β^* Homeomorphisms in Topological Spaces

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ABSTRACT

In this paper the authors define β^* homeomorphisms which are generalization of homeomorphisms and investigate some of their basic properties and also investigate generalized β^* closed maps.

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1. INTRODUCTION

Malghan[4] introduced the concept of generalized closed maps in topological spaces. Biswas[1], Mashour[5], Sundaram[9], Crossley and Hildebrand [2], and Devi[3]have introduced and studied semi-open maps, α -open maps, and generalized open maps respectively.

Several topologists have generalized homeomorphisms in topological spaces. Biswas[1], Crossley and Hildebrand[2], Sundaram[5s] have introduced and studied semi-homeomorphism and some what homeomorphism and generalized homeomorphism and gc-homeomorphism respectively.

Throughout this paper (X, τ) and (Y, σ)(or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, cl(A) and int(A) represents the closure of A and interior of A respectively.

2. PRELIMINARIES

The authors recall the following definitions

Definition [9] 2.1: A subset A of a space X is g-closed if and only if cl(A) $\subset$ G whenever A $\subset$ G and G is open.

Definition [3] 2.2: A map f : X → Y is called g-closed if each closed set F of X, f(F) is g-closed in Y .

Definition [4] 2.3: A map f : X → Y is said to be generalized continuous if $f^{-1}(V)$ is g-open in X for each set V of Y

Definition [8] 2.4: A subset A of a topological space X is said to be β^* closed set in X if cl(int(A)) contained in U whenever U is G-open

Definition 2.5[7]: Let f : X → Y from a topological space X into a topological space Y is called β^* -continuous if the inverse image of every closed set in Y is β^* closed in X.

3. β^* Closed map

Definition 3.1: A map f : X → Y is called β^* closed map if for each closed set F of X, f(F) is β^* closed set.

Theorem 3.2: Every closed map is a β^* -closed map.

Proof: Let f : X → Y be an closed map. Let F be any closed set in X. Then f(F) is an closed set in Y. Since every closed set is β^* , f(F) is a β^* -closed set. Therefore f is a β^* closed map.

Remark 3.3:The converse of the theorem 3.4 need not be true as seen from the following example.

Example 3.4: Let X = Y = {a, b, c} with topologies τ = {X, ϕ , {a},{a,b}} and σ = {Y, ϕ , {a},{c},{a,c}} Let f(a)= a, f(b) = c, f(c) = b be the map. Then f is β^* -closed but not closed, Here f is β^* -continuous. But f is not continuous since for the

closed set {b, c} in X is {a, b} which is not closed in Y.

Definition 3.5: A map f : X → Y is called β^* closed map if for each closed set F of X, f(F) is β^* closed set.

Remark 3.6: Every g-closed map is a β^* closed map and the converse is need not be true from the following example.

Example3.7: Let X = {a, b, c} and τ ={ ϕ , x, {a}, {a, b}}, τ^c = { ϕ ,X, {b,c}, {c}} be topologies

on X. f : X → Y each closed set f(F) is g-closed. Here {a, c} is g-closed but not β^* -closed.

Theorem 3.8: A map f : X → Y is β^* closed if and only if for each subset S of Y and for each open set U containing

$f^{-1}(S)$ there is a β^* -open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$

Proof: Suppose f is β^* closed. Let S be a subset of Y and U is an open set of X such that $f^{-1}(S) \subset U$, Then $V = Y - f^{-1}(X - U)$is a β^* -open set V of Y Such that $S \subset V$ such that $f^{-1}(V) \subset U$.

For the converse suppose that F is a closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F$ is open. By hypothesis there is β^* -open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore $F \subset X - f^{-1}(V)$.Hence $Y - V \subset f(F) \subset f(X - f^{-1}(V)) \subset Y - V$ which implies $f(F) = Y - V$. Since $Y - V$ is β^* -closed if f(F) is β^* -closed and thus f is a β^* -closed map.

Theorem 3.9: If $f : X \to Y$ is continuous and β^* -closed and A is a β^* -closed set of X then f(A) is β^* -closed.

Proof: Let $f(A) \subset O$ where O is an open set of Y. Since f is g-continuous, $f^{-1}(O)$ is an open set containing A. Hence $cl(int(A)) \subset f^{-1}(O)$ is A is β^* -closed set. Since f is β^* -closed, f(cl(int(A))) is a β^* -closed set contained in the open set O which implies than $cl(int(f(cl(int(A))))) \subset O$ and hence $cl(int(f(cl(int(A))))) \subset O$ .f is a β^* -closed set.

corollary 3.6: If $f : X \to Y$ is g-continuous and closed and A is g-closed set of X the f(A)is β^* -closed.

Corollary 3.10: If $f : X \to Y$ is β^* -closed and continuous and A is β^* -closed set of X then

$f_A : A \to Y$ is continuous and β^* -closed set.

Proof: Let F be a closed set of A then F is β^* closed set of X. From above theorem 3.5 follows that $f_A(F) = f(F)$ is β^* -closed set of Y. Here f_A is β^* -closed and continuous.

Theorem 3.11: If a map $f : X \to Y$ is closed and a map $g : Y \to Z$ is β^* -closed then $f : X \to Z$ is β^* -closed.

Proof : Let H be a closed set in X. Then f(H) is closed and $(g \circ F)(H) = g(f(H))$ is β^* -closed as g is β^* -closed. Thus $g \circ f$ is β^* -closed.

Theorem 3.12: If $f : X \to Y$ is continuous and β^* -closed and A is a β^* -closed set of X then $f_A : A \to Y$ is continuous and β^* -closed.

Proof: If F is a closed set of A then F is a β^* closed set of X. From Theorem 3.4, It follows that $f_A(F) = f(F)$ is a β^* -closed set of Y. Hence f_A is β^* -closed. Also f_A is continuous.

Theorem 3.13: If $f : X \to Y$ is β^* -closed and $A = f^{-1}(B)$ for some closed set B of Y then $f_A : A \to Y$.is β^* -closed .

Proof: Let F be a closed set in A. Then there is a closed set H in X such that $F = A \cap H$. Then $f_A(F) = f(A \cap H) = f(H) \cap f(B)$. Since f is β^* -closed. f(H) is β^* -closed in Y. so $f(H) \cap B$ is β^* -closed in Y. Since the intersection of a β^* -closed and a closed set is a β^* -closed set. Hence f_A is β^* -closed.

Remark 3.14: If B is not closed in Y then the above theorem does not hold from the following example.

Example 3.15: Take B = {b,c}. Then $A = f^{-1}(B)$ = {b, c} and {c} is closed in A but $f_A(\{b\}) = \{b\}$ is not β^* -closed in Y .{a} is also not β^* -closed in B.

4. β^* Homeomorphism

Definition 4.1 : A bijection $f : X \to Y$ is called β^* homeomorphism if f is both β^* continuous and β^* closed

Theorem 4.2 : Every homeomorphism is a β^* homeomorphism

Proof: Let $f : X \to Y$ be a homeomorphism. Then f is continuous and closed. Since every continuous function is β^* continuous and every closed map is β^* closed, f is β^* continuous and β^* closed. Hence f is a β^* homeomorphism.

Remark 4.3: The converse of the theorem 4.2 need not be true as seen from the following example.

Example 4.4: Let X = Y = {a, b, c} with topologies τ = {X, ϕ ,{a}, {a, b}} and σ = {Y, ϕ , {a},{c}, {a, c}}. Let $f : X \to Y$ with f(a)=a,f(b)=c,f(c)=b.Then f is β^* homeomorphism but not a homeomorphism, since the inverse image of {a, c} in Y is not closed in X.

Theorem 4.5: For any bijection $f : X \to Y$ the following statements are equivalent.

(a) Its inverse map $f-1 : Y \to X$ is β^* continuous.

(b) f is a β^* open map.

(c) f is a β^* -closed map.

Proof: (a) $\Longrightarrow$ (b)

Let G be any open set in X. Since f^{-1} is β^* continuous, the inverse image of G under f^{-1}, namely f(G) is β^* open in Y and so f is a β^* open map. .

(b) $\Longrightarrow$ (c)

Let F be any closed set in X. Then F^c open in X.Since f is β^* open, $f(F^c)$ is β^* open in Y . But $f(F^c) = Y - f(F)$ and so f(F) is β^* closed in Y. Therefore f is a β^* closed map.

(c) $\Longrightarrow$ (a)

Let F be any closed set in X. Then the inverse image of F under f^{-1}, namely f(F) is β^* closed in Y since f is a β^* closed map. Therefore f^{-1} is β^* continuous.

Theorem 4.6: Let f : X → Y be a bijective and β^* continuous map. Then, the following statements are equivalent.

(a) f is a β^* open map

(b) f is a β^* homeomorphism.

(c) f is a β^* closed map.

Proof: (a) $\Longrightarrow$ (b)

Given f : X → Y be a bijective, β^* continuous and β^* open. Then by definition, f is a β^* homeomorphism.

(b) $\Longrightarrow$ (c)

Given f is β^* open and bijective. By theorem 4.5, f is β^* closed map.

(c) $\Longrightarrow$ (a)

Given f is β^* closed and bijective. By theorem 4.5,f is a β^* open map.

Remark 4.7: The following example shows that the composition of two β^* homeomorphism is not a β^* homeomorphism.

Example 4.8: Let X = Y = Z = {a, b, c} with topologies τ = {X,ϕ, {a}, {a,b}} and σ = {Y, ϕ,{a},{c) {a, c}} and η = {Z,ϕ, {a},{b},{a,b}}. Let f : X → Y and g : Y → Z be the map with f(a)=a,f(b)=c,f(c)=b. Then both f and g are β^* homeomorphisms but their composition $g \circ f : X \to Z$ is not a β^* homeomorphism, since F = {a, c} is closed in X, but $g \circ f(F) = g \circ f(\{a, c\}) = \{a, b\}$ which is not β^* -closed in Z.

5. REFERENCES

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