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Some Higher Order Triangular Sum Labeling of Graphs

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ABSTRACT

A (p,q) graph G is said to admit n^{th} order triangular sum labeling if its vertices can be labeled by non negative integers such that the induced edge labels obtained by the sum of the labels of end vertices are the first q n^{th} order triangular numbers. A graph G which admits n^{th} order triangular sum labeling is called n^{th} order triangular sum graph. In this paper we prove that paths, combs, stars, subdivision of stars, bistars and coconut trees admit fourth, fifth and sixth order triangular sum labelings.

Keywords:

Fourth, fifth, sixth order triangular numbers, fourth, fifth, sixth order triangular sum labelings.

1. INTRODUCTION

The graphs considered here are finite, connected, undirected and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For various graph theoretic notations and terminology we follow Harary [1] and for number theory we follow Burton [2]. We will give the brief summary of definitions which are useful for the present investigations.

DEFINITION 1. *If the vertices of the graph are assigned values subject to certain conditions it is known as graph labeling.*

A dynamic survey on graph labeling is regularly updated by Gallian [3] and it is published by Electronic Journal of Combinatorics. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades.
Most important labeling problems have three important ingredients.
* A set of numbers from which vertex labels are chosen;
* A rule that assigns a value to each edge;
* A condition that these values must satisfy.
The present work is to aimed to discuss three such labelings known as fourth, fifth and sixth order triangular sum labeling.

DEFINITION 2. *A triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If the n^{th} triangular number is denoted by A_n, then*

$$\begin{aligned} A_n &= 1 + 2 + ... + n \\ &= \frac{1}{2}n(n+1). \end{aligned}$$

The triangular numbers are 1,3,6,10,15,21,28,36,45,55,66,78,...

DEFINITION 3. *A triangular sum labeling of a graph G is a one-to-one function $f : V(G) \rightarrow N$ (where N is the set of all non-negative integers) that induces a bijection $f^+ : E(G) \rightarrow \{A_1, A_2, ..., A_q\}$ of the edges of G defined by $f^+(uv) = f(u) + f(v), \forall e = uv \in E(G)$. The graph which admits such labeling is called a triangular sum graph.*

This concept was introduced by Hegde and Shankaran [4]. Obviously triangular numbers are first order triangular numbers, triangular sum labelings are first order triangular sum labelings and triangular sum graphs are first order triangular sum graphs.

Motivated by the concept of triangular sum labeling, we introduced second and third order triangular sum labeling of graphs [5].

DEFINITION 4. *A second order triangular number is a number obtained by adding all the squares of positive integers less than or equal to a given positive integer n. If the n^{th} second order triangular number is denoted by B_n, then*

$$\begin{aligned} B_n &= 1^2 + 2^2 + ... + n^2 \\ &= \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

The second order triangular numbers are 1,5,14,30,55,91,140,204, 285,385,506,650,...

DEFINITION 5. *A second order triangular sum labeling of a graph G is a one-to-one function $f : V(G) \rightarrow N$ that induces* *a bijection $f^+ : E(G) \to \{B_1, B_2, ..., B_q\}$ of the edges of G defined by $f^+(uv) = f(u)+f(v), \forall e = uv \in E(G)$. The graph which admits such labeling is called a second order triangular sum graph.*

DEFINITION 6. *A third order triangular number is a number obtained by adding all the cubes of positive integers less than or equal to a given positive integer n. If the n^{th} third order triangular number is denoted by C_n, then*

$$C_n = 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2.$$

The third order triangular numbers are 1,9,36,100,225,441,784,1296,2025,3025,4356,6084,...

DEFINITION 7. *A third order triangular sum labeling of a graph G is a one-to-one function $f : V(G) \to N$ that induces a bijection $f^+ : E(G) \to \{C_1, C_2, ..., C_q\}$ of the edges of G defined by $f^+(uv) = f(u)+f(v), \forall e = uv \in E(G)$. The graph which admits such labeling is called a third order triangular sum graph.*

In [5], we proved that paths, stars, coconut trees, bistars and $B_{m,n,k}$ admit second and third order triangular sum labelings. In this paper, we extend the labelings to fourth, fifth and sixth order triangular sum labelings.

DEFINITION 8. *A fourth order triangular number is a number obtained by adding all the fourth powers of positive integers less than or equal to a given positive integer n. If the n^{th} fourth order triangular number is denoted by D_n, then*

$$D_n = 1^4 + 2^4 + ... + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

The fourth order triangular numbers are 1,17,98,354,979,2275,4676,8772,15333,25333,39974,60710,...

DEFINITION 9. *A fourth order triangular sum labeling of a graph G is a one-to-one function $f : V(G) \to N$ that induces a bijection $f^+ : E(G) \to \{D_1, D_2, ..., D_q\}$ of the edges of G defined by $f^+(uv) = f(u)+f(v), \forall e = uv \in E(G)$. The graph which admits such labeling is called a fourth order triangular sum graph.*

DEFINITION 10. *A fifth order triangular number is a number obtained by adding all the fifth powers of positive integers less than or equal to a given positive integer n. If the n^{th} fifth order triangular number is denoted by E_n, then*

$$E_n = 1^5 + 2^5 + ... + n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1).$$

The fifth order triangular numbers are 1,33,276,1300,4425,12201,29008, 61776,120825,220825,...

DEFINITION 11. *A fifth order triangular sum labeling of a graph G is a one-to-one function $f : V(G) \to N$ that induces a bijection $f^+ : E(G) \to \{E_1, E_2, ..., E_q\}$ of the edges of G defined by $f^+(uv) = f(u)+f(v), \forall e = uv \in E(G)$. The graph which admits such labeling is called a fifth order triangular sum graph.*

DEFINITION 12. *A sixth order triangular number is a number obtained by adding all the sixth powers of positive integers less than or equal to a given positive integer n. If the n^{th} sixth order triangular number is denoted by F_n, then*

$$F_n = 1^6 + 2^6 + ... + n^6 = \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1).$$

The sixth order triangular numbers are 1,65,794,4890,20515,67171,184820,446964,978405,1978405,...

DEFINITION 13. *A sixth order triangular sum labeling of a graph G is a one-to-one function $f : V(G) \to N$ that induces a bijection $f^+ : E(G) \to \{F_1, F_2, ..., F_q\}$ of the edges of G defined by $f^+(uv) = f(u)+f(v), \forall e = uv \in E(G)$. The graph which admits such labeling is called a sixth order triangular sum graph.*

2. MAIN RESULTS

Here we prove that paths, combs, stars, subdivision of stars, bistars and coconut trees admit fourth, fifth and sixth order triangular sum labelings.

THEOREM 1. *The path P_n admits fourth order triangular sum labeling.*

PROOF. Let $P_n : u_1u_2...u_n$ be the path and let $v_i = u_iu_{i+1}(1 \le$i$\le$n-1) be the edges.
For i=1,2,...,n, define

$$f(u_i) = \frac{1}{30}(i-1)i(i+1)(3i^2-7)$$

We will prove that the induced edge labels obtained by the sum of the labels of end vertices are the first $n-1$ fourth order triangular numbers.
For $1 \le i \le n-1$,

$$\begin{aligned} f(u_i) + f(u_{i+1}) &= \frac{1}{30}(i-1)i(i+1)(3i^2-7) \\ &\quad + \frac{1}{30}i(i+1)(i+2)[3(i+1)^2-7] \\ &= \frac{1}{30}i(i+1)(2i+1)(3i^2+3i-1) \\ &= D_i \\ &= f^+(v_i) \end{aligned}$$

Thus the induced edge labels are the first $n-1$ fourth order triangular numbers.
Hence path P_n admits fourth order triangular sum labeling. □

EXAMPLE 1. *The fourth order triangular sum labeling of P_8 is shown below.*

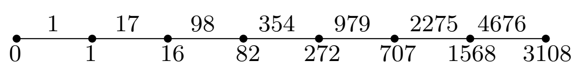

Fig 1: Fourth order triangular sum labeling of P_8

THEOREM 2. *The comb $P_n \odot K_1$ admits fourth order triangular sum labeling.*

PROOF. Let $P_n : u_1u_2...u_n$ be the path and let $v_i = u_iu_{i+1}(1 \le$i$\le$n-1) be the edges. Let $w_1, w_2, ..., w_n$ be the pendant vertices adjacent to $u_1, u_2, ..., u_n$ respectively and $t_i = u_iw_i(1 \le$i$\le$n) be the edges.
For i=1,2,...,n, define

$$f(u_i) = \frac{1}{30}(i-1)i(i+1)(3i^2-7)$$

and

$$\begin{aligned} f(w_i) = \frac{1}{30}[&3i^5 + (30n-15)i^4 + (60n^2-60n+20)i^3 \\ &+ (60n^3-90n^2+30n)i^2 \\ &+ (30n^4-60n^3+30n^2-8)i \\ &+ (6n^5-15n^4+10n^3-n)] \end{aligned}$$

Then $f(u_i) + f(u_{i+1}) = f^+(v_i)$ for $1 \le i \le n-1$

and $f(u_i) + f(w_i) = f^+(t_i)$ for $1 \le i \le n$.

Thus the induced edge labels are the first $2n-1$ fourth order triangular numbers.
Hence comb admits fourth order triangular sum labeling. □

EXAMPLE 2. *The fourth order triangular sum labeling of $P_6 \odot K_1$ is shown below.*

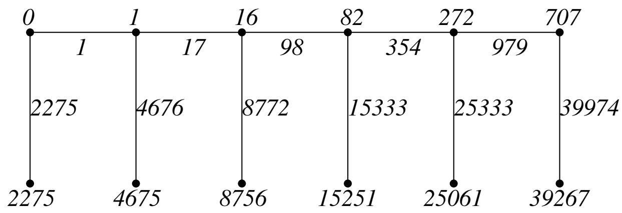

Fig 2: Fourth order triangular sum labeling of $P_6 \odot K_1$

THEOREM 3. *The star graph $K_{1,n}$ admits fourth order triangular sum labeling.*

PROOF. Let v be the apex vertex and let $v_1, v_2, ..., v_n$ be the pendant vertices of the star $K_{1,n}$.
Define

$$\begin{aligned} f(v) &= 0 \\ \text{and} f(v_i) &= \frac{1}{30}i(i+1)(2i+1)(3i^2+3i-1), 1 \le i \le n. \end{aligned}$$

We see that the induced edge labels are the first n fourth order triangular numbers.
Hence $K_{1,n}$ admits fourth order triangular sum labeling. □

EXAMPLE 3. *The fourth order triangular sum labeling of $K_{1,7}$ is shown below.*

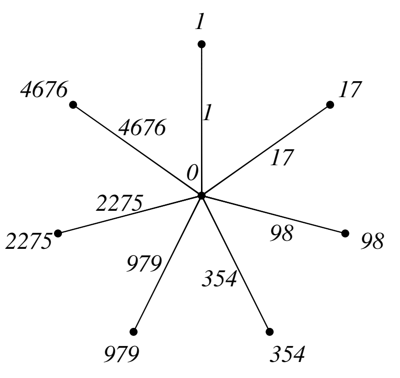

Fig 3: Fourth order triangular sum labeling of $K_{1,7}$

THEOREM 4. *$S(K_{1,n})$, the subdivision of the star $K_{1,n}$, admits fourth order triangular sum labeling.*

PROOF. Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iu_i : 1 \le i \le n\}$.
Define f by

$$\begin{aligned} f(v) &= 0, \\ f(v_i) &= \frac{1}{30}i(i+1)(2i+1)(3i^2+3i-1), 1 \le i \le n \\ \text{and} f(u_i) &= ni^4 + 4(1+2+...+n)i^3 \\ &\quad + 6(1^2+2^2+...+n^2)i^2 \\ &\quad + 4(1^3+2^3+...+n^3)i \\ &\quad + (1^4+2^4+...+n^4), 1 \le i \le n. \end{aligned}$$

We see that the induced edge labels are the first $2n$ fourth order triangular numbers.
Hence $S(K_{1,n})$ admits fourth order triangular sum labeling. □

EXAMPLE 4. *The fourth order triangular sum labeling of $S(K_{1,5})$ is shown below.*

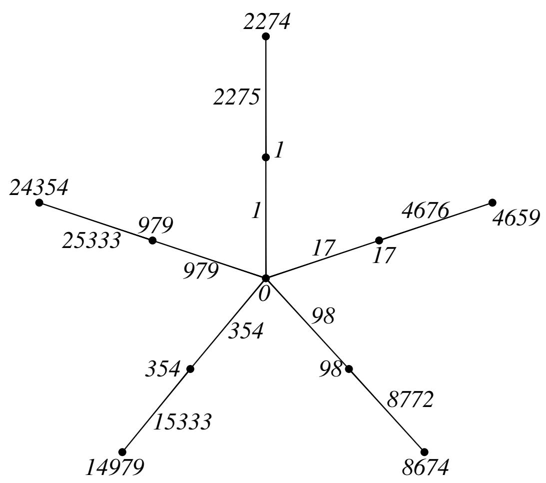

Fig 4: Fourth order triangular sum labeling of $S(K_{1,5})$

THEOREM 5. *The bistar $B_{m,n}$ admits fourth order triangular sum labeling.*

PROOF. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \le \text{i} \le \text{m}, 1 \le \text{j} \le \text{n}\}$ and
$E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \le \text{i} \le \text{m}, 1 \le \text{j} \le \text{n}\}$.
Define f by

$$\begin{aligned} f(u) &= 0, \\ f(v) &= 1, \\ f(u_i) &= \frac{1}{30}(i+1)(i+2)(2i+3)(3i^2+9i+5), 1 \le i \le m \\ \text{and} f(v_j) &= \frac{1}{30}(m+j+1)(m+j+2) \\ &\quad \times(2m+2j+3)[3j^2+(6m+9)j \\ &\quad + (3m^2+9m+5)] - 1, 1 \le j \le n. \end{aligned}$$

We see that the induced edge labels are the first $m+n+1$ fourth order triangular numbers.
Hence $B_{m,n}$ admits fourth order triangular sum labeling. □

EXAMPLE 5. *The fourth order triangular sum labeling of $B_{4,3}$ is shown below.*

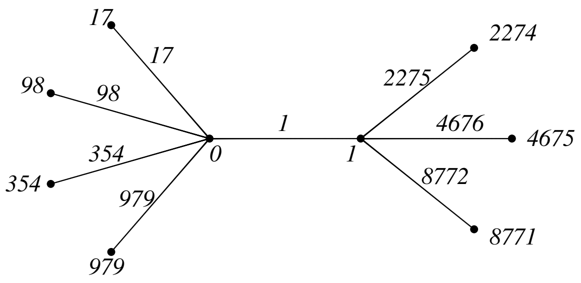

Fig 5: Fourth order triangular sum labeling of $B_{4,3}$

THEOREM 6. *Coconut tree admits fourth order triangular sum labeling.*

PROOF. Let $v_0, v_1, v_2, ..., v_i$ be the vertices of a path, having path length $i(i \geq 1)$ and $v_{i+1}, v_{i+2}, ..., v_n$ be the pendant vertices, being adjacent with v_0.
For $0 \leq j \leq i$, define

$$f(v_j) = \frac{1}{30} j(j+1)(j+2)(3j^2+6j-4)$$

and for $i+1 \leq k \leq n$, define

$$f(v_k) = \frac{1}{30} k(k+1)(2k+1)(3k^2+3k-1)$$

We see that the induced edge labels are the first n fourth order triangular numbers.
Hence coconut tree admits fourth order triangular sum labeling. □

EXAMPLE 6. *The fourth order triangular sum labeling of a coconut tree is shown below.*

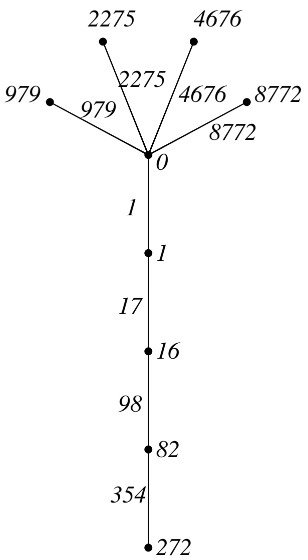

Fig 6: Fourth order triangular sum labeling of a coconut tree

THEOREM 7. *The path P_n admits fifth order triangular sum labeling.*

PROOF. Let $P_n : u_1 u_2 ... u_n$ be the path and let $v_i = u_i u_{i+1} (1 \leq$i$\leq$n-1$)$ be the edges.
For i=1,2,...,n, define

$$f(u_i) = \begin{cases} \frac{1}{12}(i-1)^2(i^4+2i^3-2i^2-6i-3) & \text{if i is odd} \\ \frac{1}{12} i^2(i^4-5i^2+7) & \text{if i is even} \end{cases}$$

Case(i): i is odd
For $1 \leq i \leq n-1$,

$$\begin{aligned} f(u_i) + f(u_{i+1}) &= \frac{1}{12}(i-1)^2(i^4+2i^3-2i^2-6i-3) \\ &\quad + \frac{1}{12}(i+1)^2[(i+1)^4 - 5(i+1)^2 + 7] \\ &= \frac{1}{12} i^2 (i+1)^2 (2i^2+2i-1) \\ &= E_i \\ &= f^+(v_{i)} \end{aligned}$$

Case(ii): i is even
For $1 \leq i \leq n-1$,

$$\begin{aligned} f(u_i) + f(u_{i+1}) &= \frac{1}{12} i^2(i^4-5i^2+7) + \frac{1}{12} i^2[(i+1)^4 \\ &\quad + 2(i+1)^3 - 2(i+1)^2 - 6(i+1) - 3] \\ &= \frac{1}{12} i^2 (i+1)^2 (2i^2+2i-1) \\ &= E_i \\ &= f^+(v_{i)} \end{aligned}$$

Thus the induced edge labels are the first $n-1$ fifth order triangular numbers.
Hence path P_n admits fifth order triangular sum labeling. □

EXAMPLE 7. *The fifth order triangular sum labeling of P_7 is shown below.*

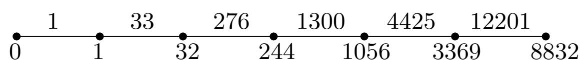

Fig 7: Fifth order triangular sum labeling of P_7

THEOREM 8. *The comb $P_n \odot K_1$ admits fifth order triangular sum labeling.*

PROOF. Let $P_n : u_1 u_2 ... u_n$ be the path and let $v_i = u_i u_{i+1} (1 \leq$i$\leq$n-1$)$ be the edges. Let $w_1, w_2, ..., w_n$ be the pendant vertices adjacent to $u_1, u_2, ..., u_n$ respectively and $t_i = u_i w_i (1 \leq$i$\leq$n$)$ be the edges.
For i=1,2,...,n, define

$$f(u_i) = \begin{cases} \frac{1}{12}(i-1)^2(i^4+2i^3-2i^2-6i-3) & \text{if i is odd} \\ \frac{1}{12} i^2(i^4-5i^2+7) & \text{if i is even} \end{cases}$$

and

$$f(w_i) = \begin{cases} \frac{1}{12}[i^6 + (12n-6)i^5 + (30n^2 - 30n + 10)i^4 \\ +(40n^3 - 60n^2 + 20n)i^3 \\ +(30n^4 - 60n^3 + 30n^2 - 8)i^2 \\ +(12n^5 - 30n^4 + 20n^3 - 2n)i \\ +(2n^6 - 6n^5 + 5n^4 - n^2 + 3)] & \text{if i is odd} \\ \frac{1}{12}[i^6 + (12n-6)i^5 + (30n^2 - 30n + 10)i^4 \\ +(40n^3 - 60n^2 + 20n)i^3 \\ +(30n^4 - 60n^3 + 30n^2 - 8)i^2 \\ +(12n^5 - 30n^4 + 20n^3 - 2n)i \\ +(2n^6 - 6n^5 + 5n^4 - n^2)] & \text{if i is even.} \end{cases}$$

Then $f(u_i) + f(u_{i+1}) = f^+(v_i)$ for $1 \leq i \leq n-1$

and $f(u_i) + f(w_i) = f^+(t_i)$ for $1 \leq i \leq n$.

Thus the induced edge labels are the first $2n-1$ fifth order triangular numbers.
Hence comb admits fifth order triangular sum labeling. □

EXAMPLE 8. *The pentatopic sum labeling of $P_5 \odot K_1$ is shown below.*

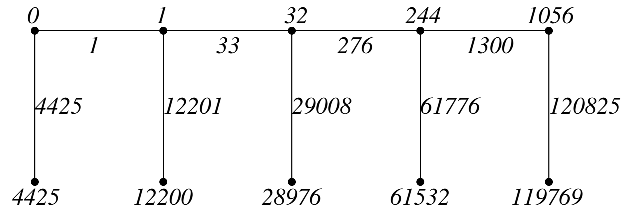

Fig 8: Fifth order triangular sum labeling of $P_5 \odot K_1$

THEOREM 9. *The star graph $K_{1,n}$ admits fifth order triangular sum labeling.*

PROOF. Let v be the apex vertex and let $v_1, v_2, ..., v_n$ be the pendant vertices of the star $K_{1,n}$.
Define

$$f(v) = 0$$
$$\text{and} f(v_i) = \frac{1}{12} i^2(i+1)^2(2i^2+2i-1), 1 \leq i \leq n.$$

We see that the induced edge labels are the first n fifth order triangular numbers.
Hence $K_{1,n}$ admits fifth order triangular sum labeling. □

EXAMPLE 9. *The fifth order triangular sum labeling of $K_{1,6}$ is shown below.*

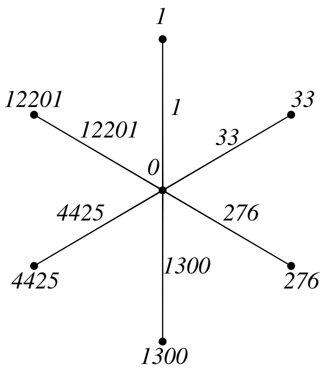

Fig 9: Fifth order triangular sum labeling of $K_{1,6}$

THEOREM 10. *$S(K_{1,n})$, the subdivision of the star $K_{1,n}$, admits fifth order triangular sum labeling.*

PROOF. Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iu_i : 1 \leq i \leq n\}$.
Define f by

$$f(v) = 0,$$
$$f(v_i) = \frac{1}{12} i^2(i+1)^2(2i^2+2i-1), 1 \leq i \leq n$$
$$\text{and} f(u_i) = ni^5 + 5(1+2+...+n)i^4 + 10(1^2+2^2+...+n^2)i^3 + 10(1^3+2^3+...+n^3)i^2 + 5(1^4+2^4+...+n^4)i + (1^5+2^5+...+n^5), 1 \leq i \leq n.$$

We see that the induced edge labels are the first $2n$ fifth order triangular numbers.
Hence $S(K_{1,n})$ admits fifth order triangular sum labeling. □

EXAMPLE 10. *The fifth order triangular sum labeling of $S(K_{1,4})$ is shown below.*

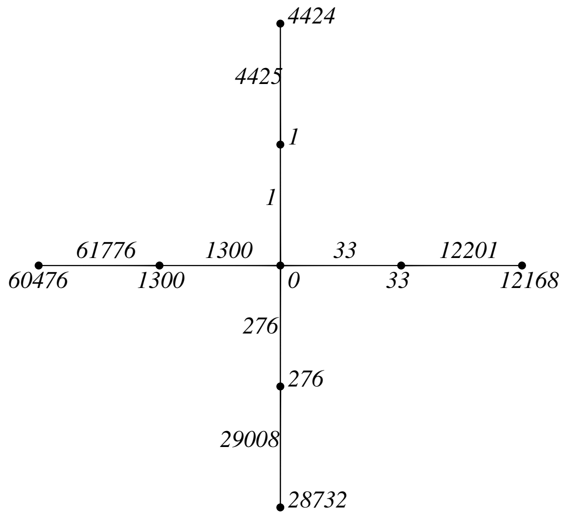

Fig 10: Fifth order triangular sum labeling of $S(K_{1,4})$

THEOREM 11. *The bistar $B_{m,n}$ admits fifth order triangular sum labeling.*

PROOF. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and
$E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.
Define f by

$$f(u) = 0,$$
$$f(v) = 1,$$
$$f(u_i) = \frac{1}{12}(i+1)^2(i+2)^2(2i^2+6i+3), 1 \leq i \leq m$$
$$\text{and} f(v_j) = \frac{1}{12}(m+j+1)^2(m+j+2)^2[2j^2+(4m+6)j + (2m^2+6m+3)] - 1, 1 \leq j \leq n.$$

We see that the induced edge labels are the first $m+n+1$ fifth order triangular numbers.
Hence $B_{m,n}$ admits fifth order triangular sum labeling. □

EXAMPLE 11. *The fifth order triangular sum labeling of $B_{3,3}$ is shown below.*

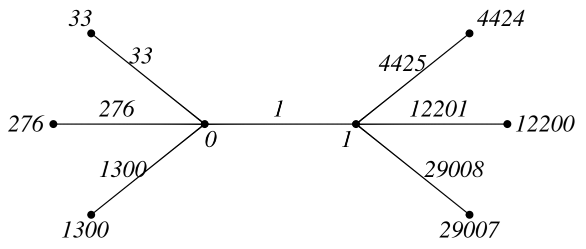

Fig 11: Fifth order triangular sum labeling of $B_{3,3}$

THEOREM 12. *Coconut tree admits fifth order triangular sum labeling.*

PROOF. Let $v_0, v_1, v_2, ..., v_i$ be the vertices of a path, having path length $i(i \geq 1)$ and $v_{i+1}, v_{i+2}, ..., v_n$ be the pendant vertices, being adjacent with v_0.
For $0 \leq j \leq i$, define

$$f(v_j) = \begin{cases} \frac{1}{12}j^2(j^4 + 6j^3 + 10j^2 - 8) & \text{if j is even} \\ \\ \frac{1}{12}(j+1)^2(j^4 + 4j^3 + j^2 - 6j + 3) & \text{if j is odd} \end{cases}$$

and for $i + 1 \leq k \leq n$, define

$$f(v_k) = \frac{1}{12}k^2(k+1)(2k^3 + 4k^2 + k - 1)$$

We see that the induced edge labels are the first n fifth order triangular numbers.
Hence coconut tree admits fifth order triangular sum labeling. □

EXAMPLE 12. *The fifth order triangular sum labeling of a coconut tree is shown below.*

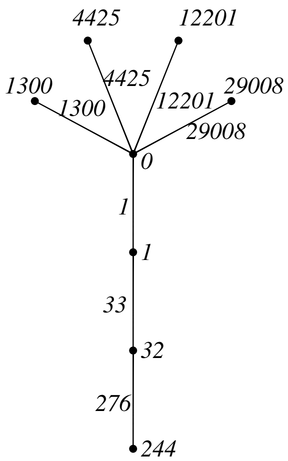

Fig 12: Fifth order triangular sum labeling of a coconut tree

THEOREM 13. *The path P_n admits sixth order triangular sum labeling.*

PROOF. Let $P_n : u_1u_2...u_n$ be the path and let $v_i = u_iu_{i+1}(1 \leq$i$\leq$n-1) be the edges.

For i=1,2,...,n, define

$$f(u_i) = \frac{1}{42}(i-1)i(i+1)(3i^4 - 18i^2 + 31).$$

Then for $1 \leq i \leq n - 1$,

$$\begin{aligned} f(u_i) + f(u_{i+1}) &= \frac{1}{42}(i-1)i(i+1)(3i^4 - 18i^2 + 31) \\ &\quad + \frac{1}{42}i(i+1)(i+2) \\ &\quad \times[3(i+1)^4 - 18(i+1)^2 + 31] \\ &= \frac{1}{42}i(i+1)(2i+1)(3i^4 + 6i^3 - 3i + 1) \\ &= F_i \\ &= f^+(v_{i)} \end{aligned}$$

Thus the induced edge labels are the first $n - 1$ sixth order triangular numbers.
Hence path P_n admits sixth order triangular sum labeling. □

EXAMPLE 13. *The sixth order triangular sum labeling of P_6 is shown below.*

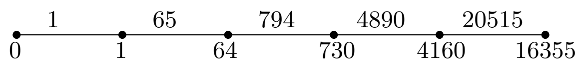

Fig 13: Sixth order triangular sum labeling of P_6

THEOREM 14. *The comb $P_n \odot K_1$ admits sixth order triangular sum labeling.*

PROOF. Let $P_n : u_1u_2...u_n$ be the path and let $v_i = u_iu_{i+1}(1 \leq$i$\leq$n-1) be the edges. Let $w_1, w_2, ..., w_n$ be the pendant vertices adjacent to $u_1, u_2, ..., u_n$ respectively and $t_i = u_iw_i(1 \leq$i$\leq$n) be the edges.
For i=1,2,...,n, define

$$f(u_i) = \frac{1}{42}(i-1)i(i+1)(3i^4 - 18i^2 + 31)$$

and

$$\begin{aligned} f(w_i) = \frac{1}{42}[&3i^7 + (42n - 21)i^6 + (126n^2 - 126n + 42)i^5 \\ &+ (210n^3 - 315n^2 + 105n)i^4 \\ &+ (210n^4 - 420n^3 + 210n^2 - 56)i^3 \\ &+ (126n^5 - 315n^4 + 210n^3 - 21n)i^2 \\ &+ (42n^6 - 126n^5 + 105n^4 - 21n^2 + 32)i \\ &+ (6n^7 - 21n^6 + 21n^5 - 7n^3 + n)] \end{aligned}$$

Then $f(u_i) + f(u_{i+1}) = f^+(v_i)$ for $1 \leq i \leq n - 1$

and $f(u_i) + f(w_i) = f^+(t_i)$ for $1 \leq i \leq n$.

Thus the induced edge labels are the first $2n - 1$ sixth order triangular numbers.
Hence comb admits sixth order triangular sum labeling. □

EXAMPLE 14. *The sixth order triangular sum labeling of $P_4 \odot K_1$ is shown below.*

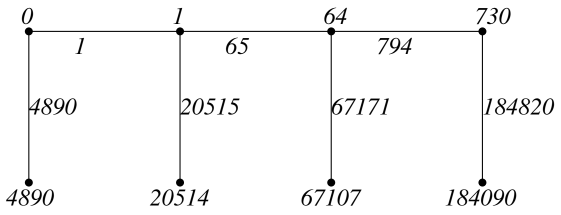

Fig 14: Sixth order triangular sum labeling of $P_4 \odot K_1$

THEOREM 15. *The star graph $K_{1,n}$ admits sixth order triangular sum labeling.*

PROOF. Let v be the apex vertex and let $v_1, v_2, ..., v_n$ be the pendant vertices of the star $K_{1,n}$.
Define

$$f(v) = 0$$
$$\text{and} f(v_i) = \frac{1}{12} i^2(i+1)^2(2i^2+2i-1), 1 \le i \le n.$$

We see that the induced edge labels are the first n sixth order triangular numbers.
Hence $K_{1,n}$ admits sixth order triangular sum labeling. □

EXAMPLE 15. *The sixth order triangular sum labeling of $K_{1,5}$ is shown below.*

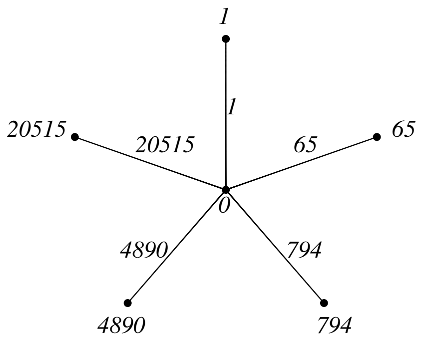

Fig 15: Sixth order triangular sum labeling of $K_{1,5}$

THEOREM 16. *$S(K_{1,n})$, the subdivision of the star $K_{1,n}$, admits sixth order triangular sum labeling.*

PROOF. Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iu_i : 1 \le i \le n\}$.
Define f by

$$f(v) = 0,$$
$$f(v_i) = \frac{1}{42} i(i+1)(2i+1)(3i^4+6i^3-3i+1), 1 \le i \le n$$
$$\begin{aligned}\text{and} f(u_i) = & ni^6 + 6(1+2+...+n)i^5 \\ & + 15(1^2+2^2+...+n^2)i^4 \\ & + 20(1^3+2^3+...+n^3)i^3 \\ & + 15(1^4+2^4+...+n^4)i^2 \\ & + 6(1^5+2^5+...+n^5)i \\ & + (1^6+2^6+...+n^6), 1 \le i \le n.\end{aligned}$$

We see that the induced edge labels are the first $2n$ sixth order triangular numbers.
Hence $S(K_{1,n})$ admits sixth order triangular sum labeling. □

EXAMPLE 16. *The sixth order triangular sum labeling of $S(K_{1,3})$ is shown below.*

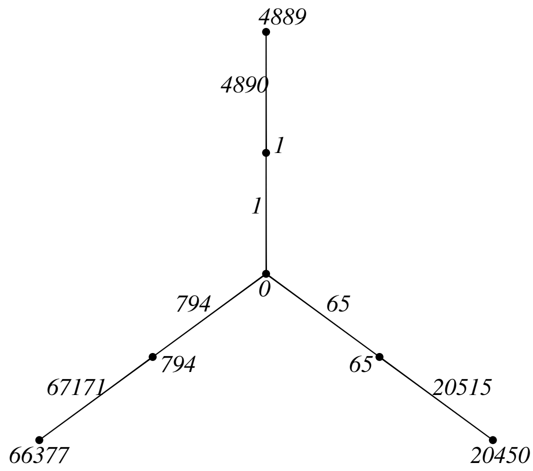

Fig 16: Sixth order triangular sum labeling of $S(K_{1,3})$

THEOREM 17. *The bistar $B_{m,n}$ admits sixth order triangular sum labeling.*

PROOF. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \le \text{i} \le \text{m}, 1 \le \text{j} \le \text{n}\}$ and
$E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \le \text{i} \le \text{m}, 1 \le \text{j} \le \text{n}\}$.
Define f by

$$f(u) = 0,$$
$$f(v) = 1,$$
$$f(u_i) = \frac{1}{12}(i+1)^2(i+2)^2(2i^2+6i+3), 1 \le i \le m$$
$$\begin{aligned}\text{and} f(v_j) = & \frac{1}{12}(m+j+1)^2(m+j+2)^2[2j^2+(4m+6)j \\ & + (2m^2+6m+3)] - 1, 1 \le j \le n.\end{aligned}$$

We see that the induced edge labels are the first $m+n+1$ sixth order triangular numbers.
Hence $B_{m,n}$ admits sixth order triangular sum labeling. □

EXAMPLE 17. *The sixth order triangular sum labeling of $B_{3,2}$ is shown below.*

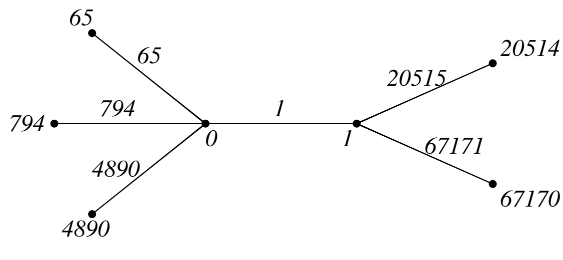

Fig 17: Sixth order triangular sum labeling of $B_{3,2}$

THEOREM 18. *Coconut tree admits sixth order triangular sum labeling.*

PROOF. Let $v_0, v_1, v_2, ..., v_i$ be the vertices of a path, having path length $i(i \ge 1)$ and $v_{i+1}, v_{i+2}, ..., v_n$ be the pendant vertices, being adjacent with v_0.
For $0 \le j \le i$, define

$$f(v_j) = \begin{cases} \frac{1}{2}j^2(j^4+6j^3+10j^2-1) & \text{if j is even} \\ \frac{1}{2}(j+1)^2(j^4+4j^3+j^2-6j+3) & \text{if j is odd} \end{cases}$$

and for $i+1 \leq k \leq n$, define

$$f(v_k) = \frac{1}{12}k^2(k+1)(2k^3+4k^2+k-1)$$

We see that the induced edge labels are the first n sixth order triangular numbers.
Hence coconut tree admits sixth order triangular sum labeling. □

EXAMPLE 18. *The sixth order triangular sum labeling of a coconut tree is shown below.*

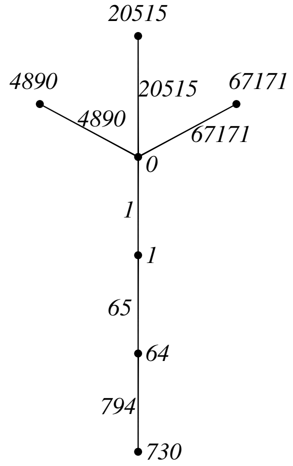

Fig 18: Sixth order triangular sum labeling of a coconut tree

3. CONCLUDING REMARKS

All the graphs that satisfy triangular sum labelings will satify higher order triangular sum labelings also. In the present work,we have tried to investigate the fourth, fifth and sixth order triangular sum labeling of standard graphs only. To investigate analogous results for different graphs and to extend the labelings furthermore towards a generalization is an open area of research.

4. ACKNOWLEDGEMENT

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