International Journal of Computer Applications (0975 – 8887)
Volume 69– No.10, May 2013

Prime Graph of Cartesian Product of Rings

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ABSTRACT

Let $\boldsymbol{R}$ be a commutative ring. The prime graph of the ring $\boldsymbol{R}$ is defined as a graph whose vertex set consists of all elements of $\boldsymbol{R}$ and any two distinct vertices x and y are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. In this paper we investigate some relations between the chromatic number of prime graph of finite product of commutative rings and the chromatic number of prime graph of these rings. We also obtain some results on the chromatic number of prime graph of the ring $\mathbb{Z}_m \times \mathbb{Z}_n$.

Keywords

Prime Graph, Chromatic Numbers, Rings, Product of rings.

1. INTRODUCTION

Beck [2] introduced a new graph concept called zero-divisor graph. This graph concept is associated to a commutative ring with unity and the work was mostly concerned with colouring of rings. Anderson and Livingston [1] modified the concept of zero-divisor graph and defined the Zero-divisor graph as a simple graph $\Gamma(R)$ associated to a commutative ring R with unity whose vertices are $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. The paper concentrated on the interplay between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma(R)$.

Another graph structure associated to a ring called prime graph was introduced by Bhavanari et al [3] which can be considered as an extension of Beck's work, where all elements of the ring are considered as the vertices of the graph and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. The work was related to the study of the basic properties of prime graph of a ring.

In our present paper we investigate some relations between the chromatic number of prime graph of finite product of rings and the chromatic numbers of prime graph of these rings.

Definition 1.1[3]*: Let R be a ring. A graph $G = (V, E)$ is said to be a prime graph of the ring R if $V = R$ and $E = \{\{x, y\} : xRy = 0$ or $yRx = 0, x \neq y\}$. This graph is denoted by $PG(R)$.*

Example 1.1: Let $R = \mathbb{Z}_{14}$. Then $PG(R)$ is shown in Fig 1

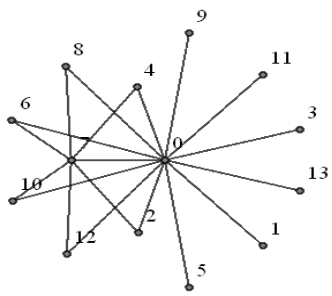

Fig 1: Prime Graph of $\mathbb{Z}_{\mathbf{14}}$

2. THE RING $\boldsymbol{R} = \boldsymbol{R_1} \times \boldsymbol{R_2} \times \ldots \times \boldsymbol{R_n}$

Let us consider the ring $\boldsymbol{R}$, where $\boldsymbol{R} = \boldsymbol{R_1} \times \boldsymbol{R_2} \times \ldots \times \boldsymbol{R_n}$.

Let $\boldsymbol{a} = (\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n})$ and $\boldsymbol{b} = (\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n}) \in \boldsymbol{R}$. Then a and b are adjacent in $\boldsymbol{PG(R)}$ if and only if

$$(\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n})\boldsymbol{R}(\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_n}) = (\boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0})$$

i.e. $\quad (\boldsymbol{a_1 R_1 b_1}, \boldsymbol{a_2 R_2 b_2}, \ldots, \boldsymbol{a_n R_n b_n}) = (\boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0})$

i.e. $\quad \boldsymbol{a_i R_i b_i} = \boldsymbol{0}$ for all $\boldsymbol{1} \leq \boldsymbol{i} \leq \boldsymbol{n}$.

Here we tried to find a relation between the $\boldsymbol{\chi PG(R_i)}$ and $\boldsymbol{\chi PG(R)}$.

***Example* 2.1:** Let $\boldsymbol{R} = \mathbb{Z}_{\mathbf{4}} \times \mathbb{Z}_{\mathbf{6}}$. Then *PG(R)* is shown in

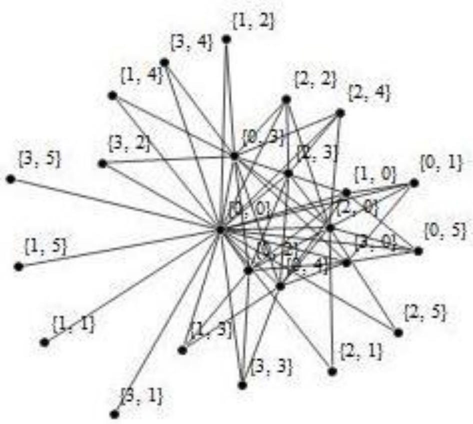

Fig2.

Fig 2: Prime Graph of $\mathbb{Z}_4 \times \mathbb{Z}_6$

Theorem 2.2: *Let $\boldsymbol{R = R_1 \times R_2 \times \ldots \times R_n}$ be a commutative ring, where every $\boldsymbol{R_i}$ is a prime ring. Then $\boldsymbol{PG(R)}$ consists of a complete n-partite subgraph whose vertex set is a subset of non-zero elements of R. Also $\boldsymbol{\chi PG(R) = n + 1}$.*

Proof: $R = R_1 \times R_2 \times \ldots \times R_n$ be a ring where every R_i is a commutative prime ring. Therefore for any two elements $a, b \in R_i$ for each i, if aR_ib=0 then a=0 or b=0.

Let
$V_i = \{r_i \in R :\ r_i = (0, 0, \ldots, a, \ldots, 0),\ a \in R_i,\ a \neq 0\}$.

Since each R_i is prime ring, any two elements $r_i, r'_i \in V_i$ are not adjacent to each other.

But any element of V_i is adjacent to all elements of V_j for all $i \neq j$. Therefore the elements of $\bigcup_{i=1}^{n} V_i$ induce a complete n-partite subgraph of $PG(R)$. This subgraph is n-colourable.

Let $r = (a_1, a_2, \ldots, a_n) \in R$ such that at least two entries are non-zero, let these be a_i and a_j. Then r is adjacent to the elements of all $V_k, k \neq i, j$. So no other point of R is adjacent to elements of all V_i in $PG(R)$. Since $0 \in R$ is adjacent to all these elements, so $\chi PG(R) = n + 1$. ♦

Theorem 2.3: *Let $R' = R \times R \times \ldots \times R$(n copies of R) be a ring where R is any commutative ring. Then $\chi PG(R') = n(\chi PG(R) - 1) + 1$ if there is no element $a \in R$ such that $aRa = 0$.*

Proof: Let $\chi PG(R) = k + 1$. Let $a_1, a_2, \ldots, a_k \in R$ such that $a_i \neq 0$ and $a_iRa_j = 0, i \neq j, 1 \leq i,\ j \leq k$.

Let for $1 \leq i \leq n$,

$$V_i = \{r_i^j \in R' : r_i^j = (0, 0, \ldots, \underbrace{a_j}_{i^{th}}, \ldots, 0), a_j \in R,$$
$$a_j \neq 0, 1 \leq j \leq k\}.$$

Let $r_i^j,\ r_i^{j'} \in V_i$ then

$$r_i^j R' r_i^{j'} = (0, 0, \ldots, \underbrace{a_j}_{i^{th}}, \ldots, 0)R'(0, 0, \ldots, \underbrace{a_{j'}}_{i^{th}}, \ldots, 0)$$
$$= (0, 0, \ldots, \underbrace{a_jR'a_{j'}}_{i^{th}}, \ldots, 0)$$
$$= 0,\ for\ all\ j \neq j',\ 1 \leq j,\ j' \leq k.$$

Therefore elements of $V_i$for each i, are adjacent to each other. Also the elements of each V_i are adjacent to all elements of $V_j, i \neq j$.

$\therefore$ all elements of $\bigcup_{i=1}^{n} V_i$ induce a complete subgraph of $PG(R')$ whose vertices are non-Zero elements of R'. Therefore the elements of $\bigcup_{i=1}^{n} V_i$ along with $0 \in R'$ induce the maximal clique in (R') .

$$\therefore \chi PG(R') = |\bigcup_{i=1}^{n} V_i| + 1 = nk + 1$$
$$= n(\chi PG(R) - 1) + 1. \quad ♦$$

Theorem 2.4: *Let $R' = R \times R \times \ldots \times R$ (n copies of R) be a ring. Let $\{a_1, a_2, \ldots, a_k\} \subseteq R$ such that $a_iRa_j = 0,$ $for\ all\ i \neq j$ and $\{a'_1, a'_2, \ldots, a'_{k'}\} \subseteq \{a_1, a_2, \ldots, a_k\}$ such that $a'_iRa'_i = 0,\ for\ all\ i \neq j$, where $k' \leq k$, then*

$$\chi PG(R') = n(k - k') + (k' + 1)^n.$$

Proof:- Since *Let $\{a_1, a_2, \ldots, a_k\} \subseteq R$ such that $a_iRa_j = 0,$ $for\ all\ i \neq j$, $\chi PG(R) = k + 1$.*

Let for $1 \leq i \leq n$,

$$V_i = \{r_i^j \in R' : r_i^j = (0, 0, \ldots, \underbrace{a_j}_{i^{th}}, \ldots, 0), a_j \in R, a_j \neq 0,$$
$$1 \leq j \leq k\}.$$

The elements of V_i for each i, are adjacent to each other and elements of each V_i are adjacent to all elements of V_j for all $i \neq j$. So $\bigcup_{i=1}^{n} V_i$ induces a complete graph of order nk.

Now any element $r \in R'$, whose all non-zero entries are a'_i, $1 \leq i \leq k'$ is adjacent to all elements of $\bigcup_{i=1}^{n} V_i$. Also all those elements whose non-zero entries are a'_i, $1 \leq i \leq k'$ are adjacent to each other, so $PG(R')$ consists of an induced subgraph of order $\geq nk$. Total number of such elements with at least two non-zero entries whose entries are elements of the set $\{a'_1, a'_2, \ldots, a'_{k'}\}$ is $\sum_{i=2}^{n} \binom{n}{i} (k')^i$.

$\therefore$ $PG(R')$ contains a maximal clique of order

$$1 + nk + \sum_{i=2}^{n} \binom{n}{i} (k')^i = n(k - k') + \sum_{i=0}^{n} \binom{n}{i} (k')^i$$
$$= n(k - k') + (k' + 1)^n$$

Therefore $\chi PG(R') = n(k - k') + (k' + 1)^n$. ♦

Theorem 2.5: *Let $R = R_1 \times R_2 \times \ldots \times R_n$ be a ring where every R_i is a commutative ring. Let for $1 \leq i \leq m$, each R_i is a prime ring. For $m + 1 \leq i \leq n$, let $a_{i,1}, a_{i,2}, \ldots, a_{i,k_i} \in R_i$ such that $a_{i,l}R_ia_{i,l'} = 0, 1 \leq l,\ l' \leq k_i$. Also let $\{a'_{i,1}, a'_{i,2}, \ldots, a'_{i,k'_i}\}$ is subset of $\{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$ such that $k'_i \leq k_i$ and $a'_{i,l}R_ia'_{i,l} = 0, 1 \leq l \leq k'_i$. Then $\chi PG(R) = m + \sum_{i=m+1}^{n}(k_i - k'_i) + \prod_{i=m+1}^{n}(k'_i + 1)$*

Proof: For $m + 1 \leq i \leq n$,

$$V_i = \{r_i \in R :\ r_i = (0, 0, \ldots, a_{i,l}, \ldots, 0), a_{i,l} \in R_i, a_{i,l} \neq 0,$$
$$1 \leq l \leq k_i\}.$$

Then all the elements of $\bigcup_{i=m+1}^{n} V_i$ induce a complete graph of order $|\bigcup_{i=m+1}^{n} V_i| = \sum_{i=m+1}^{n} |V_i| = \sum_{i=m+1}^{n} k_i$ $(V_i \cap V_j = \phi)$. For $1 \leq i, j \leq m$, the elements $r_i = (0, 0, \ldots, a, \ldots, 0)$, $a \in R_i$ and $r_j = (0, 0, \ldots, b, \ldots, 0)$, $b \in R_j$ are adjacent to each other and these elements are also adjacent to all the elements of set $\bigcup_{i=m+1}^{n} V_i$. So we have a complete subgraph of order $m + \sum_{i=m+1}^{n} k_i$.

Now let $\{a'_{i,1}, a'_{i,2}, \ldots, a'_{i,k'_i}\} \subseteq \{a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}\}$ is such that $a'_{i,l}R_ia'_{i,l} = 0$, $1 \leq l \leq k'_i$. Let us consider the non-zero elements whose first m entries are 0 and rest of the *(n-m)* entries are either 0 or $a'_{i,l} \in R_i$, $1 \leq l \leq k'_i$. Then these elements are adjacent to each other and also adjacent to all elements of the set $\bigcup_{i=m+1}^{n} V_i$. Number of such elements is $\prod_{i=m+1}^{n}(k'_i + 1)$. But out of these elements we have already considered $\sum_{i=m+1}^{n} k'_i$ elements in the set $\bigcup_{i=m+1}^{n} V_i$. Therefore the number of elements that induce the complete subgraph is $m + \sum_{i=m+1}^{n} k_i + \prod_{i=m+1}^{n}(k'_i + 1) - \sum_{i=m+1}^{n} k'_i$ $= m + \sum_{i=m+1}^{n}(k_i - k'_i) + \prod_{i=m+1}^{n}(k'_i + 1)$.

No other elements can be adjacent to all of these elements, so this is the complete subgraph with maximum order.

Therefore

$$\chi PG(R) = m + \sum_{i=m+1}^{n}(k_i - k'_i) + \prod_{i=m+1}^{n}(k'_i + 1). \quad ♦$$

Corollary 2.6: *Let $R' = R \times R \times \ldots \times R$(n copies of R) be a ring such that for all a_i, $1 \leq i \leq k$, $a_iRa_i = 0$, then $\chi PG(R') = \left(\chi PG(R)\right)^n$.*

Proof: Putting $k = k'$ in Theorem 2.5 we get $\chi PG(R') = (k+1)^n = (\chi PG(R))^n$.

***Example* 2.7:** Let $R = \mathbb{Z}_6 \times \mathbb{Z}_6$, then $\chi PG(R) = 5$.

Proof: We have $\chi PG(\mathbb{Z}_6) = 3$. Now 2 and 3 of $\mathbb{Z}_6$ are such that $2\mathbb{Z}_6 3 = 0$. So $k = 2$ and $k' = 0$. Since $\mathbb{Z}_6$ is not prime ring so *m*=0. Therefore $\chi PG(R) = 0 + (2+2) + 1 \times 1 = 5$.

***Example* 2.8:** Let $R = \mathbb{Z}_6 \times \mathbb{Z}_8$, then $\chi PG(R) = 5$.

Proof: We have $\chi PG(\mathbb{Z}_6) = 3$ and $\chi PG(\mathbb{Z}_8) = 3$. Now 2 and 3 of $\mathbb{Z}_6$ are such that $2\mathbb{Z}_6 3 = 0$ and 2 and 4 of $\mathbb{Z}_8$ are such that $2\mathbb{Z}_8 4 = 0$. So $k_1 = 2, k'_1 = 0$ and $k_2 = 2, k'_2 = 1$ $(4\mathbb{Z}_8 4 = 0)$. Since $\mathbb{Z}_6$ and $\mathbb{Z}_8$ are not prime rings so $m = 0$. Therefore $\chi PG(R) = \{(2-0) + (2-1)\} + (0+1) \times (1+1) = 5$.

***Example* 2.9:** Let $R = \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_8$ then $\chi PG(R) = 6$.

Proof: We have $\chi PG(\mathbb{Z}_3) = 2$, $\chi PG(\mathbb{Z}_6) = 3$ and $\chi PG(\mathbb{Z}_8) = 3$. Since $\mathbb{Z}_3$ is a prime ring we have $m = 1$, so $k_2 = 2, k'_2 = 0$ and $k_3 = 2, k'_3 = 1$ $(4\mathbb{Z}_8 4 = 0)$. Therefore $\chi PG(R) = 1 + \{(2-0) + (2-1)\} + (0+1) \times (1+1) = 6$.

3. THE RING $R = \mathbb{Z}_m \times \mathbb{Z}_n$

In this section we study the chromatic number of the $PG(R)$, where $R = \mathbb{Z}_m \times \mathbb{Z}_n$.

Theorem 3.1: *Let $R = \mathbb{Z}_m \times \mathbb{Z}_n$ then*

(i) $\chi PG(R) = \chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1$ *if there is no $a \in \mathbb{Z}_m$ such that $a\mathbb{Z}_m a = 0$ or $b \in \mathbb{Z}_n$ such that $b\mathbb{Z}_n b = 0$.*

(ii) $\chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1 \leq \chi PG(R) \leq \chi PG(\mathbb{Z}_m)\chi PG(\mathbb{Z}_n)$ *if there is elements $a \in \mathbb{Z}_m$ and $b \in \mathbb{Z}_n$ such that $a\mathbb{Z}_m a = 0$ and $b\mathbb{Z}_n b = 0$.*

Proof:

(i) Let $R = \mathbb{Z}_m \times \mathbb{Z}_n$ $(m \neq n)$. Let $(a_1, b_1), (a_2, b_2) \in R$. Then $(a_1, b_1), (a_2, b_2)$ are adjacent in *PG(R)* if $a_1\mathbb{Z}_m a_2 = 0$ *and* $b_1\mathbb{Z}_n b_2 = 0, a_1, a_2 \in \mathbb{Z}_m, b_1, b_2 \in \mathbb{Z}_n$.

Case I: Let *m* and *n* are primes. Then $\mathbb{Z}_m$ and $\mathbb{Z}_n$ both are prime rings. So $\chi PG(\mathbb{Z}_m) = 2$ and $\chi PG(\mathbb{Z}_n) = 2$. Also for a_1 and a_2 of $\mathbb{Z}_m$ if $a_1\mathbb{Z}_m a_2 = 0$ then $a_1 = 0$ or $a_2 = 0$ and for b_1 and b_2 of $\mathbb{Z}_n$ if $b_1\mathbb{Z}_n b_2 = 0$ then $b_1 = 0$ or $b_2 = 0$.

Therefore the elements adjacent in $PG(R)$ are of the forms $(a, 0)$ and $(0, b)$ and no other elements are adjacent in $PG(R)$. So $(0,0)$ $(a, 0)$ and $(0, b)$ form a triangle and $\chi PG(R) = 3$.

Therefore $\chi PG(R) = \chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1$

Case II: Let $\mathbb{Z}_m$or $\mathbb{Z}_n$ is not prime ring. Let $\mathbb{Z}_n$ is not a prime ring.

Let $\chi PG(\mathbb{Z}_n) = k + 1$ and $b_1, b_2, \dots, b_k \in \mathbb{Z}_n (b_i \neq 0)$ such that $b_i\mathbb{Z}_n b_j = 0$ for all $i \neq j$.

Then all the elements of the form $(0, b_i)$ are adjacent to each other in $PG(R)$. Also all of them are adjacent to each of elements $(0,0)$ and $(a, 0)$. So $\chi PG(R) = k + 2$.

Again if $b_j\mathbb{Z}_n b_j = 0$ for $b_j \in \mathbb{Z}_n$ then each of (a, b_j) is adjacent to $(0, b_j)$ and $(0, b_i)$ for $i \neq j$ but not adjacent to $(a, 0)$. So we can colour (a, b_j) for all j and $(a, 0)$ with the same colour. Therefore the chromatic number remains unaltered that is $k + 2$.

Now $\chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) = 2 + (k+1) = k + 3$.

Therefore $\chi PG(R) = \chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1$.

Case III: Let $\mathbb{Z}_m$ and $\mathbb{Z}_n$ both are not prime rings i.e. m, n are not prime and let $\chi PG(\mathbb{Z}_m) = s + 1$ and $\chi PG(\mathbb{Z}_n) = r + 1$. Let $a_1, a_2, \dots, a_s \in \mathbb{Z}_m (a_i \neq 0)$ so that $a_i\mathbb{Z}_m a_j = 0$ for all $i \neq j$ and $b_1, b_2, \dots, b_r \in \mathbb{Z}_n (b_k \neq 0)$ so that $b_k\mathbb{Z}_n b_l = 0$ for all $k \neq l$. Therefore every element of type $(a_i, 0)$ for all i, are adjacent to each other and every pair of type $(0, b_k)$ for all k are adjacent to each other.

Also every element $(a_i, 0)$ is adjacent to each of the elements $(0, b_k)$. So all these elements of type $(a_i, 0)$ and $(0, b_k)$ induce a complete sub graph K_{s+r}. If for all $a \in \mathbb{Z}_m$, $a\mathbb{Z}_m a \neq 0$ and for all $b \in \mathbb{Z}_n$, $b\mathbb{Z}_n b \neq 0$ then (a_i, b_k) is adjacent to each of (a_j, b_l) where $i \neq j, k \neq l$.

Let $S_i = \{(a_i, b_k) : a_i \in \mathbb{Z}_m, b_k \in \mathbb{Z}_n, k = 1, 2, \dots, r\}$ *for* $i = 1, 2, \dots, s$

$S'_k = \{(a_i, b_k) : a_i \in \mathbb{Z}_m, b_k \in \mathbb{Z}_n, i = 1, 2, \dots, s\}$ *for* $k = 1, 2, \dots, r$

Now for each i, the elements of S_i are not adjacent to each other but they are adjacent to the element of the sets S_j for $i \neq j$. So they induce an s-partite graph and therefore we can assign minimum s colours to elements of these s sets.

Similarly for each k, the elements of S'_k are not adjacent to each other but they are adjacent to the element of the sets S'_l for $k \neq l$. So they induce a r-partite graph and therefore we can assign minimum r colours to elements of these r sets.

Since the sets S_i for all i and the sets S'_k for all k represents the same set of elements so both the colorings are colouring of same set. Thus possible minimum number of colours assigned to these vertices is $min(s, r)$. But we already had shown that we need $s + r$ colours to colour the vertices of type $(a_i, 0)$ and $(0, b_k)$ which is greater than both s and r.

Thus $\chi PG(R) = s + r + 1 = \chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1$.

(ii) Let $\mathbb{Z}_m$ and $\mathbb{Z}_n$ be not prime rings i.e. m, n are not prime and let $\chi PG(\mathbb{Z}_m) = s + 1$ and $\chi PG(\mathbb{Z}_n) = r + 1$.. Let $a_1, a_2, \dots, a_s \in \mathbb{Z}_m (a_i \neq 0)$ so that $a_i\mathbb{Z}_m a_j = 0$ for all $i, j \in \{1, 2, \dots, s\}$ and $b_1, b_2, \dots, b_r \in \mathbb{Z}_n (b_k \neq 0)$ so that $b_k\mathbb{Z}_n b_l = 0$ for all $k, l \in \{1, 2, \dots, r\}$. Thus as in case III of (i) all the elements of type $(a_i, 0)$ and $(0, b_k)$ induce a complete sub graph K_{s+r}. Let $(a, b) \in R$ such that $a\mathbb{Z}_m a = 0$ *and* $b\mathbb{Z}_n b = 0$. If a and b are distinct from all $a_i \in \mathbb{Z}_m$ and $b_k \in \mathbb{Z}_n$ then (a, b) is not adjacent to any of the vertices $(a_i, 0)$ and $(0, b_k)$.

Therefore $\chi PG(R) = \chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1$ ……... (1)

If $a = a_i$, for some $i, 1 \leq i \leq s$ and $b = b_k$, for some $k, 1 \leq k \leq r$ then (a, b) is adjacent to all elements of type $(a_i, 0)$ and $(0, b_k)$. So we obtain a complete subgraph K_{s+r+1}.

Thus $\chi PG(R) = s + r + 2$ that is

$$\chi PG(R) = \chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) \qquad \dots\dots\dots\dots (2)$$

Now let $a_1, a_2, \dots, a_{s'} \in \mathbb{Z}_m (a_j \neq 0, s' \leq s)$, $a_j\mathbb{Z}_m a_j = 0$ and $b_1, b_2, \dots, b_{r'} \in \mathbb{Z}_n (b_l \neq 0, r' \leq r)$ so that $b_l\mathbb{Z}_n b_l = 0$, then each vertex (a_j, b_l) is adjacent to all the vertices $(a_i, 0)$ and $(0, b_k)$. Also all the elements (a_j, b_l) are adjacent to each other. Thus we obtain a complete subgraph $K_{s+r+s'r'}$.

Therefore $\chi PG(R) = s + r + s'r' + 1$

$$= (s - s') + (r - r') + (s' + 1)(r' + 1) \quad ..... (3)$$

If $s = s'$ and $r = r'$, we have from (3),

$$\chi PG(R) = (s+1)(r+1) = \left(\chi PG(\mathbb{Z}_m)\right)\left(\chi PG(\mathbb{Z}_n)\right) .....(4)$$

Since $s \leq s'$ and $r \leq r'$, we have from (1), (2), (3) and (4),

$$\chi PG(\mathbb{Z}_m) + \chi PG(\mathbb{Z}_n) - 1 \leq \chi PG(R) \leq \chi PG(\mathbb{Z}_m)\chi PG(\mathbb{Z}_n).$$

♦

Corollary 3.2: *Let* $R = \mathbb{Z}_n \times \mathbb{Z}_n$. *Let* $\chi PG(\mathbb{Z}_n) = r + 1$ *and* $a_1, a_2, \ldots, a_r \in \mathbb{Z}_n (a_i \neq 0)$ *such that* $a_i \mathbb{Z}_n a_j = 0$ for all $i \neq j$. *Then*

- *(i)* $\chi PG(R) = 2\chi PG(\mathbb{Z}_n) - 1$ *if for no* $a \in \mathbb{Z}_n$, $a\mathbb{Z}_n a = 0$.
- *(ii)* $\chi PG(R) = 3$ *if n is prime.*
- *(iii)* $\chi PG(R) = 2\chi PG(\mathbb{Z}_n)$ *if there is only one* $a_i \in \mathbb{Z}_n$ *such that* $a_i \mathbb{Z}_n a_i = 0, 1 \leq i \leq r$.
- *(iv)* $\chi PG(R) = (\chi PG(\mathbb{Z}_n))^2$, *if* $a_i \mathbb{Z}_n a_j = 0$ *and* $a_i \mathbb{Z}_n a_i = 0 \ \forall i, j$.

Proof: Let $R = \mathbb{Z}_n \times \mathbb{Z}_n$, $\chi PG(\mathbb{Z}_n) = r + 1$ and $a_1, a_2, \ldots, a_r \in \mathbb{Z}_n (a_i \neq 0)$ such that $a_i \mathbb{Z}_n a_j = 0$ for all $i \neq j$.

- *(i)* Taking $m = n$ in *Theorem 2.1 (i)* we get

  $\chi PG(R) = \chi PG(\mathbb{Z}_n) + \chi PG(\mathbb{Z}_n) - 1 = 2\chi PG(\mathbb{Z}_n) - 1.$
- *(ii)* Let $n = p$ (a prime), then $\chi PG(\mathbb{Z}_n) = 2$. So from *(i)* $\chi PG(R) = 2\chi PG(\mathbb{Z}_n) - 1 = 3$.
- *(iii)* Since $a_i \in \mathbb{Z}_n$ is only element such that $a_i \mathbb{Z}_n a_i = 0$, taking *s = r, s′= r′=1* in (3) of *Theorem* 2.1 *(ii)* we get $\chi PG(R) = (r-1) + (r-1) + (1+1)(1+1) = 2(r+1) = 2\chi PG(\mathbb{Z}_n)$.
- *(iv)* Taking $m = n$ in (4) of *Theorem* 2.1(*ii*) we get $\chi PG(R) = (\chi PG(\mathbb{Z}_n))^2$. ♦

4. CONCLUSION

From the above discussion we observed that for the ring $R = \prod_{i=1}^{n} R_i$, the chromatic number of *PG(R)* lies between (n+1) and $\prod_{i=1}^{n} \chi PG(R_i)$ i.e.

$$n + 1 \leq \chi PG(R) \leq \prod_{i=1}^{n} \chi PG(R_i).$$

The lower bound is obtained when each R_i is a prime ring. The upper bound is obtained if for every $R_i, 1 \leq i \leq n$, $a_i R_i a_i = 0$ and $a_i R_i b_i = 0$ for all $a_i, b_i \in R_i$.

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