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Constrained control of C^2 rational interpolant with multiple shape parameter

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ABSTRACT

A C^2 cubic rational spline with cubic numerator and linear denominator has been constructed .This rational spline belongs to C^2 in the interpolating interval.By selecting the suitable value of shape parameters,it is easy to find the constrains for the shape of interpolating curve to lie above,below or between the given straight lines.Also the error bound for interpolating function is discussed.

Keywords:

Rational cubic spline,shape parameters,constrained interpolation,Error estimation.

1. INTRODUCTION

The most commonly used splines are [see,3,4,11] rational splines, specially the rational cubic spline with quadratic and linear
denominator. Rational splines are the powerful tools for designing curves,surfaces and for geometric shapes such as controlling the curve to be in a given region. Since the suitable parameters in
interpolating function can be selected according to the needs of controlling ,the constrained control of the shapes becomes possible. Many authors [see,7,8,10] are studying about it. Most of the problems of engineering and mathematical physics are related to C^2 continuity. M.Sarfraz[10] has worked on rational cubic spline interpolation with shape control. Duan[11] has constructed a C^2rational cubic spline interpolation based on function values and constrained control by taking quadratic denominator. In this paper, we have extended the idea of Duan[11] and constructed rational cubic spline with linear denominator using multiple shape parameters.

This paper is arranged as follows. In Section 2, a rational cubic spline and its interpolation is discussed.InSection 3, C^2 continuity of the curve is discussed. In Section 4, the region control of the interpolant curves have been discussed and the sufficient condition for constraining the interpolating curves to be bounded between two straight lines is derived . The linear inequality involving the parameters has been found which can be solved easily. In Section 5,approximation of the such rational cubic spline is discribed.

2. C^1RATIONAL CUBIC SPLINE

Let $t_0 < t_1 < ... < t_n < t_{n+1}$ are the knots , $h_i = t_{i+1} - t_i$ and f_i for $i = 0, 1, 2, 3, ..., n, n+1$ are the function values defined at the knots. Let $\theta = (t - t_i)/h_i$ for $t \in [t_i, t_{i+1}]$ and let α_i and β_i be positive parameters.

Now consider,the rational cubic function which is defined by

$$P(t) = \frac{p_i(t)}{q_i(t)}, t \in [t_{i+1}, t_i] \quad (1)$$

for $i = 0, 1, ..., n-1$,
where

$$p_i(t) = (1-\theta)^3\alpha_i A_i + \theta(1-\theta)^2(\alpha_i+\gamma_i)B_i + \theta^2(1-\theta)(\beta_i+\gamma_i)C_i + \theta^3\beta_i D_i$$

$$q_i(t) = (1-\theta)\alpha_i + \theta\beta_i,$$

This rational cubic spline satisfying the following interpolatory condition

$$P(t_i) = f_i,\ P(t_{i+1}) = f_{i+1},\ P^{'}(t_i) = \Delta_i,\ P^{'}(t_{i+1}) = \Delta_{i+1} \quad (2)$$

Thus we have,

$$A_i = f_i\ ,\ D_i = f_{i+1},$$

$$B_i = f_i + \frac{\alpha_i \Delta_i h_i}{\alpha_i + \gamma_i},$$

$$C_i = f_{i+1} - \frac{\beta_i \Delta_{i+1} h_i}{\beta_i + \gamma_i},$$

where

$$\Delta_i = \frac{(f_{i+1} - f_i)}{h_i}.$$

and $\gamma_i = \alpha_i + \beta_i$
It is easy to prove that the interpolation exists and is unique for the given data and parameters α_i and β_i.

3. C^2- RATIONAL CUBIC SPLINE

Applying the following C^2-continuity on the interpolant defined in(1) for $[t_0, t_n]$

$$P^{''}(t_i+) = P^{''}(t_i-), i = 1, 2, ..., n-1,$$

then we get following equation concerning the parameters $\alpha_i$andβ_i

$$h_i \frac{\gamma_{i-1}}{\beta_{i-1}}(\Delta_i - \Delta_{i-1}) + h_{i-1}\frac{\beta_i}{\alpha_i}(\Delta_{i+1} - \Delta_i) = 0, i = 1, 2, ..., n-1, \quad (3)$$

It can be seen that the parameter $\alpha_i$could be any positive real number to keep the interpolating curve C^2-continuous for α_{i-1}, β_iandβ_{i-1}.

4. SHAPE PRESERVING RATIONAL CUBIC SPLINE

Consider a straight line g(t) or piecewise linear curve defined on $[t_0, t_n]$ with joints at the partition $\Delta : t_0 < t_1 < ... < t_n < t_{n+1}$and a data set$(t_i, f_i) : i = 0, 1, ..., n, n+1$ with $f_i \geq (\leq) g_i$ .Let P(t) be a rational cubic interpolatory function defined by (1);if $P(t) \geq g(t)$ then P(t) is called a constrained interpolation above and if $P(t) \leq g(t)$ then it is called below $g(t)$ for all $t \in [t_0, t_n]$ First, consider $P(t) \geq g(t)$ and $q_i(t) \geq 0$.For $t \in [t_i, t_{i+1}]$,and from (1) we have

$$P(t) = \frac{p_i(t)}{q_i(t)} \geq g(t)$$

is equivalent to

$$p_i(t) - q_i(t)g(t) \geq 0$$

and suppose that

$$U_i(t) = p_i(t) - q_i(t)g(t)$$

therefore,

$$\begin{aligned} U_i(t) = (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2(\alpha_i + \gamma_i)B_i + \theta^2(1-\theta) \\ (\beta_i + \gamma_i)C_i + \theta^3\beta_i f_{i+1} - ((1-\theta)^2\alpha_i + \\ \theta(1-\theta)\gamma_i + \theta^2\beta_i)((1-\theta)g_i + \theta g_{i+1}) \geq 0 \end{aligned} \quad (4)$$

g_i , g_{i+1},denotes the value of g at t_i and t_{i+1} respectively.
Since

$$\begin{aligned} ((1-\theta)^2\alpha_i + \theta(1-\theta)\gamma_i + \theta^2\beta_i)((1-\theta)g_i + \theta g_{i+1}) \\ = (1-\theta)^2\alpha_i g_i + \theta(1-\theta)^2(\gamma_i g_i + \alpha_i g_{i+1}) + \\ \theta^2(1-\theta)(\beta_i g_i + \gamma_i g_{i+1}) + \theta^3 g_{i+1}\beta_i \end{aligned}$$

then (4) becomes

$$\begin{aligned} U_i(t) = (1-\theta)^3\alpha_i(f_i - g_i) \\ +\theta(1-\theta)^2 M_i + \theta^2(1-\theta)N_i \\ +\theta^3\beta_i(f_{i+1} - g_{i+1}) \geq 0 \end{aligned}$$

where

$$\begin{aligned} M_i = \alpha_i(f_{i+1} - g_{i+1}) + \gamma_i(f_i - g_i) \\ N_i = \gamma_i(f_{i+1} - g_{i+1}) + \beta_i(f_{i+1} - \Delta_{i+1}h_i - g_i) \end{aligned} \quad (5)$$

Since $f_i - g_i \geq 0$ and $f_{i+1} - g_{i+1} \geq 0$,there is a sufficient condition for the rational cubic curve P(t) lie above the straight line g(t) in$[t_i, t_{i+1}]$.It is given in the following theorem:
theorem 1. Given $(t_i, f_i, g_i), i = 0, 1, ..., n, n+1$with $f_i \geq g_i, i = 0, 1, ..., n$,the sufficient condition for the rational cubic curve P(t) defined by (1) to lie above the straight line g(t) in $[t_i, t_{i+1}]$ is that the positive parameters α_i, β_i satisfy

$$\gamma_i(f_{i+1} - g_{i+1}) + \beta_i(f_{i+1} - \Delta_{i+1}h_i - g_i) \geq 0$$

If the knots are equally spaced then we have the following corollary of theorem(1):

Corollary 1. The sufficient condition for the rational cubic curve P(t) defined by (1) to lie above the straight line g(t) in$[t_i, t_{i+1}]$ for the equally spaced partition

$$\gamma_i(f_{i+1} - g_{i+1}) + \beta_i(2f_{i+1} - f_{i+2} - g_i) \geq 0$$

α_i, β_i are the positive parameters.
for a given data set$(t_i, f_i, g_i), i = 0, 1, ..., n, n+1$ the corresponding numbers defined by (5) are called criterion numbers for the rational cubic interpolant above the straight line in $[t_i, t_{i+1}]$.
Similarly,the sufficient condition for the rational cubic curve P(t) defined by(1) to lie below the below the straight line $g^*(t)$ in $[t_i, t_{i+1}]$ is that the parameters α_i, β_isatisfy

$$\gamma_i(f_{i+1} - g^*_{i+1}) + \beta_i(2f_{i+1} - f_{i+2} - g^*_i) \leq 0$$

where $f_j \leq g^*$,$j = i, i+1$and g^*_i, g^*_{i+1} represent g^* at t_i and t_{i+1} ,respectively.Thus the above discussion gives the following theorem:

Theorem 2. Given $(t_i, f_i, g_i, g^*_i), i = 0, 1, ..., n, n+1$ with $g_i \leq f_i \leq g^*_i$,the sufficient condition for the rational cubic curve P(t)
defined by (1)to lie above the straight line g(t) and below the straight line $g^*(t)$ in $[t_i, t_{i+1}]$ is that the parameters $\alpha_i > 0, \beta_i > 0$ satisfying the following linear system of inequalities :

$$\gamma_i(f_{i+1} - g_{i+1}) + \beta_i(2f_{i+1} - f_{i+2} - g_i) \geq 0 \quad (6)$$

$$\gamma_i(g^*_{i+1} - f_{i+1}) + \beta_i(f_{i+2} + g^*_i - 2f_{i+1}) \geq 0 \quad (7)$$

Above theorem shows that P(t) defined by (1) bounded between two straight lines $g(t)$ and$g^*(t)$ in $[t_i, t_{i+1}]$
We can consider the two cases :
(a)The first case when $f_i = g_i < g^*_i, i = 1, 2, ..., n+1$. In this case ,the inequality given in theorem (2) becomes

$$\beta_i(2f_{i+1} - f_{i+2} - g_i) \geq 0$$

$$\gamma_i(g^*_{i+1} - f_{i+1}) + \beta_i(f_{i+2} + g^*_i - 2f_{i+1}) \geq 0 \quad (8)$$

Thus, we get the following existence theorem:

Theorem3. Given $(t_i, f_i, g_i, g^*_i), i = 0, 1, ..., n+1$ with $f_i = g_i < g^*_i$, if $(2f_{i+1} - f_{i+2} - g_i) \geq 0$, then there must exist the parameters $\alpha > 0$,$\beta_i > 0$ such that the rational cubic spline P(t) defined by (1)lies above the line $g(t)$ and below the straight line $g^*(t)$ in $[t_i, t_{i+1}]$.
(b)The second case :$g_i < f_i < g^*_i$,$i = 1, 2, ..., n+1$.
In this case ,the constraint interpolation problem can be solved completely¡.In fact ,science $f_{i+1} - g_{i+1} > 0$ and $g^*_{i+1} - f_{i+1} > 0$,the following theorem follows:

Theorem 4. Given $(t_i, f_i, g_i, g^*_i), i = 0, 1, ..., n+1$ with $g_i < f_i < g^*_i$,then there exist parameters $\alpha_i > 0$,$\beta_i > 0$ such that the rational cubic spline P(t) defined by (1) lies above the straight line $g(t)$ and below the straight line g^*_t in $[t_i, t_{i+1}]$.

5. ERROR ESTIMATION

To find the error estimation we consider that the given function f(t)$\in C^2$ and p(t) is the interpolating function of f(t) in$[t_i, t_{i+1}]$,$i = 0, 1, ..., n + 1$ If the knots are equally spaced,Eq.(1) becomes

$$P(t) = \frac{p_i(t)}{q_i(t)}$$

$t \in [t_i, t_{i+1}]$.
where,

$$p_i(t) = ((1-\theta)^2((1-\theta)\alpha_i + \theta\gamma_i))f_i + \theta[(1-\theta)^2\alpha_i + \theta(1-\theta)(2\beta_i+\gamma_i) + \theta^2\beta_i]f_{i+1} - \theta^2(1-\theta)\beta_i f_{i+2}$$

$$q_i = (1-\theta)\alpha_i + \theta\beta_i,$$

$hi = h = \frac{t_n - t_0}{n}$for all $i = 1, 2, ..., n$. Using the Peano-Kernel Theorem [9] gives the following

$$R[F] = f(t) - p(t) = \int f^2(\tau)R_t[(t-\tau)_+]d\tau, t \in [t_i, t_{i+1}], \quad (9)$$

where,

$$R_t[(t-\tau)_+] = \begin{cases} (t-\tau) - \frac{(\theta(1-\theta)^2\alpha_i + \theta^2(1-\theta)(2\beta_i+\gamma_i))(t_{i+1}-\tau) - \theta^2(1-\theta)(t_{i+2}-\tau)\beta_i}{(1-\theta)\alpha_i+\beta_i\theta} \\ t_i < \tau < t \\ -\frac{(\theta(1-\theta)^2\alpha_i + \theta^2(1-\theta)(2\beta_i+\gamma_i))(t_{i+1}-\tau) - \theta^2(1-\theta)(t_{i+2}-\tau)\beta_i}{(1-\theta)\alpha_i+\beta_i\theta} \\ t < \tau < t_{i+1} \\ \frac{\theta^2(1-\theta)\beta_i(t_{i+2}-\tau)}{(1-\theta)\alpha_i+\beta_i\theta} \\ t_{i+1} < \tau < t_{i+2} \end{cases} = \begin{cases} p(\tau), t_i < \tau < t \\ q(\tau), t < \tau < t_{i+1} \\ r(\tau), t_{i+1} < \tau < t_{i+2} \end{cases}$$

Thus from(9)

$$|R[f]| \le \|f^{(2)}(t)\|[\int_{t_i}^{t} |p(\tau)|d\tau + \int_{t}^{t_{i+1}} |q(\tau)|d\tau + \int_{t_{i+1}}^{t+2} |r(\tau)|d\tau] \quad (10)$$

Next, calculate the following terms in (10)

observe that

$p(t) = q(t) \le 0, p(t_i) \ge 0$

Therefore p(t) is having roots at τ_*

so that,

$\int_t^{t_{i+1}} |p(\tau)|d\tau = \int_t^{\tau_*} -p(\tau)d\tau + \int_{\tau_*}^{t+1} p(\tau)d\tau$

Similarly

$q(t) \le 0$ and $q(t_{i+1}) = \frac{\theta^2(1-\theta)h_i\beta_i}{\alpha_i(1-\theta)+\theta\beta_i} \ge 0$

Therefore we find the root of $q(\tau)$ at τ^*

$\tau^* = t_{i+1} - \frac{\theta(1-\theta)\beta_i h_i}{(1-\theta)^2\alpha_i + \theta(1-\theta)(2\beta_i+\gamma_i)+\theta^3\beta_i}$

Thus,

$\int_t^{t_{i+1}} |q(\tau)|d\tau = \int_t^{\tau^*} -q(\tau)d\tau + \int_{\tau^*}^{t+1} q(\tau)d\tau$

from the above calculations it can be shown that

$$|R[f]| \le \|f^2(t)\|h^2 w(\theta, \alpha_i, \beta_i) \quad (11)$$

where $w(\theta, \alpha_i, \beta_i)$is a constant depending upon $\theta, \alpha_i, \beta_i$

6. CONCLUSION

*In this paper a rational cubic spline with cubic numerator and linear denominator was constructed.This rational cubic spline with suitable choice of multiple parameters isC^2 continuous.
*In section 3 , the sufficient condition for interpolating curve to be bounded between two straight lines was obtained.In this method simple linear inequality gives the solution for shape preserving spline. This method can be used to control the shape of different
interpolating surface.

7. REFERENCES

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