On γ-Induced Fuzzy Supra Topological Spaces and Fuzzy Supra γ-S-closed Spaces

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ABSTRACT
The aim of this paper is to study some properties of γ-induced fuzzy supra topological spaces by utilizing γ-lower semi-continuous functions. Next the connection between properties of a topological space (X,T) and γ-induced fuzzy supra topological space $(X,S_\gamma(\tau))$ are studied. Finally the concept of fuzzy supra γ-S-closed space with the notion of fuzzy supra γ-semi open set are introduced.

General Terms
Fuzzy Topological Spaces

Keywords
Topology, Induced fuzzy topology, γ-induced fuzzy supra topological space, fuzzy supra γ-semi open set, fuzzy supra γ-S-closed space.

1. INTRODUCTION
In 1975, Weiss introduced the concept of induced fuzzy topological space with the notions of a lower semi-continuous function. The concept of induced fuzzy supra topological space introduced by Bhaumik and Mukherjee [7], was defined with the notion of s-lower semi continuous functions. The class of b-open sets in the sense of Andrijevic [2] was discussed by El-Atik [10] under the name of γ-open sets. The family of all γ-open sets of X will denoted by γO (*X*).

The aim of this paper is to introduce a new fuzzy supra topological space (γ-IFST space) which is defined with the generalized concept of γ-continuous functions. The relation between the IFTS (X, $\omega(T)$) and the γ-IFST space $(X,S_\gamma(T))$ are also studied. In this paper 1_A denotes the characteristic function of an ordinary subset A. The following definitions and results are studied for ready refferences :

(a) A subset *A* of a topological space *X* is said to be γ-open [8] if $A\subset Cl(IntA)\cup Int(ClA)$.

(b) A function $f:(X,T)\to(Y,G)$ from a topological space (*X*,T) to another topological space (Y,G) is said to be γ-continuous [10] if the preimage of every open subset of Y is γ-open in X.

(c) A function $f:(X,T_1)\to(Y,T_2)$ is called γ-irresolute [6] iff the inverse image $f^{-1}(A)$ is γ-open in (X,T_1) for every γ-open in *A* in Y.

(d) A function $f:(X,T)\to(R,u)$ is said to be lower semi continuous [25] at a point x_0 of X iff for each ε > 0, there exists an open neighbourhood $N(x_0)$ such that for every $x\in N(x_0)$ implies $f(x)>f(x_0)-\varepsilon$

(e) Let (*X*,T) be a topological space. The collection $\omega(T)$ of all lower semi continuous functions $f:(X,T)\to I$ forms a fuzzy topology on X. Then (X, $\omega(T)$) is known as induced fuzzy topological space (IFTS)[11].

(f) If $A\in T$ then $1_A\in\omega(T)$ [25].

(g) $\lambda\in\omega(T)$ iff for each $r\in I$ the strong r-cut $\sigma_r(\lambda)\in T$ [25] where $\sigma_r(\lambda)=\{x:\lambda(x)>r\}$.

In [12] Mukherjee and Ghosh defined a fuzzy topological space X to be fuzzy S-closed iff every cover of X by fuzzy semiopen sets admits a finite subfamily whose fuzzy closures cover the space.

(h) A topological space (*X*,T) is S-closed [24] iff every semi-open cover of X has a finite subcover.

(i) A collection U of fuzzy sets in a fuzzy topological space (*X*,τ) is said to be a fuzzy cover [4] of X iff $\vee U = 1_X$.

(j) A fuzzy topological space (*X*,τ) is said to be a fuzzy S-closed [12] iff every cover of X by fuzzy semi open [3] sets admits a finite subfamily whose fuzzy closures covers the space.

(k) Let (*X*,τ) be a topological space. The family of all s-lower semi continuous functions from this topological space (*X*,τ) to the closed unit interval I form a fuzzy supra topology on X [7].

(l) The fuzzy supra topology obtained in above is called induced fuzzy supra topology [7] and the space (X,S (τ)) is called the induced fuzzy supra topological space. The members of S (τ) are called fuzzy supra open subsets.

(m) Let α belongs to the induced fuzzy supra topological space (X,S (τ)) i.e α∈(X,S (τ)). Then α is called fuzzy supra semi-open [21] iff there exists a fuzzy supra open subset β of (X, S (τ)) such that $\beta\subset\alpha\subset cl\,\beta$.

If A is semi-open in (*X*,τ) then 1_A is fuzzy supra semi-open in (X, S (τ)) [7].

2. γ-INDUCED FUZZY SUPRA TOPOLOGICAL SPACES
In this section a new concept of fuzzy supra topological space (γ-IFST space) is introduced by utilizing γ-lower semi continuous (γ-LSC) functions. In ordinary topological space (X,T) Bhattacharya and Biswas [6] defined γ-open neighbourhood as follows :

Let *p* be a point in (*X*,*T*). A subset *N* of *X* is a γ-open neighbourhood of *p* iff *N* is a superset of a γ-open set *S* containing *p* : $p\in S\subset N$, where *S* is a γ-open set in (*X*,*T*).

2.1 Definition
A function $f:X\to R$ from a topological space to the real number space is said to be γ-lower semi continuous (γ-LSC) [

γ-upper semi continuous (γ-USC)] at a point x_0 of X iff for each ε > 0, there exists an γ-open neighbourhood $N(x_0)$ such that $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$ [resp. $f(x) < f(x_0) + \varepsilon$].

2.2 Result

(i) The necessary and sufficient condition for a real valued function f to be γ-LSC is that for all $r \in R$, the set $\{x \in X : f(x) > r\}$ is γ-open.

Proof

Let f be γ-lower semi continuous and $x_0 \in X$. $N(x_0)$ be a γ-open neighbourhood of $x_0 \in X$. Then $f(x_0)$ is a real number and so that $f(x_0) - \varepsilon$ is a fixed real number in R for a point $x_0 \in X$. By definition $f(x) > f(x_0) - \varepsilon = r$ (say) for all $x \in N(x_0)$.

Now the set of all points x for which $f(x) > r$ is γ-open. Consequently $\{x \in X : f(x) > r\}$ is γ-open.

Conversely, let $\{x \in X : f(x) > r\}$ is γ-open. Let x_0 be any point in X. Let us choose a real number ε such that $f(x_0) - \varepsilon = r \in R$. From the given condition $\{x \in X : f(x) > f(x_0) - \varepsilon\}$ is γ-open which implies that if $\{x \in A \subseteq X\}$ is γ-open, the condition $f(x) > f(x_0) - \varepsilon$ is true which shows that f is γ-lower semi continuous.

(ii) The necessary and sufficient condition for a real valued function f to be γ-LSC is that for all $r \in R$, the set $\{x \in X : f(x) \le r\}$ is γ-closed, being the complement of γ-open.

(iii) The function f from a space (X,T) to a space (R, σ') where $\sigma' = (r, \infty) : r \in R$ is γ-LSC iff the inverse image of every open subset of (R, σ') is γ-open in (X,T).

(iv) The characteristic function of a γ-open set is γ-LSC.

Proof

Let A be a γ-open set. The characteristic function 1_A of A is defined as $1_A(x) = \begin{cases} 1, & if \quad x \in A \\ 0, & if \ x \in X - A \end{cases}$

We have to show that 1_A is γ-lower semi continuous that is $\{x : 1_A(x) = r\}$ is γ-closed for each r in R.

For $r<0$ the set $\{x : 1_A(x) = r\} = \varphi$ which is closed and hence γ-closed.

For $0 = r < 1$ the set $\{x : 1_A(x) = r\}$ = X-A which is γ-closed being the complement of a γ-open set A.

For $r = 1$, $\{x : 1_A(x) = r\}$ = X which is closed and hence γ-closed.

Hence the theorem.

(v) If f is an arbitrary family of γ-LSC functions then the function g, defined $g(x) = \sup_i f_i(x)$ is γ-LSC.

Proof

Let x_0 be any point of X. Since each f_i is γ-LSC for any $\varepsilon > 0$ there exists a γ-open neighbourhood $N_i(x_0)$ such that $x \in N_i(x_0) \Rightarrow f_i(x) > f_i(x_0) - \varepsilon$. Since the arbitrary union of γ-open sets is γ-open. We have, $\bigcup_{i=1}^{n} N_i(x_0)$ is γ-open neighbourhood of x_0.

Now, if $x \in \cup N_i(x_0)$ then for all i, $f_i(x) > f_i(x_0) - \varepsilon$ whence $g(x) = \sup_i f_i(x) > \sup_i f_i(x_0 - \varepsilon) = g(x_0) - \varepsilon$. This proves g is γ-LSC.

The intersection of two γ-open sets may not be γ-openas shown in the following example.

Example.

Let $X = \{a,b,c\}$ and $T = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$.

We observe that $\{b,c\}$ and $\{a,c\}$ are γ-open sets. But their intersection $\{c\}$ is not γ-open.

By the help of above example we can prove the following result.

(vi) If $f_1, f_2, ..., f_n$ are γ-lower semi continuous functions, then h, defined by $h(x) = \inf_i f_i(x)$, where i = 1,2,….,n, is not γ-lower semi-continuous.

Since every open set is γ-open thus every LSC function is γ-LSC but the converse is not true.

2.3 Example

Let $X = \{a,b,c,d\}$ and $Y = \{0,1\}$, $T = \{\phi, \{c\}, \{d\}, \{a,c\}, \{c,d\}, \{a,c,d\}, X\}$ and $T_1 = \{\phi, \{1\}, Y\}$ be two topologies on X and Y respectively.

We define a function $f : (X,T) \to (Y,T_1)$ by $f(a) = 0$, $f(b) = f(c) = f(d) = 1$. Here $f^{1}(0) = \{a\}$, $f^{1}(1) = \{b,c,d\}$, $f^{1}(Y) = X$, we also observe that $\{c,d\} \subseteq \{b,c,d\} \subseteq Cl\ Int\{c,d\} \cup Int\ Cl\{c,d\}$ = X.

Thus $\{b,c,d\}$ is γ-open in (X,T).

Now, we fix $r = \frac{1}{2}$ and using the result 2.2 (i) $f^{-1}(1) = \{b,c,d\} = \{x : f(x) > \frac{1}{2}\}$ is γ-open in (X,T) but not open in (X,T). Thus f is γ-LSC but f is not LSC.

2.4 Theorem

Let (X,T) be a topological space. The family of all γ-LSC functions from the space (X,T) to the unit closed interval I=[0,1] forms a fuzzy supra topology on X.

Proof

Let $S_\gamma(T)$ be the collection of all γ-LSC functions from the space (X,T) to the unit closed interval I. Now it can be prove that $S_\gamma(T)$ forms a fuzzy supra topology on X.

(a) Since X is open, it is γ-open and thus 1_X is γ-LSC i.e $1_X \in S_\gamma(T)$.

(b) ϕ is γ-open and thus 1_ϕ is γ-LSC i.e $1_\phi \in S_\gamma(T)$.

(c) Let $\{\lambda_i\}$ be any arbitrary family of γ-LSC functions. Thus $\sup\{\lambda_i\}$ is also γ-LSC. Hence $\vee \lambda_j \in S_\gamma(T)$.

Thus $S_\gamma(T)$ satisfies condition (i) – (iii) of fuzzy supra topology.

2.5 Definition

The fuzzy supra topology, obtained in above, is called γ-induced fuzzy supra-topology (γ-IFST) and the space (X, $S_\gamma(T)$) is called the γ-induced fuzzy supra-topological space(γ-IFST space). The members of $S_\gamma(T)$ are called fuzzy supra γ-open subsets.

2.6 Theorem

A fuzzy set λ in an γ-IFST space (X, $S_\gamma(T)$) is fuzzy supra γ-open iff for each $r \in I$, the strong r-cut $\sigma_r(\lambda)$ (resp. weak r-cut $w_r(\lambda)$) is γ-open in the topological space (X,T).

Proof

A fuzzy subset λ is fuzzy supra γ-open in (X, $S_\gamma(T)$) if λ∈ $S_\gamma(T)$ iff λ is γ-lower semi continuous iff for each $r\in I$, {x∈ X : λ(x) > r} is γ-open in *(X,T)* (by result 2.2(i)). That is $\sigma_r(\lambda)$ is γ-open in the topological space (X,T).

2.7 Result

If A is γ-open in (X,T) then 1_A is fuzzy γ-open in (X, $\omega(T)$) and by theorem 2.6, if A is γ-open in (X,T) then $1_A \in S_\gamma(T)$.

2.8 Theorem

If $\omega(T)$ is an induced fuzzy topology and $S_\gamma(T)$ is an γ-induced fuzzy supra-topology on X then $\omega(T)\subset S_\gamma(T)$.

Proof

Let λ∈ $\omega(T)$ i.e λ is a lower semi continuous function. Since every lower semi continuous function is γ-lower semi continuous function, λ is γ-lower semi continuous function i.e. λ∈ $S_\gamma(T)$. Hence the theorem.

2.9 Definition

A function $f:(X,\tau)\to(Y,\tau_1)$ from an γ-induced fuzzy topological space (X,τ) to the another γ-induced fuzzy topological space (Y,τ_1) is said to be fuzzy supra γ-continuous if the inverse image of every fuzzy supra γ-open subset of Y is also fuzzy supra γ-open in X.

2.10 Theorem

A function $f:(X,S_\gamma(T))\to(Y,S_\gamma(G))$ is fuzzy supra γ-continuous iff $f:(X,T)\to(Y,G)$ is γ-irresolute[6].

Proof

Let f be fuzzy supra γ-continuous and A is γ-open in (Y,G) then

$$f^{-1}(A)=\{x\in X:1_A(f(x))=1\}$$
$$=\{x\in X: f^{-1}(1_A(x))>r, 0<r\le 1\}$$
$$=\sigma_r(f^{-1}(1_A))$$

Since A is γ-open in (Y,G), $1_A\in S_\gamma(T)$ (by result 2.7)

Thus $f^{-1}(1_A)$ is fuzzy supra γ-open in (X, $S_\gamma(T)$) (since f is fuzzy supra γ-continuous). By theorem 2.6 $\sigma_r(f^{-1}(1_A))$ is γ-open in the topological space (X,T). Thus f is γ-irresolute.

On the other hand, it is consider that $f:(X,T)\to(Y,G)$ is γ-irresolute and β is fuzzy supra γ-open subset in $(Y,S_\gamma(G))$. Then for any p > 0,

$$\sigma_p(f^{-1}(\beta))=\{x\in X: f^{-1}(\beta(x))>p\}$$
$$=\{x\in X:\beta(f(x))>p\}$$
$$=(\beta f)^{-1}(p,\infty)$$
$$=f^{-1}(\beta^{-1}(p,\infty))$$

Now, $\beta\in S_\gamma(G)$. Thus β is γ-LSC. Therefore, $\beta^{-1}(p,\infty)$ is γ-open. Also by hypothesis $f^{-1}(\beta^{-1}(p,\infty))$ is γ-open in (X,T) i.e. $\sigma_p(f^{-1}(\beta))$ is γ-open in (X,T) which implies $f^{-1}(\beta)\in S_\gamma(T)$. Hence f is fuzzy supra γ-continuous.

3. FUZZY SUPRA γ-S-CLOSED SPACES

3.1 Definition

A subset A of X is said to be γ-semiopen set if there exists a γ-open set U of X such that U⊂ A ⊂ Cl(U). The complement of such set is called γ-semiclosed.

3.2 Definition

A topological space (X,τ) is said to be γ-S-closed iff every γ-semiopen cover of X has a finite subfamily whose closures covers X.

3.3 Definition

If α belongs to the γ-induced fuzzy supra topological space (X, $S_\gamma(T)$) i.e. α∈(X, $S_\gamma(T)$). Then α is called fuzzy supra γ-semi open if there exists a fuzzy supra γ-open subset β of (X, $S_\gamma(T)$) such that $\beta\subset\alpha\subset Cl\,\beta$.

3.4 Definition

A γ-induced fuzzy supra-topological space (X, $S_\gamma(T)$) is said to be fuzzy supra γ-S-closed iff for each fuzzy supra γ-semi open family *B* such that $\sup_{\lambda\in B}\lambda>\alpha$, α∈(0,1] there exists a sub family $B_0\subset B$ such that $\sup_{\lambda\in B_0} Cl\lambda>\alpha-\varepsilon$, where ε∈(0,α].

3.5 Theorem

A γ-induced fuzzy supra-topological space (X, $S_\gamma(T)$) is fuzzy supra γ-S-closed iff (X,τ) is γ-S-closed.

Proof

Let (X, $S_\gamma(T)$) be a fuzzy supra γ-S-closed space. Let $\{A_j : j\in\Lambda\}$ be a γ-semi open cover of (X,τ). Then the family of fuzzy supra γ-semi open subsets $\{1_{A_j}\}$ in $S_\gamma(T)$ satisfies sup 1_{A_j}=1. Now, since (X, $S_\gamma(T)$) is fuzzy supra γ-S-closed then for all ε,0<ε≤1, there exists a finite family $J_1, J_2, \ldots, J_n$, such that $\sup ClA_j>1-\varepsilon$. But $Cl1_{A_j}=1_{ClA_j}=1_{A_j}=1$.

Thus $\cup ClA_j=X$. Then (X,τ) is γ-S-closed.

Conversely, let $\beta\in S_\gamma(T)$ be a collection of fuzzy supra γ-semi open subsets. Now for each t, 0 < t < α, b is taken in such a way that t < b < a and for each λ∈ B and $\mu\in S_\gamma(T)$, we have, $\mu<\lambda<Cl\mu$. Then $\mu^{-1}(b,1]\subset\lambda^{-1}(b,1]\subset(Cl\mu)^{-1}(b,1]\subset Cl_{\mu^{-1}}(b,1]$. Since $\lambda^{-1}(b,1]$ is γ-semi open in (X,τ), $\{\lambda^{-1}(b,1]\}$ is γ-semi open in X and $\{\lambda^{-1}(b,1]\}$ covers X. So there exists a finite sub collection $\{\lambda_j, i=1,2,\ldots,n\}$ such that $(Cl\mu)^{-1}(b,1]$ covers X. So (X, $S_\gamma(T)$) is fuzzy supra γ-S-closed.

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5. CONCLUSION

Connection between properties of a topological space (X,T) and γ-induced fuzzy supra topological space $(X,S_{\gamma}(\tau))$ are established. Finally the concept of fuzzy supra γ-S-closed space with the notion of fuzzy supra γ-semi open set are introduced. There is a future scope to study γ-generalized closed set in L-fuzzy topological space.

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