, \Psi_2)|| \leq H_f\left[||\varphi - \Psi||_\Lambda + ||\varphi_1 - \Psi_1|| + ||\varphi_2 - \Psi_2||\right].$$

**(H2)** $K = \sup_{t\in[0,T]} K(t),\ M_1 = \sup_{t\in[0,T]} M(t),\ N_s = \sup_{0<t<T} ||S_\alpha(t)||$ and $N_T = \sup_{0<t<T} Ce^{\omega t}(1 + t^{1-\alpha})$.

**(H3)** The function $I_k : X \to X$ are continuous and there exist constant d such that $||I_k(x)|| \leq d,\ k = 1, 2, \cdots, m$, for each $x \in X$.

(**H4**) There exist $\rho$ such that

$$||I_k(x) - I_k(y)|| \leq \rho||x-y||,\ k = 1,2,\cdots,m,$$

for each $x, y \in X$.

(**H5**) For each $(t,s) \in D$ the function $h(t,s,.) : D \times \Lambda \to X$, is continuous and for each $x \in \Lambda,\ h(t,s,.) : D \times \Lambda \to X$, is strongly measurable. There exists an integrable function $p : J \to [0,\infty)$ and a constant $\gamma > 0$, such that

$$||h(t,s,x)|| \leq \gamma p(s) W(||x||),$$

where $W : [0,\infty) \to [0,\infty)$ is continuous nondecreasing function.

(**H6**) For each $(t,s) \in D$ the function $k(t,s,.) : D \times \Lambda \to X$, is continuous and for each $y \in \Lambda,\ k(t,s,.) : D \times \Lambda \to X$, is strongly measurable. There exists an integrable function $q : J \to [0,\infty)$ and a constant $\gamma_1 > 0$, such that

$$||k(t,s,y)|| \leq \gamma_1 q(s) W_1(||y||),$$

where $W_1 : [0,\infty) \to [0,\infty)$ is continuous nondecreasing function.

(**H7**) The function $f : J \times \Lambda \times X \times X \to X$ satisfies the following Caratheodory conditions:
(a)$t \to f(t,x,y,z)$ is measurable for each $(x,y,z) \in \Lambda \times X \times X$,
(b)$(x,y) \to f(t,x,y,z)$ is continuous for almost all $t \in J$.

(**H8**) $||f(t,x,y,z)|| \leq m(t)\Psi(||x||_\Lambda + ||y|| + ||z||)$ for almost all $t \in J$ and all $x \in \Lambda,\ y,z \in X$, where $m \in L^1(J,R_+)$ *and* $\Psi : R_+ \to (0,\infty)$ is continuous and increasing with

$$C_2\int_0^T \widehat{m}(s)ds \leq \int_{C_1}^\infty \frac{ds}{\Psi(s)+W(s)+W_1(s)},$$

where

$$C_1 = \frac{N_s(||x_0||+d)}{1-N_S},\ C_2 = \frac{N_T T^\alpha}{\alpha(1-N_s)},$$
$$\widehat{m}(t) = max\{C_2 m(t), \gamma p(t), \gamma_1 q(t)\}.$$

(**H9**) $h : D \times \Lambda_h \to X$, and there exist a constant $H_h > 0$, such that

$$\left\|\int_0^t (h(t,s,x_s) - h(t,s,y_s))ds\right\| \leq H_h||x_s - y_s||_\Lambda.$$

(**H10**) $k : D \times \Lambda_h \to X$, and there exist a constant $H_k > 0$, such that

$$\left\|\int_0^t (k(t,s,x_s) - k(t,s,y_s))ds\right\| \leq H_k||x_s - y_s||_\Lambda.$$

**Remark 1.**
For each $l > 0$, we define $\Lambda_l = \{x \in \Lambda_h,\ ||x|| \leq l\}$, then for each $x, y \in \Lambda_h$, we have

$$\begin{aligned}&\left\|f\left(t,x_s,\int_0^t h(t,s,x_s)ds,\int_0^T k(t,s,x_s)ds\right)\right.\\ &\left.-f\left(t,y_s,\int_0^t h(t,s,y_s)ds,\int_0^T k(t,s,y_s)ds\right)\right\|\\ &\leq H_f\left[||x_s-y_s||_\Lambda + H_h||x_s-y_s||_\Lambda + H_k||x_s-y_s||_\Lambda\right]\\ &\leq H_f[1+H_h+H_k]||x_s-y_s||_\Lambda.\end{aligned}$$

We have the following theorem regarding the existence and uniqueness of mild solution for the IVP (1.1)

THEOREM 7. *Assume conditions (A), (H1), (H2), (H4), (H9) and (H10) are satisfied, then the problem (1.1) has a unique mild solution provided that*

$$N_s(\rho+1) + \frac{1}{\alpha}N_T H_f(1+H_h+H_k)KT^\alpha < 1$$

**Proof:** Transform the problem (1.1) into a fixed point problem. Consider the operator $F : \Lambda_h \to \Lambda_h$ defined by

$$Fx(t) = \begin{cases} S_\alpha(t)\phi + \int_0^t T_\alpha(t-s) \\ \quad f\left(s,x_s,\int_0^s h(s,\theta,x_\theta)d\theta,\right. \\ \quad\quad \left.\int_0^T k(s,\theta,x_\theta)d\theta\right)ds,\ t \in [0,t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ +\int_{t_1}^t T_\alpha(t-s) f\left(s,x_s,\int_0^s h(s,\theta,x_\theta)d\theta,\right. \\ \quad\quad \left.\int_0^T k(s,\theta,x_\theta)d\theta\right)ds,\ t \in (t_1,t_2]; \\ \vdots \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-))] \\ +\int_{t_m}^t T_\alpha(t-s) f\left(s,x_s,\int_0^s h(s,\theta,x_\theta)d\theta,\right. \\ \quad\quad \left.\int_0^T k(s,\theta,x_\theta)d\theta\right)ds,\ t \in [t_m,T]; \\ \phi(t), \quad\quad -\tau \leq t \leq 0. \end{cases}$$

Let $x, y \in \Lambda_h$, then for each $t \in (0,t_1]$, we have

$$\begin{aligned}||Fx(t) - Fy(t)|| \leq\ & N_s||x_s - y_s||_\Lambda \\ & +N_T\int_0^t (t-s)^{\alpha-1} \\ & H_f[1+H_h+H_k] \\ & \quad ||x_s-y_s||_\Lambda ds \\ \leq\ & \left[N_s + \frac{1}{\alpha}N_T H_f\right. \\ & \left.[1+H_h+H_k]KT^\alpha\right] \\ & ||x-y||.\end{aligned}$$

For $t \in (t_1,t_2]$, we have

$$\begin{aligned}||Fx(t) - Fy(t)|| \leq\ & N_s[||x(t_1^-) - y(t_1^-)|| \\ & +\rho||x(t_1^-) - y(t_1^-)||] \\ & +N_T\int_0^t (t-s)^{\alpha-1} \\ & H_f[1+H_h+H_k]||x_s-y_s||_\Lambda ds \\ \leq\ & \left[N_s(\rho+1) + \frac{1}{\alpha}N_T H_f\right. \\ & \left.(1+H_h+H_k)KT^\alpha\right] \\ & ||x-y||.\end{aligned}$$

Similarly, we have

$$\begin{aligned}||Fx(t) - Fy(t)|| \leq\ & \left[N_s(\rho+1) + \frac{1}{\alpha}N_T H_f\right. \\ & \left.(1+H_h+H_k)KT^\alpha\right]||x-y||, \\ & \quad\quad t \in (t_i,t_{i+1})\end{aligned}$$

and

$$\begin{aligned}||Fx(t) - Fy(t)|| \leq\ & \left[N_s(\rho+1) + \frac{1}{\alpha}N_T H_f\right. \\ & \left.(1+H_h+H_k)KT^\alpha\right]||x-y||, \\ & \quad\quad t \in (t_m,T].\end{aligned}$$

Then, for each $t \in [-\tau,T]$, we have

$$\begin{aligned}||Fx(t) - Fy(t)|| \leq\ & \left[N_s(\rho+1) + \frac{1}{\alpha}N_T H_f\right. \\ & \left.(1+H_h+H_k)KT^\alpha\right]||x(t)-y(t)||.\end{aligned}$$

Therefore, $F$ is a contraction operator, hence $F$ has a unique fixed point by the Banach contraction principle. That is problem (1.1) has a unique mild solution.

Next we give an existence result based on nonlinear alternative of Leray-Schauder applied to completely continuous operator.

THEOREM 8. *If $N_s < 1$ and conditions (A), (H2), (H3), (H5)-(H8) are satisfied, then the problem (1.1) has at least one mild solution.*

**Proof:** Transform the problem (1.1) into a fixed point problem. Consider the operator $F : \Lambda_h \to \Lambda_h$ define by

$$Fx(t) = \begin{cases} S_\alpha(t)\phi + \int_0^t T_\alpha(t-s) \\ \quad f\left(s, x_s, \int_0^s h(s,\theta,x_\theta)d\theta,\right. \\ \quad\quad \left.\int_0^T k(s,\theta,x_\theta)d\theta\right) ds, \ \ t \in [0,t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ + \int_{t_1}^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h(s,\theta,x_\theta)d\theta,\right. \\ \quad\quad \left.\int_0^T k(s,\theta,x_\theta)d\theta\right) ds, \ \ t \in (t_1,t_2]; \\ \vdots \\ S_\alpha(t-t_m)[x(t_m^-) + I_m(x(t_m^-))] \\ + \int_{t_m}^t T_\alpha(t-s) f\left(s, x_s, \int_0^s h(s,\theta,x_\theta)d\theta,\right. \\ \quad\quad \left.\int_0^T k(s,\theta,x_\theta)d\theta\right) ds, \ \ t \in [t_m,T]; \\ \phi(t), \quad\quad\quad\quad -\tau \le t \le 0. \end{cases}$$

The proof is given in several steps.
**Step 1** $F$ is continuous.
Let $x_n$ be a sequence such that $x_n \to x$ in $\Lambda_h$, then for $t \in [0,T]$ we have

$$||f(s, x_{n_s}, \cdot, \cdot) - f(s, x_s, \cdot, \cdot)|| \le \epsilon, \quad n \to \infty,$$

become the function f is continuous. Now, for every $t \in [0,t_1]$, we have

$$\begin{aligned} ||F(x_n)(t) - F(x)(t)|| &\le N_T \int_0^t (t-s)^{\alpha-1} \\ &\quad ||f(s, x_{n_s}, \cdot, \cdot) \\ &\quad\quad - f(s, x_s, \cdot, \cdot)|| \\ &\le \frac{\epsilon T^\alpha N_T}{\alpha}. \end{aligned}$$

Similarly, for $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} ||Fx_n(t) - Fx(t)|| &\le N_s[||x_n(t_i^-) - x(t_i^-)|| \\ &\quad + \rho||(x_n(t_i^-)) - (x(t_i^-))||] \\ &\quad + \frac{\epsilon T^\alpha N_T}{\alpha} \to 0, \end{aligned}$$

as $n \to \infty$, and for each $t \in (t_m, T]$, we have

$$\begin{aligned} ||Fx_n(t) - Fx(t)|| &\le N_s[||x_n(t_m^-) - x(t_m^-)|| \\ &\quad + \rho||(x_n(t_m^-)) - (x(t_m^-))||] \\ &\quad + \frac{\epsilon T^\alpha N_T}{\alpha} \to 0, \end{aligned}$$

as $n \to \infty$. Since f and $I_k, \ k = 1, 2, \cdots, m$ are continuous, we have $F$ is continuous.
**Step 2** $F$ maps bounded sets into bounded sets in $\Lambda_h$.
It is enough to show that for any $r > 0$, there exists a positive constant $l$, such that for each $x \in B_r = \{x \in \Lambda_h, \ ||x|| \le r\}$ we have $||F(x)|| \le l$.
Since f is continuous, there exist a constant $M_r$, such that

$$||f(t,u,v,w)|| \le M_r, \quad u \in \Lambda_h, v, w \in X, \ \ t \in [\tau, T].$$

Then, for any $x \in B_r, \ t \in [0,t_1]$, we have

$$||Fx(t)|| \le N_s r + \frac{T^\alpha N_T M_r}{\alpha}.$$

Similarly, for each $t \in (t_i, t_{i+1}], \ i = 1, 2, \cdots, m$, we have

$$||Fx(t)|| \le N_s(r+d) + \frac{T^\alpha N_T M_r}{\alpha}.$$

Therefore, for each $x \in B_r, \ t \in [-\tau, T]$, we have

$$||Fx(t)|| \le N_s(r+d) + \frac{T^\alpha N_T M_r}{\alpha} =: l.$$

**Step 3** $F$ maps bounded sets into equicontinuous sets in $\Lambda_h$.
Let $B_r$ is a bounded set of $\Lambda_h$ as in Step 2. Then, for each $s_1, s_2 \in [0, t_1], \ s_1 < s_2$, we have

$$\begin{aligned} ||F(x)(s_2) - F(x)(s_1)|| &\le M||\phi|| \, ||e^{\omega s_2} - e^{\omega s_1}|| \\ &\quad + N_T M_r \left( \int_0^{s_2} (s_2 - s)^{\alpha-1} ds \right. \\ &\quad \left. - \int_0^{s_1} (s_1 - s)^{\alpha-1} ds \right) \\ &\le M||\phi|| \, |e^{\omega s_2} - e^{\omega s_1}| \\ &\quad + \frac{N_T M_r (s_2^\alpha - s_1^\alpha)}{\alpha}. \end{aligned}$$

Similarly, for each $s_1, s_2 \in [t_i, t_{i+1}], \ s_1 < s_2$, we have

$$\begin{aligned} ||F(x)(s_2) - F(x)(s_1)|| &\le M(r+d)e^{-\omega t_i}|e^{\omega s_2} - e^{\omega s_1}| \\ &\quad + \frac{N_T M_r (s_2^\alpha - s_1^\alpha)}{\alpha} \end{aligned}$$

As $s_2 \to s_1$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $s_1 < s_2 \le 0$ and $s_1 \le 0 \le s_2$ is obvious.
As a consequence of steps 1-3, together with Arzela-Ascoli theorem, we can conclude that $F : \Lambda_h \to \Lambda_h$ is continuous and completely continuous.
**Step 4** A Priori bounds.
We now show there exists an open set $U \subset \Lambda_h$ with $x \neq \lambda F(x)$ for $\lambda \in (0,1)$ and $x \in \partial U$.
Let $x_\lambda \in U$ and $x_\lambda(t) = \lambda F(x_\lambda)(t)$ for $0 < \lambda < 1$, we have

$$||x_\lambda(t)|| = \begin{cases} \lambda[N_s||x_0|| + N_T \int_0^t (t-s)^{\alpha-1} m(s) \\ \quad \Psi\left(||x_\lambda(t)|| + \int_0^s \gamma p(\tau) W(x_\tau) d\tau\right. \\ \quad \left. + \int_0^T \gamma_1 q(\tau) W_1(x_\tau) d\tau\right) ds], \\ \quad\quad\quad\quad t \in [0, t_1]; \\ \lambda[N_s(||x_\lambda(t)|| + d) \\ \quad + N_T \int_{t_1}^t (t-s)^{\alpha-1} m(s) \\ \quad \Psi\left(||x_\lambda(t)|| + \int_0^s \gamma p(\tau) W(x_\tau) d\tau\right. \\ \quad \left. + \int_0^T \gamma_1 q(\tau) W_1(x_\tau) d\tau\right) ds], \\ \quad\quad\quad\quad t \in (t_1, t_2]; \\ \vdots \\ \lambda[N_s(||x_\lambda(t)|| + d) \\ \quad + N_T \int_{t_m}^t (t-s)^{\alpha-1} m(s) \\ \quad \Psi\left(||x_\lambda(t)|| + \int_0^s \gamma p(\tau) W(x_\tau) d\tau\right. \\ \quad \left. + \int_0^T \gamma_1 q(\tau) W_1(x_\tau) d\tau\right) ds], \\ \quad\quad\quad\quad t \in (t_m, T]; \\ \phi(t), \quad\quad\quad -\tau \le t \le 0. \end{cases}$$

By the young inequality, for $t \in (t_i, t_{i+1}],\ i = 1, 2, \cdots, m$, we get that

$$\begin{aligned} ||x_\lambda(t)|| \ \le\ & N_s||x_\lambda(t)|| + N_s d \\ & + \frac{T^\alpha N_T}{\alpha} \int_{t_1}^{t} m(s)\Psi\Big(||x_\lambda(t)|| \\ & + \int_0^s \gamma p(\tau) W(x_\tau) d\tau \\ & + \int_0^T \gamma_1 q(\tau) W_1(x_\tau) d\tau \Big) ds \end{aligned}$$

and for all $t \in (0, t_1]$, we have

$$\begin{aligned} ||x_\lambda(t)|| \ \le\ & N_s||x_0|| + \frac{T^\alpha N_T}{\alpha} \int_0^t m(s)\Psi\Big(||x_\lambda(t)|| \\ & + \int_0^s \gamma p(\tau) W(x_\tau) d\tau \\ & + \int_0^T \gamma_1 q(\tau) W_1(x_\tau) d\tau \Big) ds. \end{aligned}$$

Then, for all $t \in [0, T]$, we have

$$\begin{aligned} ||x_\lambda(t)|| \ \le\ & \beta_\lambda(t) = C_1 + C_2 \int_0^t m(s)\Psi\Big(||x_\lambda(t)|| \\ & + \int_0^s \gamma p(\tau) W(x_\tau) d\tau \\ & + \int_0^T \gamma_1 q(\tau) W_1(x_\tau) d\tau \Big) ds, \end{aligned}$$

where $C_1 = \frac{N_s(||x_0||+d)}{1-N_s},\ \ C_2 = \frac{N_T T^\alpha}{\alpha(1-N_s)}$.
Computing $\beta_\lambda^{'}(t)$ for all $t \in [0, T]$, we arrive at

$$\begin{aligned} \beta_\lambda^{'}(t) \ \le\ & C_2 m(t)\Psi\Big(||x_\lambda(t)|| + \int_0^t \gamma p(\tau) W(||x_\lambda(t)||) d\tau \\ & + \int_0^T \gamma_1 q(\tau) W_1(||x_\lambda(t)||) d\tau \Big), \\ \le\ & C_2 m(t)\Psi\Big(\beta_\lambda(t) + \int_0^t \gamma p(\tau) W(||x_\lambda(t)||) d\tau \\ & + \int_0^T \gamma_1 q(\tau) W_1(||x_\lambda(t)||) d\tau \Big). \end{aligned}$$

Let

$$\begin{aligned} \omega(t) \ =\ & \beta_\lambda(t) + \int_0^t \gamma p(\tau) W(||x_\lambda(t)||) d\tau \\ & + \int_0^T \gamma_1 q(\tau) W_1(||x_\lambda(t)||) d\tau \end{aligned}$$

then $\omega(0) = \beta_\lambda(0)\ and\ \beta_\lambda(t) \le \omega(t)$,

$$\begin{aligned} \omega^{'}(t) \ \le\ & \beta_\lambda^{'}(t) + \gamma p(t) W(||x_\lambda(t)||) \\ & + \gamma_1 q(t) W_1(||x_\lambda(t)||) \\ \le\ & C_2 m(t)\Psi(\omega(t)) + \gamma p(t) W(\omega(t)) \\ & + \gamma_1 q(t) W_1(\omega(t)) \\ \le\ & \widehat{m}(t)[\Psi(\omega(t)) + W(\omega(t)) + W_1(\omega(t))]. \end{aligned}$$

This implies that

$$\begin{aligned} \int_{\omega(0)}^{\omega(t)} \frac{ds}{\Psi(s) + W(s) + W_1(s)} & \le \int_0^T \widehat{m}(s) ds \\ & \le \int_0^\infty \frac{ds}{\Psi(s) + W(s) + W_1(s)}, \end{aligned}$$

where we have used the fact: $\beta_\lambda(0) = C_1,\ \beta_\lambda(t)$ is positive and non-decreasing. Hence, by the above inequality, we conclude that the set of functions $\{\beta_\lambda(t) : \lambda \in (0, 1)\}$ is bounded. This implies that $U = \{x \in \Lambda_h : x = \lambda F(x),\ \lambda \in (0, 1)\}$ is bounded in X. Since $F : \Lambda_h \to \Lambda_h$ is continuous and completely continuous. As consequence of the nonlinear alternative of Leray-Schauder type, we deduce that $F$ has a fixed point x in $\Lambda_h$. This completes the proof.

**4. Conclusion**

In this paper, the existence and uniqueness of mild solutions for the semilinear fractional order functional integrodifferential equations with impulses are discussed by using phase space axioms. We applied the concepts of fractional calculus together with fixed point theorems to establish the existence results.

**5. Acknowledgement**

The authors wish to thank the referees for their comments and suggestions.

## 4. REFERENCES

[1] A. Anguraj and M. Mallika Arjunan, Existence and uniqueness of mild and classical solutions of impulsive evolution equations, *Electron. J. Differ. Equ.* 2005(2005), 111, 1-8.

[2] A. Anguraj, M. Mallika Arjunan and E. Hernandez, Existence results for impulsive neutral functional differential equations with state dependent delay, *Appl. Anal.,* 26(7)(2007), 861-872.

[3] K. Balachandran, S. Kiruthika and J.J. Trujillo, Existence results for fractional impulsive integrodifferential equations in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.*, (2010).

[4] K. Balachandran, D.G. Park and S.M. Anthoni, Existence of solutions of abstract nonlinear second order neutral functional integrodifferential equations, *Comput. Math. Appl.* 46(2003), 1313.

[5] K. Balachandran, J.Y. Park and M. Chandrasekaran, Nonlocal Cauchy problem for delay integrodifferential equations of Sobolov type in Banach spaces, *Appl. Math. Lett.* 15(7)(2002), 845-854.

[6] E. Bazhlekova, *Fractional evolution equations in Banach spaces,* Ph.D. Thesis, *Eindhoven University of Technology,* 2001.

[7] M. Benchora, J. Henderson and S.K. Ntouyas, *Impulsive differential equations and inclusions,* Hindawi publishing corporation, Newyork, 2006.

[8] M. Benchora, BA. Slimani, Existence and Uniqueness of solutions to impulsive fractional differential equations, *Electron. J. Differ. Equ.*, 2009(10), 1-11.

[9] Y.K. Chang, A. Anguraj and M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, *Nonlinear Anal. , Hybrid Syst.*, 2(1)(2008), 209-218.

[10] Y.K. Chang, V. Kavitha and M. Mallika Arjunan, Existence results for impulsive neutral differential and integrodifferential equations with nonlocal conditions via fractional operators, *Nonlinear Anal. , Hybrid Syst.*, 4(1)(2010), 32-43.

[11] J. Dabas, A. Chauhan and M. Kumar, Existence of the mild solutions for impulsive fractional equations with infinite delay, *Int.J. Differ. Equations*, 2011(2011).

[12] D. Delbosco, L. Rodino, Existence and uniqueness for a fractional differential equation, *J. Math. Anal. Appl.,* 204(1996), 609-625.

[13] M.B. Dhakne and S.D. Kender, On abstract Nonlinear Mixed Volterra-Fredholm Integrodifferential equations, *Comm. Appl. Nonlinear Anal.*, 13(4), (2006), 101-112.

[14] T. Diagana, G. M. Mophou and G. M. Guerekata, On the existence of mild solutions to some semilinear fractional integrodifferential equations, *Electron. J. Qual. Theory Differ. Equ.,* 58(2010), 1-17.

[15] A. Granas, J. Dugundji, *Fixed Point Theory,* Springer-Verlag, New York, 2003.

[16] Heping Jiang, Existence results for fractional order functional differential equations with impulse, *Comput. Math. Appl.*, 64(2012), 3477-3483.

[17] JinRong Wang, Michal Feckan, Yong Zhou. On the new concept of solutions and existence results for impulsive fractional evolution equations, *Dyn. Partial Differ. Equ.*, 8(2011), 345-361.

[18] JinRong Wang, Yong Zhou, A class of fractional evolution equations and optimal controls, *Nonlinear Anal.*, RWA 12(2011), 262-272.

[19] JinRong Wang, Yong Zhou, Analysis of nonlinear fractional control systems in Banach spaces, *Nonlinear Anal.*, TMA 74(2011), 5929-5942.

[20] V.Kavitha, M. Mallika Arjunan and C. Ravichandran, Existence results for impulsive systems with nonlocal conditions in Banach Spaces, *J. Nonlinear Sci. Appl.* 4(2)(2011), 138-151.

[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, [in:]*North Holland Math. stud.*, vol.204, *Elsevier Science*, 2006.

[22] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal. 69(2008), 3337-3343.

[23] V. Lakshmikantham, A.S. Vastala, Basic theory of fractional differential equations, *Nonlinear Anal.* 69(2008), 2677-2682

[24] V. Lakshmikantham, J. Vasundhara Devi, Theory of fractional differential equations in Banach Spaces, *Eur. J. Pure Appl. Math.* , 1(2008), 38-45.

[25] K.S. Miller, B. Ross, *An introduction to the Fractional Calculus and Differential equations*, John Wiley, New York, 1993.

[26] G.M. Mophou, G.M.N'Gurerekata, Existence of mild solution for some fractional differential equations with nonlocal conditions, *Semigroup Forum*, 79(2009), 315-322.

[27] G.M. Mophou, Existence and Uniqueness of mild solutions of mild solution to impulsive fractional differential equations, *Nonlinear Anal.*, 72(2010), 1604-1615.

[28] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[29] H.L. Tidke and M.B. Dhakne, Existence of solutions for nonlinear mixed type integrodifferential equation of second order, *Surv. Math. Appl.* 5(2010), 61-72.

[30] J. Wang and W. Wei, A class of nonlocal impulsive problems for integrodifferential equations in Banach spaces, *Results Math.*, 58(2010), 379-397.

[31] Xiao-Bao Shu, Yongzeng Lai, Yuming Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal.*, TMA (74)(2011), 2003-2011.

[32] Yong Zhou, Existence and Uniqueness of fractional functional differential equations with unbounded delay, *Int. J. Dyn. Syst. Differ. Equ.,* 1(2008), 239-244.

[33] Yong Zhou, Feng Jiao, Jing Li, Existence and uniqueness for fractional neutral differential equations with infinite delay, *Nonlinear Anal.,* TMA 71(2009), 3249-3256.