A Comment on Djoudi's Fixed Point Theorem

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ABSTRACT
The aim of this paper is to establish the generalization of common fixed point theorem proved by A.Djoudi by using weakly compatible mappings.

Key words:
Compatible mappings, Compatible mappings of type (A), Compatible mappings of type(B), weakly Compatible mappings ,Common fixed points.

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1. INTRODUCTION

G. Jungck [1] introduced more generalized commuting mappings called compatible mappings, which are more general than commuting and weakly commuting maps.

Further ,G.Jungck, P.P.Murthy and Y.J.Cho [3] gave the generalization of compatible mappings called compatible mappings of type (A) which is equivalent to the concept of compatible maps under some conditions.

H.K.Pathak and M.S.Khan [4] introduced the concept of compatible mappings of type (B) as a generalization of compatible mappings of type (A).

Later Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps.

The present paper is to prove a common fixed point theorem for weakly compatible mappings. This theorem generalizes the result of Djoudi.

Definition 1.1:
Let S and T be mappings from a metric space (X,d) in to itself. The mappings S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n,TSx_n) = 0$, whenever $\langle x_n\rangle$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z\in X$.

Definition 1.2:
Let S and T be mappings from a metric space (X,d) in to itself. The mappings S and T are said to be compatible mapping of type (B), if

$$\lim_{n\to\infty} d(STx_n,TTx_n)\le \frac{1}{2}\left[\lim_{n\to\infty} d(STx_n,Sz)+ \lim_{n\to\infty} d(Sz,SSx_n)\right]$$

and

$$\lim_{n\to\infty} d(TSx_n,SSx_n)\le \frac{1}{2}\left[\lim_{n\to\infty} d(TSx_n,Tz)+ \lim_{n\to\infty} d(Tz,TTx_n)\right]$$

whenever $\langle x_n\rangle$is a sequence in X such that $\lim_{n\to\infty} Sx_n= \lim_{n\to\infty} Tx_n = z$ for some $z\in X$.

Definition 1.3:
Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e if Sx=Tx for some $x\in X$ then STx=TSx.

2. A Common fixed point theorem

Let R_+ be the set of non negative real numbers and let $\varphi : R^5_+ \to R_+$ be a function satisfying the following conditions:

φ is upper semi continuous in each coordinate variable and non decreasing.

$\phi(t) = \max\{\varphi (0,t,0,0,t), \varphi (t,0,0,t,t), \varphi (t,t,t,2t,0), \varphi(0,0,t,t,0) \} < t$ for any $t > 0$.

The following is the theorem proved by A.Djoudi [6].

2.1 Theorem:
Let I, J, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions

(2.1.1) $S(X)\subset J(X)$ and $T(X)\subset I(X)$

(2.1.2)$d(Sx,Ty)\le\max\{\varphi(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty), d(Ix,Ty),d(Jy,Sx))\}$ for all $x,y\in X$.

(2.3) one of S,I,T and J is continuous

(2.4) the pairs (S,I) and (T,J) are compatible mappings of type(B)

Then S,I,T and J have a unique common fixed point z. Furthermore z is the unique common fixed point of both mappings.

Then by condition (2.1), $S(X) \subset J(X)$, for an arbitrary $x_0 \in X$ there exist a point $x_1\in X$ such that $Sx_0 = Jx_1$. Also since $T(X) \subset I(X)$, for this point x_1 we can choose a point x_2 in X such that $Tx_1 = Ix_2$. Continuing in this way, one can construct a sequence $\langle y_n\rangle$ in X such that $y_{2n} = Sx_{2n} = Jx_{2n+1}$ and $y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}$ for n = 0,1,2,.......... -------(2.5)

Lemma 2.3. [6] Let I, J, S and T be mappings from a metric space (X,d) into itself satisfying (2.1) and (2.2). Then the sequence $\{y_n\}$defined by (2.3) is a cauchy sequence in X.

The conclusion of Djoudi's theorem is established using the weaker condition weakly compatible in place of compatible mappings of type (B).

3. Main Theorem:

Let I, J, S and T be mappings from a complete metric space (X,d) into itself satisfying (2.1),(2.2) and that the pairs (S, I) and (J, T) are weakly compatible . Then I, J, S and T have a common fixed point 'z'. Furthermore 'z' is the unique fixed point of both mappings.

Proof: Let $\{y_n\}$ be the sequence in X defined in (2.5), then by Lemma 2.3 of [8], $\{y_n\}$ is a Cauchy sequence in X and so it converges to some element 'z' in X. Consequently, subsequences (Jx_{2n+1}), (Sx_{2n}), (Ix_{2n}) and (Tx_{2n+1}) of $\{y_n\}$ are also converge to z as $n\to\infty$. --------------(3.1)

Since $T(X)\subset I(X)$ there exist a point $u\in X$ such that z=Iu. Now to prove Su=z.

By (2.2),

$d(Su, Tx_{2n+1}) \le \max\{\varphi\{d(Iu, Jx_{2n+1}), d(Iu, Su), d(Jx_{2n+1}, Tx_{2n+1}), d(Iu, Tx_{2n+1}), d(Jx_{2n+1}, Su)\}$.

Using (3.1) we obtain

$d(Su, z) \le \max\{\varphi\{0, d(z, Su), 0, 0, d(z, Su)\}$

this gives $d(Su, z) \le \varphi\{d(Su, z)\} < d(Su, z)$, a contradiction, if $Su\neq z$ by the definition of φ. Thus Su =z. Hence Su =Iu=z.

Since the pair (S,I) is weakly compatible, we get SIu=ISu or Sz=Iz.

Also since $S(X)\subset J(X)$, there exists a point $v\in X$ such that z =Jv. We prove z=Tv.

Again by (2.2), we have

$d(Sx_{2n}, Tv) \le \max\{\varphi\{d(Ix_{2n}, Jv), d(Ix_{2n}, Sx_{2n}), d(Jv, Tv), d(Ix_{2n}, Tv), d(Jv, Sx_{2n})\}$

Using (3.1), z =Jv and Su =Iu=z, we obtain

$d(z, Tv) \le \max\{\varphi\{d(z,Jv), 0, d(Jv,Tv), d(z,Tv), d(Jv,z)\}\}$

$\quad =\max\{\varphi\{0, 0, d(z,Tv),0,0\}\}$

$d(z, Tv) \le \varphi\{d(z, Tv)\} < d(z,Tv)$, a contradiction if $z\neq Tu$.

This implies that z = Tv. Hence Tv=Jv=z.

Also Since the pair (J,T)is weakly compatible, we get TJv = JTv or Tz=Jz.

Now we prove Sz=z.

By (2.2),

$d(Sz, Tx_{2n+1}) \le \max\{\varphi\{d(Iz,Jx_{2n+1}), d(Iz,Sz), d(Jx_{2n+1},Tx_{2n+1}), d(Iz,Tx_{2n+1}),$

$d(Jx_{2n+1}, Sz)\}$

Letting $n\to\infty$, using (3.1) and Sz=Iz, we obtain

$d(Sz, z) \le \max\{\varphi\{d(Sz,z),0,0,d(Sz,z), d(z,Sz)\}$.

This gives $d(Sz,z) \le \varphi\{d(Sz, z)\} < d(Sz, z)$, a contradiction, if $Sz\neq z$ by the definition of φ. Thus Sz =z. Hence Sz =Iz=z, showing that z is a common fixed point of S and I.

Now we prove Tz=z.

By (2.2),

$d(Sx_{2n}, Tz) \le \max\{\varphi\{d(Ix_{2n},Jz), d(Ix_{2n},Sx_{2n}), d(Jz,Tz), d(Ix_{2n},Tz), d(Jz, Sx_{2n})\}\}$

Letting $n\to\infty$, using (3.1) and Jz=Tz, we obtain

$d(z, Tz) \le \max\{\varphi\{d(z,Tz),0,0,d(z,Tz), d(Tz,z)\}$.

This gives $d(z,Tz) \le \varphi\{d(z,Tz)\} < d(z,Tz)$, a contradiction, if $Tz\neq z$ by the definition of φ. Thus Tz =z. Hence Tz =Jz=z, showing that z is a common fixed point of T and J.

Since Sz=Iz=Tz =Jz=z, showing that z is a common fixed point S,I,T and J.

Uniqueness:

Now if z^1 is another fixed point for J,I, T and S, then

$d(z^1, z) = d(Sz^1,Tz)$

$\le \varphi\{d(Iz^1,Jz), d(Iz^1, Sz^1), d(Jz, Tz)\ d(Iz^1,Tz), d(Jz, Sz^1)\}$

$\le \varphi\{d(z^1,z)\ 0,0\ d(z^1,z), d(z^1,z)\} < d(z, z^1)$

Hence $z^1= z$. Showing that z is the unique common fixed point of S,I,T and J.

Now an example is given to justify the above result.

4. Example:

Let X=[-1,1] with $d(x,y)=|x-y|$

$$Sx = Tx = \begin{cases} \dfrac{1}{20} & if\ -1 < x < \dfrac{1}{6} \\ \dfrac{1}{6} & if\ \dfrac{1}{6} \le x < 1 \end{cases}$$

$$Ix = Jx = \begin{cases} \dfrac{1}{5} & if\ -1 < x < \dfrac{1}{6} \\ \dfrac{1}{3} - x & if\ \dfrac{1}{6} \le x < 1 \end{cases}$$

Then $S(X) = T(X) = \left\{\dfrac{1}{20}, \dfrac{1}{6}\right\}$ while $I(X)=J(X) = \left\{\dfrac{1}{5} \cup \left[\dfrac{1}{6}, \dfrac{-2}{3}\right)\right\}$ so that $S(X)\subset J(X)$ and $T(X)\subset I(X)$ proving the condition (1.5). If $x_n=\left(\dfrac{1}{6}+\dfrac{1}{6n}\right)$ for $n\ge1$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = \frac{1}{6}$. It can be easily verified that

$\lim_{n\to\infty} SIx_n = \frac{1}{20}$, $\lim_{n\to\infty} SSx_n = \frac{1}{20}$ and $\lim_{n\to\infty} IIx_n = \frac{1}{20}$,

Now $\lim_{n\to\infty} d(SIx_n, IIx_n) = \frac{3}{20} \geq \frac{1}{2}\,[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, SSx_n)] = \frac{7}{120}$ failing to satisfy the compatibility of type(B) condition. It is interesting to note that the pairs (S,I) and (T,J) are weakly compatible as they commute at coincident point $\frac{1}{6}$. More over $\frac{1}{6}$ is the unique common fixed point of P,Q,S and T.

6. References

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