I-Continuity in Topological Spaces due to Martin: A Counter-example

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ABSTRACT

Martin (I-continuity in topological spaces, Acta Mathematica, Faculty of Natural Sciences Constantine the Philosopher University Nitra, 6 (2003), 115-122.) has introduced an interesting concept of I-continuity of a function f . In this paper, a counter example to the assertion of Martin has been discussed which he has established in his result (Theorem 2.2), stating that continuity implies I-continuity. It has been noticed that only the homeomorphism of f implies I-continuity of f.

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Keywords

Ideal, I-convergence, I-continuity.

1. INTRODUCTION

The notion of I-convergence was introduced by Kostyrko et al. in [7] (see also [3], [5]) for sequences in metric spaces. This concept of I-convergence was used in [1] (see also [2], [4], [6]) for introducing I-convergence in case of real functions. These ideas have been further studied by Martin [10] for verifying certain properties of I-continuous functions and their relationship with the classical definition of continuous functions.

In the present paper, it has been concluded that the concepts of continuity (cf. [9]) and I-continuity are independent. This assertion contradicts the claim of Martin established in his paper (cf. [10]) that continuity of a function implies I-continuity. In support of this claim, Several elaborative examples have been constructed .

Finally, a result which states that "If f is a homeomorphism then f is I-continuous" has been proved .Which is a direct consequence of the fact that the homeomorphism carries a topological property from one space to its homeomorphic image.

2. Priliminaries.

Throughout this paper (X, τ_X) denotes a topological space, and F_X denotes the collection of all **closed subsets** of the topological space (X, τ_X). Also N denotes a set of **natural numbers**, and $\{x_n\}^{\infty}_{n=1}$ be a **sequence** of elements of the topological space (X, τ_X). The set of all subsets of a given set S is called the **power set** of S and is denoted by P(S).

Definition 2.1. (cf.[10]) A family I of subsets of N is an **ideal** in N if

- $A, B \in I \Rightarrow A \cup B \in I$,
- $A \in I$ and $B \subset A \Rightarrow B \in I$.

Remark 2.1. If $N \in I$, then the Ideal is said to be **improper**.

Example 2.1. N = {1, 2, ...} be a set of natural numbers. Consider I = {φ, {1}, {2},{1, 2}}. It is easy to see that I is an ideal in N. (refer Definition 2.1)

Definition 2.2. (cf.[10]) Let I be an ideal in N. A sequence $\{x_n\}^{\infty}_{n=1}$ in a topological space X is said to be I−convergent to a point $x \in X$ if

$$A(U) = \{n : x_n \notin U\} \in I \quad (2.1)$$

holds for each open set U containing x. It has been denoted by

$$I - \lim x_n = x.$$

Remark 2.2. The $I - \lim x_n$ is not unique in general.

Example 2.2. Let X = {a, b, c} be a topological space with the topology $\tau_X = \{X, \varphi, \{a\}, \{a, c\}\}$, $F_X = \{\{\varphi, X, \{b, c\}, \{b\}\}$. Consider a sequence $\{x_n\}^{\infty}_{n=1}$ in X as follows:

$$\{x_n\} = \begin{cases} c, \text{ for } n = 1, 2, ..., 60 \\ b, \text{ for } n = 61, ..., 120 \\ a, \text{ for } n = 121, ... \end{cases} \quad (2.2)$$

Consider $I = \{ \varphi, P(S)\}$ in N (2.3)
where $S = \{1, 2, ...120\}$

By construction, the sequence $\{x_n\}$ converges to **a** in the classical sense. The I−limits of the sequence $\{x_n\}$ has been verified as follows:

• **Claim. a** is an I−limit of $\{x_n\}$.
$a \in X$, {a}, {a, c}. In view of relation (2.1). Now consider
$A(X) = \{ \varphi: x_n \notin X\}$.
$A(\{a\}) = \{1, ...120 : x_n \notin \{a\}\}$.
$A(\{a, c\}) = \{61, ...120 : x_n \notin \{a, c\}\}$.

Reffering relation (2.3), it is clear that A(X), A{a}, A{a, c}∈ I. Hence, it has been concluded that **a** is an I−limit of $\{x_n\}$. (cf. Definition 2.2).
Similarly, it may be checked easily that **b** and **c** are also the I−limits of $\{x_n\}$. Therefore, $I - \lim x_n$ = a, b, c.

Definition 2.3. (cf.[10]) Let I be an ideal in N and X, Y be topological spaces. A map $f : X \rightarrow Y$ is called I−continuous if for each sequence $\{x_n\}^{\infty}_{n=1}$ in X

$I - \lim x_n = x_0$, $I - \lim f(x_n) = f(x_0)$.

Example 2.3. Let X = {a, b, c} be a topological space with the topology τ_X = {X, φ , {a}, {a, c}}, F_X = { φ, X, {b, c}, {b}}. We Define a sequence $\{x_n\}^{\infty}_{n=1}$ in X as follows:

$$\{x_n\} = \begin{cases} c, & \text{for } n = 1, 2, ..., 60 \\ b, & \text{for } n = 61, ..., 120 \\ a, & \text{for } n = 121, ... \end{cases} \quad (2.4)$$

Consider I = { φ, P(S)} in N (2.5)

Where S = {1, 2, ...120}

It has been already shown in example 2.2, that

$I - \lim x_n$ = a, b, c.

Next, consider Y = {p, q, r} with the topology T_Y = {Y, φ, {p}, {r}, {p, r}}, F_Y = { φ, Y, {q, r}, {p, q}, {q}}. Define a function f : X → Y as follows :

$$\left.\begin{matrix} f(c) \\ f(a) \end{matrix}\right\} = p; \quad (2.6)$$

f(b) = q

It may be verified easily that f is a continuous function. In view of (2.4) and (2.6),
the sequence $\{f(x_n)\}^{\infty}_{n=1}$ in Y turns out to be,

$$\{f(x_n)\} = \begin{cases} f(c) = p, \text{ for } n = 1, 2, ..., 60 \\ f(b) = q, \text{ for } n = 61, ..., 120 \\ f(a) = p, \text{ for } n = 121, ..., ... \end{cases} \quad (2.7)$$

The I−limits of the sequence $\{f(x_n)\}$ have been checked as follows:

• **Claim. p** is an I−limit of $\{f(x_n)\}$.
p ∈Y, {p}, {p, r}.
A(Y) = { φ : $f(x_n) \notin$ Y }
A({p}) = {61, ..., 120 : $f(x_n) \notin$ {p}}
A({p, r}) = {61, ..., 120 : $f(x_n) \notin$ {p, r}}
Reffering relation (2.5) it is clear that A(Y), A{p}, A{p, r} ∈ I. Hence, it may be concluded that **p** is an I−limit of $\{f(x_n)\}$. (cf. Definition 2.2).
Similarly, it may be checked that **q** is also the I−limit of $\{f(x_n)\}$.

• **Claim. r** is not an I−limit of {f(xn)}
r ∈ Y, {r}, {p, r}
A(Y) = { φ : $f(x_n) \notin$ Y } ∈ I cf. relation (2.1)
A({r}) = {N : $f(x_n) \notin$ {r}} ∈ I cf. relation (2.1)
A({p, r}) = {61, ..., 120 : $f(x_n) \notin$ {p, r}} ∈ I cf. relation (2.1)
Hence**,** **r** is not an I−limit of $\{f(x_n)\}$. Therefore,
$I - \lim f(x_n)$ = p, q.
So we get, $I - \lim x_n$ = a, b, c , $I - \lim f(x_n)$ = p, q. Thus, f is I−continuous, in view of definition 2.3 and the equation (2.7).

3. Main result due to Martin.

In this section, the basic result due Martin which he established by considering the relationship between the concept of continuity and I−continuity of a function f has been considered. The authors found a small error in the proof of this result which disproves the claim. In support of this investigation, a counter-example has been discussed in details.

Theorem 3.1. (Due to Martin) Let X and Y be topological spaces and let I be an arbitrary ideal in N. If f : X → Y is continuous then f is I−continuous. (cf. Theorem 2.2 [10])

Proof. (Due to Martin): Let f : X→ Y be continuous function and $I-\lim x_n = x$. Then, for each neighborhood V of f(x) there exists a neighborhood U of x such that, f(U) ⊆V . Hence,
$\{n \in N : f(x_n) \notin V\} \subseteq \{n \in N : x_n \notin U\} \in I$ and
$I - \lim f(x_n) = f(x)$.

Error in the last step. Let f : X → Y be continuous and $I - \lim x_n = x$. Then, for each neighborhood V of f(x) there exists a neighborhood U of x such that f(U) ⊆ V . (cf[8]) Since $I - \lim x_n = x$ then,
A(U) = {n : $x_n \notin$ U} ∈ I ⇒ f(xn) ∉ f(U). Since f(U) ⊆ V , it is clear that $f(x_n)$ may or may not belong to V ie. f(xn) ∉ f(U) ⇒ f(xn) ∉ V Hence, f is not I−continuous necessarily.

Therefore, it may be concluded that if the function is continuous then it is not necessary that function is I−continuous.

Counter-examples.

Example 3.1. Let X = {a, b, c} be a topological space with the Topology τ_X = {X, φ , {a}, {a, c}}, F_X = { φ, X, {b, c}, {b}}. Define a Sequence $\{x_n\}^{\infty}_{n=1}$ X as follows:

$$\{x_n\} = \begin{cases} c, & \text{for } n = 1, 2, ..., 60 \\ b, & \text{for } n = 61, ..., 120 \\ a, & \text{for } n = 121, ... \end{cases} \quad (3.1)$$

Consider

I = { φ , P(S)} in N (3.2)

Where S = {1, 2, ...120}

It has been already checked in example 2.2 that, $I - \lim x_n$ = a, b, c.
Now , consider Y = {p, q, r} with the topology T_Y= {Y, φ, {p}}, F_Y = { φ, Y, {q, r}}. Define a function f : X → Y as follows :

$$\left.\begin{matrix} f(c) \\ f(a) \end{matrix}\right\} = p \quad (3.3)$$

f(b) = q

It may be verified easily that f is into and continuous. In view of (3.1) and (3.3) the sequence $\{f(x_n)\}^{\infty}_{n=1}$ in Y turns out to be,

$$\{f(x_n)\} = \begin{cases} f(c) = p, \text{ for } n = 1, 2, ..., 60 \\ f(b) = q, \text{ for } n = 61, ..., 120 \\ f(a) = p, \text{ for } n = 121, ..., ... \end{cases} \quad (3.4)$$

The I−limits of the sequence {f(xn)} has been verified as follows:

• **Claim. p** is an I−limit of $\{f(x_n)\}$.
p ∈ Y, {p}.
A(Y) = { φ : $f(x_n) \notin$ Y }
A({p}) = {61, ..., 120 : $f(x_n) \notin$ {p}}
Reffering relation (3.2) it is clear that A(Y), A{p} ∈ I. Hence, it has Been concluded that **p** is an I−limit of $\{f(x_n)\}$. (cf. Definition 2.2).
Similarly, it may be checked that **q** and **r** are also the I−limits of $\{f(x_n)\}$.
Therefore, $I - \lim f(x_n)$ = p, q, r
But, if $I - \lim x_n$ = a, b, c , then $I - \lim f(x_n)$ = p, q.
Referring the result of Martin (Theorem 2.2 of [10]), the conclusion is:
$I - \lim f(x_n)$ = p, q which is a contradiction as by applying the

definition directly, we get I − lim $f(x_n)$ = p, q, r.

Remark 3.1. By considering one-one onto function, instead of many one function (as considered in Example 3.1) it has observed in the following example that the result due to Martin does not hold.

Example 3.2. Let X = {a, b, c} be a topological space with the topology τ_X = {X, φ, {a}, {b}, {a, b}}, F_X = { φ, X, {b, c}, {a, c}, {c}}. We define a sequence $\{x_n\}^{\infty}_{n=1}$ in X as follows:

$$\{x_n\} = \begin{cases} c, & \text{for } n = 1, 2, ..., 50 \\ b, & \text{for } n = 61, ..., 100 \\ a, & \text{for } n = 101, ... \end{cases} \quad (3.5)$$

Consider

I = { φ, P(S)} in N (3.6)

where S = {1, 2, ...100}

It may be checked easily that I − lim x_n = a, c.

Next, consider Y = {1, 2, 3} with the topology T_Y = {Y, φ, {1, 2}}, F_Y = { φ, Y, {3}}.

Define a function f : X →Y as follows:

f(a) = 1 f(b) = 2 f(c) = 3 (3.7)

Following the technique of earlier examples, it may be verified easily that f is continuous but not I−continuous.

In the next Example, it has been verified that for a given non Continuous function f, the property of I−continuity holds.

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Example 3.3. Let X = {a, b, c} be a topological space with the topology τ_X = {X, φ, {a, b}}, F_X = {φ ,X, {c}}. Define a sequence $\{x_n\}^{\infty}_{n=1}$ in X as follows:

$$\{x_n\} = \begin{cases} c, & \text{for } n = 1, 2, ..., 50 \\ b, & \text{for } n = 61, ..., 100 \\ a, & \text{for } n = 101, ... \end{cases} \quad (3.8)$$

Consider

I = { φ, P(S)} in N (3.9)

Where S = {51, ..., ...}

It may be checked easily that I − lim x_n = c.

Consider Y = {1, 2, 3} with the topology T_Y = {Y, φ, {1}, {2}, {1, 2}} , F_Y = { φ, Y, {2, 3}, {1, 3}, {3}}.

Now define a function f : X → Y The I−limits of the sequence have been computed as follows:

f(a) = 1, f(b) = 2, f(c) = 3 (3.10)

It may be verified easily that f is not continuous function. In view of (3.8) and (3.10) the sequence $\{f(x_n)\}^{\infty}_{n=1}$ in Y turns out to be,

$$\{f(x_n)\} = \begin{cases} f(c) = p, \text{ for } n = 1, 2, ..., 50 \\ f(b) = q, \text{ for } n = 61, ..., 100 \\ f(a) = p, \text{ for } n = 101, ..., ... \end{cases} \quad (3.11)$$

• **Claim**. **1** is not an I−limit of {f(xn)} .

1 ∈ Y, {1}, {1, 2}.

A(Y) = { φ : $f(x_n) \notin$ Y } ∈ I (cf. relation (2.1))

A({1}) = {1, ..., 100 : $f(x_n) \notin$ {1}} ∉ I (cf. relation (2.1))

A({1, 2}) = {1, ..., 50 : $f(x_n) \notin$ {1, 2}} ∉ I (cf. relation (2.1))

Hence, **1** is not an I−limit of f(xn). Similarly, it may be checked easily that **2** is not the I−limit of $\{f(x_n)\}$.

• **Claim. 3** is an I−limit of $\{f(x_n)\}$

3 ∈ Y .

A(Y) = { φ : $f(x_n) \notin$ Y } ∈ I (see relation (2.1))

Hence, **3** is an I−limit of $f(x_n)$. Therefore, I − lim $f(x_n)$ = 3.

Thus, we get I − lim x_n = c ⇒ I − lim $f(x_n)$ = 3 (cf.(3.11)).

Hence, f is I−continuous.

4. Main result.

Remark 4.1. Any property of the domain which has been described in terms of open sets also holds in its homeomorphic image.

Theorem 4.1. Let X and Y be topological spaces and let I be an arbitrary ideal in N. If f : X → Y is a homeomorphism then f is I−continuous.

Proof. Let f :X →Y be a homeomorphism (cf. [8]) and I − lim x_n = x.

Reffering relation (2.1),

A(U) = {n : $x_n \notin$ U for each nbd U of x} ∈ I (4.1)

Since f is a homeomorphism it is an open mapping and f(U) is an open set in Y whenever U is open in X. Since x ∈ U, f(x) ∈ f(U).

Claim. $X_n \notin U \Rightarrow f(x_n) \notin f(U)$ for each nbd f(U) of f(x).

Let if possible there exits an open set V in Y containing f(x) which is not equal to any of f(U). Then x ∈ f^{-1}(V) since, f is continuous f^{-1} (V) is an open set containing x. Hence, V must be of the form f(U) for some nbd U of x. Therefore, each nbd of f(x) is of the form f(U). Hence, I − lim $f(x_n)$ = f(x).

Conclusion.

Finally the following conclusion have been noted:

1. Continuity of f does not imply I−continuity of f.
2. Considering even one-one onto continuous function then also above assertions (1) holds.
3. I−continuity of function does not imply continuity of a function.
4. If f is a homeomorphism then f is I−continuous.

4. ACKNOWLEDGMENTS

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5. REFERENCES

[1] Balaz V., Cervenansky J., Kostyrko Salat T., I-convergence and I-continuity of real functions,Acta Mathematica, Faculty of Natural Sciences Constantine the Philosopher University Nitra 5 (2002), 43-50.

[2] Cintura J., Heredity and Coreflective Subcategories of the category of the topological Spaces, Applied categorical structures 9 (2001), 131- 138.

[3] Engelking R., General Topology, PWN, Warsaw, 1977.

[4] Franclin S.P., Rajagopalan M., On subsequential spaces Topology Appl. 35 (1990), 1-19.

[5] Franclin S.P., Spaces in which sequences suffice, Fund. Math. 57 (1967), 107-115.

[6] Herrlich H., Topologische Reflexionen and Coreflexionen, Springer Verlag, Barlin, 1968.

[7] Kostyrko P., Salat T., Wilczynski W., I-convergence, Real Anal.Exch. 26 (2) (2000/2001), 669-686.

[8] Munkers J. R., Topology, Second Edition, Pearson Education Asia (1988).

[9] Power P.L., Rajak K., Some new concepts of continuity in generalized topological space, International Journal of Computer Application, volume 38 NO.5, January 2012, ISSN NO. 0975-8887 (online).

[10] Sleziak Martin, I-continuity in topological spaces, Acta Mathematica, Faculty of Natural Sciences Constantine the Philosopher University Nitra, 6 (2003) , 115-122.