$M^{[x]}/G/1$ Queue with Two Phase of Service and Optional Server Vacation

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ABSTRACT
In this paper we analyze a single server queue with batch arrival Poisson input, two heterogeneous service with different general (arbitrary) service time distributions and two phase (compulsory and optional) server vacations with general (arbitrary) vacation period. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability (1- θ), he leaves the system. After completion of each service, the server will take compulsory vacation. The vacation period of the server has two heterogeneous phases. However, after returning from first compulsory vacation the server may take one more optional vacation with probability p or return back to the system with probability (1- p). No server can take more than two vacations at a time. The probability generating function for the number of customers in the queue is found using the supplementary variable technique. The mean number of customers and the mean waiting time in the queue also found. Some particular cases are discussed. Numerical results are also obtained.

AMS Subject Classification: 60K25, 60K30

KEYWORDS
Batch Arrival, Optional service, Optional Vacation, Probability Generating Function, Stability Condition, Mean Queue Size.

1. INTRODUCTION
In recent years, queues with server vacations have emerged as an important area of queueing theory and have been studied extensively and successfully due to their various applications in computer and communication networks, manufacturing and production systems, inventory system, bank services etc. For overhauling or maintenance the system the server may go to vacation. The server works continuously as long as there is at least one customer in the system. When the server finishes serving a customer and finds the system empty, he goes away for a length of time called a vacation. For example, maintenance activities, telecommunication networks, production systems, etc.

Recently, there have been several contributions considering queueing system of M/G/1 type in which the server may provide a second phase service. Such queueing situations occur in day-to-day life, for example in many applications such as hospital services, production systems, bank services, computer and communication networks there is two phase of services such that the first phase is essential for all customers, but as soon as the essential services completed, it may leave the system or may immediately go for the second phase of service. One may refer to Kleinrock [11], Medhi [15], Choudhury [5], [4], Choudhury and Paul [7], Kalyanaraman and Pazhani Bala Mururgan [9], Badamchi and Shankar [3], Madan et al. [14]. Madan and Choudhury [13] proposed an $M^x/G/1$ queueing system, assuming batch arrivals with restricted admissibility of arriving batches and Bernoulli schedule server vacation. Earlier, Madan et al.[12] studied this type of model and studied some aspects of batch arrivals Bernoulli vacation models with restricted admissibility, where all arriving batches are not allowed into the system at all time.

The concept of vacation in the M/G/1 queueing system was studied by Keilson and Servi in [10]. For the systems with batch arrival the vacation time were analyzed by Baba in [1], Doshi [8], Takagi [16] presented an excellent survey of queueing system with server vacations. Choudhury and Madan [6] have studied a queueing system modified Bernoulli schedule with vacation under N- policy. Badamchi Zadeh [2] studied a batch arrival queueing system with two phases of heterogeneous service with optional second service and restricted admissibility with single vacation policy.

In this paper we consider batch arrival queue with two phases of service and optional server vacation. The first phase of service is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability (1- θ), he leaves the system. After completion of each service, the server will take compulsory vacation. The vacation period of the server has two heterogeneous phases. However, after returning from first compulsory vacation the server may take one more optional vacation with probability p or return back to the system with probability (1- p).

This paper is organized as follows. The mathematical description of our model is given in section 2. Definitions and notations are given in section 3. Equations governing the model are given in section 4. In section 5 supplementary variables technique has been used to obtain steady state results in explicit and closed form in terms of the probability generating functions for the number of customers in the queue. The mean queue size and mean waiting time are found in section 6. Some particular cases are discussed in section 7. Numerical results and conclusion are given in section 8 and 9 respectively.

2. MATHEMATICAL DESCRIPTION OF THE MODEL

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a ‘first come - first served basis'. Let $\lambda_i C_i dt$ (i = 1, 2, 3, . . .) be the first order probability that a batch of i customers arrives at the system during a short interval of time (t, t + dt], where $0 \le C_i \le 1$ and $\sum_{i=1}^{\infty} C_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides the first phase of service which is essential for all customers, as soon as the first service of a customer is completed, then with probability θ, he may opt for the second service or else with probability (1- θ), he leave the system.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)$ dx be the conditional probability density of service completion during the interval (x, x + dx], given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}, \quad i=1, 2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i=1, 2$$

d) After completion of each service, the server will take compulsory vacation of random length. The vacation time has two phases with phase one is compulsory. However, after phase one vacation, the server takes phase two optional vacation with probability p or may return back to the system with probability 1 – p.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\beta_i(x)dx$ be the conditional probability of a completion of a vacation during the interval (x, x + dx) given that the elapsed vacation time is x, so that

$$\beta_i(x) = \frac{v_i(x)}{1-v_i(x)}, \quad i=1, 2,$$

and therefore,

$$v_i(t) = \beta_i(t) e^{-\int_0^t (\beta_i(x)dx)} \quad i=1, 2.$$

3. DEFINITIONS AND NOTATIONS

Define $N_Q(t)$ denote the queue size (excluding one in service) at time t. We introduce the random variable Y (t) as follows

$$Y(t) = \begin{cases} 0 & \text{if the server is idle at time t} \\ 1 & \text{if the server is busy with first phase of service at time t} \\ 2 & \text{if the server is busy with second phase of service at time t} \\ 3 & \text{if the server is on the first phase vacation at time t} \\ 4 & \text{if the server is on the second phase vacation at time t} \end{cases}$$

Let L (t) be the queue size at time t. we introduce the supplementary variable

$$L(t) = \begin{cases} B_1^0(t) & if \quad Y(t) = 1 \\ B_2^0(t) & if \quad Y(t) = 2 \\ V_1^0(t) & if \quad Y(t) = 3 \\ V_2^0(t) & if \quad Y(t) = 4 \end{cases}$$

where

B_1^0 (t) = elapsed service time of the customer at the first phase of the service at time t,

B_2^0 (t) = elapsed service time of the server at the second phase of the service at time t,

V_1^0 (t) = elapsed vacation time of the server at the first phase of the vacation at time t,

V_2^0(t) = elapsed vacation time of the server at the second phase of the vacation at time t.

The process $\{(N_Q(t), L(t)), t \ge 0\}$ is a continuous time Markov process.

We define the probabilities for i=1, 2

$P_{00}(x,t) = \text{Prob}\{N_Q(t) = 0, L(t) = 0\}$

$P_{in}(x,t) = \text{Prob}\{N_Q(t) = n, L(t) = B_i^0(t); \quad x < B_i^0(t) \le x + dx\}$

$x > 0, n \ge 0$

$V_{in}(x,t) = \text{Prob}\{N_Q(t) = n, L(t) = V_i^0(t); \quad x < V_i^0(t) \le x + dx\}$

$x > 0, n \ge 0$

with assumption that steady state exist, we have

$$P_{00} = \lim_{t\to\infty} P_{00}(t)$$

$$P_{i,n}(x)dx = \lim_{t\to\infty} P_{i,n}(x,t)dx \quad i = 1,2 \quad x > 0, n \ge 0$$

$$V_{i,n}(x)dx = \lim_{t\to\infty} V_{i,n}(x,t)dx \quad i = 1,2 \quad x > 0, n \ge 0$$

we define

$P_{n1}(x,t)$: Pr {at time t, the server is active providing service and there are n (n ≥ 0) customers in the queue excluding the one customer in the first phase of service being served and the elapsed service time for this customer is x}. P_{n1} (t)= $\int_0^\infty P_{n1}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the first phase of service irrespective of the value of x.

$P_{n2}(x,t)$: Pr {at time t, the server is active providing service and there are n (n ≥ 0) customers in the queue excluding the one customer in the second phase of service being served and the elapsed service time for this customer is x}. P_{n2}(t)= $\int_0^\infty P_{n1}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the second phase of service irrespective of the value of x.

$V_{n1}(x,t)$: Pr {at time t, the server is on vacation with elapsed vacation time x and there are n (n ≥ 0) customers in the queue}. V_{n1}(t)= $\int_0^\infty V_{n1}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is on vacation irrespective of the value of x.

$V_{n2}(x,t)$: Pr {at time t, the server is on vacation with elapsed vacation time x and there are n (n ≥ 0) customers in the queue}. V_{n2}(t) = $\int_0^\infty V_{n2}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is on vacation irrespective of the value of x.

$P_{00}(t)$: Pr {at time t, there are no customers in the system and the server is idle}.

4. EQUATIONS GOVERNING THE SYSTEM

Now, analysis of this queueing model can be performed with the help of the following Kolmogorov forward equations.

$$\frac{d}{dx}P_{10}(x) + [\lambda + \mu_1(x)]P_{10}(x) = 0 \tag{1}$$

$$\frac{d}{dx}P_{1n}(x) + [\lambda + \mu_1(x)]P_{1n}(x) = \lambda\sum_{k=1}^{n} C_k P_{1n-k}(x), \quad n \geq 1 \tag{2}$$

$$\frac{d}{dx}P_{20}(x) + [\lambda + \mu_2(x)]P_{20}(x) = 0 \tag{3}$$

$$\frac{d}{dx}P_{2n}(x) + [\lambda + \mu_2(x)]P_{2n}(x) = \lambda\sum_{k=1}^{n} C_k P_{2n-k}(x), \quad n \geq 1$$
(4) (4)

$$\frac{d}{dx}V_{10}(x) + [\lambda + \beta_1(x)]V_{10}(x) = 0 \tag{5}$$

$$\frac{d}{dx}V_{1n}(x) + [\lambda + \beta_1(x)]V_{1n}(x) = \lambda\sum_{k=1}^{n} C_k V_{1n-k}(x), \quad n \geq 1 \tag{6}$$

$$\frac{d}{dx}V_{20}(x) + [\lambda + \beta_2(x)]V_{20}(x) = 0 \tag{7}$$

$$\frac{d}{dx}V_{2n}(x) + [\lambda + \beta_2(x)]V_{2n}(x) = \lambda\sum_{k=1}^{n} C_k V_{2n-k}(x), \quad n \geq 1 \tag{8}$$

$$\lambda P_{00} = (1-p)\int_0^\infty \beta_1(x)V_{10}(x)\,dx + \int_0^\infty \beta_2(x)V_{20}(x)\,dx \tag{9}$$

Equation (1) and (9) are to solved subject to the following boundary conditions:

$$P_{1n}(0) = \lambda C_{n+1}P_{00} + (1-p)\int_0^\infty \beta_1(x)V_{1n+1}(x)\,dx + \int_0^\infty \beta_2(x)V_{2n+1}(x)\,dx \quad n \geq 0 \tag{10}$$

$$P_{2n}(0) = \theta\int_0^\infty P_{1n}(x)\mu_1(x)\,dx, \; n \geq 0 \tag{11}$$

$$V_{1n}(0) = (1-\theta)\int_0^\infty \mu_1(x)P_{1n}(x)\,dx + \int_0^\infty \mu_2(x)P_{2n}(x)\,dx, n \geq 0 \tag{12}$$

$$V_{2n}(0) = p\int_0^\infty \beta_1(x)V_{1n}(x)\,dx, \quad n \geq 0 \tag{13}$$

The normalization condition is given by

$$P_{00} + \sum_{i=1}^{2}\sum_{n=0}^{\infty}\int_0^\infty P_{in}(x)dx + \sum_{i=1}^{2}\sum_{n=0}^{\infty} V_{in}(x)dx = 1 \tag{14}$$

5. QUEUE SIZE DISTRIBUTION AT RANDOM EPOCH

Let us define the following probability generating functions as

$$\left.\begin{array}{ll} P_i(x,z) = \sum_{n=0}^{\infty} z^n P_{in(x)} & ; \quad |z| \leq 1, x > 0 \\ P_i(0,z) = \sum_{n=0}^{\infty} z^n P_{in}(0) & ; \quad |z| \leq 1 \\ P_i(x,z) = \sum_{n=0}^{\infty} z^n V_{in}(x) & ; \quad |z| \leq 1, x > 0 \\ P_i(0,z) = \sum_{n=0}^{\infty} z^n V_{in}(0) & ; \quad |z| \leq 1 \end{array}\right\} \tag{15}$$

Now multiplying equation (2), (4), (6), (8) and adding (1), (3), (5), (7) by suitable powers of z summing over n and using the generating functions defined in (15) and we get

$$\frac{d}{dx}P_1(x,z) + \left(\lambda + \mu_1(x) - \lambda C(z)\right)P_1(x,z) = 0 \tag{16}$$

$$\frac{d}{dx}P_2(x,z) + \left(\lambda + \mu_2(x) - \lambda C(z)\right)P_2(x,z) = 0 \tag{17}$$

$$\frac{d}{dx}V_1(x,z) + \left(\lambda + \beta_1(x) - \lambda C(z)\right)V_1(x,z) = 0 \tag{18}$$

$$\frac{d}{dx}V_2(x,z) + \left(\lambda + \beta_2(x) - \lambda C(z)\right)V_2(x,z) = 0 \tag{19}$$

Next we multiply both sides of equation (10) by z^n and sum over n from 0 to ∞ using (15) we have

$$zP_1(0,z) = \lambda C(z)P_{00} + (1-p)V_1(0,z)V_1^*(R) + V_2(0,z)V_2^*(R) \tag{20}$$

Performing similar operation on equation (11), (12) and (13) using (15) we have

$$P_2(0,z) = \theta P_1(0,z)B_1^*(R) \tag{21}$$

$$V_1(0,z) = (1-\theta)P_1(0,z)B_1^*(R) + P_2(0,z)B_2^*(R) \tag{22}$$

$$V_2(0,z) = PV_1(0,z)V_1^*(R) \tag{23}$$

where R = λ (1-C (z))

Using equations (22) and (23) in (20) , we get

$$P_1(0,z) = \frac{\lambda(C(z)-1)P_{00}}{[Z - V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]} \tag{24}$$

Similarly using (24) in (21), we get (13)

$$P_2(0,z) = \frac{\lambda\theta B_1^*(R)(C(z)-1)P_{00}}{[Z - V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]} \tag{25}$$

Similarly using (24) and (25) in (22), we get

$$V_1(0,z) = \frac{\lambda(C(z)-1)B_1^*(R)(1-\theta+\theta B_2^*(R))P_{00}}{[Z - V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]} \tag{26}$$

Finally using (25) in (22), we get

$$V_2(0,z) = \frac{\lambda P(C(z)-1)V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))P_{00}}{[Z-V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]} \quad (27)$$

Integrating equations (16), (17), (18) and (19) we have

$$P_1(x,z) = P_1(0,z)e^{-Rx}(1-B_1(x)) \quad (28)$$

$$P_2(x,z) = P_2(0,z)e^{-Rx}(1-B_1(x)) \quad (29)$$

$$V_1(x,z) = V_1(0,z)e^{-Rx}(1-V_1(x)) \quad (30)$$

$$V_2(x,z) = V_2(0,z)e^{-Rx}(1-V_2(x)) \quad (31)$$

where $P_1(0,z)$, $P_2(0,z)$, $V_1(0,z)$ and $V_2(0,z)$ have been obtained in equations (24), (25), (26) and (27) respectively.

After integrating equation (28) with respect to x, we have

$$P_1(z) = \frac{\lambda(C(z)-1)P_{00}}{[Z-V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]}\left(\frac{1-B_1^*(R)}{R}\right) \quad (32)$$

where $B_1^*(R) = \int_0^\infty e^{-Rx}\,dB_1(x)$ is the Laplace transform of first phase of service time.

Now integrate equation (29) with respect to x, we have

$$P_2(z) = \frac{\lambda\theta B_1^*(R)(C(z)-1)}{[Z-V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]}\left(\frac{1-B_2^*(R)}{R}\right) \quad (33)$$

where $B_2^*(R) = \int_0^\infty e^{-Rx}\,dB_2(x)$ is the Laplace transform of second phase of service time.

Integrate equation (30) with respect to x, we have

$$V_1(z) = \frac{\lambda(C(z)-1)B_1^*(R)(1-\theta+\theta B_2^*(R))P_{00}}{[Z-V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R)]}\left(\frac{1-V_1^*(R)}{R}\right) \quad (34)$$

where $V_1^*(R) = \int_0^\infty e^{-Rx}\,dV_1(x)$ is the Laplace steiltjes transform of first phase of vacation time.

In the same way, we integrate equation (31) with respect to x, we have

$$V_2(z) = \frac{\lambda P(C(z)-1)V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))P_{00}}{[Z-V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]}\left(\frac{1-V_2^*(R)}{R}\right) \quad (35)$$

where $V_2^*(R) = \int_0^\infty e^{-Rx}\,dV_2(x)$ is the Laplace steiltjes transform of the optional vacation time.

In order to determine $P_1(z), P_2(z), V_1(z)$ and $V_2(z)$ completely, we have to determine the only unknown P_{00} which appears in the numerators of the right hand sides of equations (32), (33), (34) and (35) respectively. For that purpose, we shall use the normalizing condition

$$P_{00} + P_1(1) + P_2(1) + V_1(1) + V_2(1) = 1 \quad (36)$$

$P_1(1), P_2(1), V_1(1)$ and $V_2(1)$ are the steady state probabilities that the server is providing first phase of service, second optional service, the server under first phase of vacation and the server under optional vacation respectively without regard to the number of customers in the queue.

$$P_1(1) = \frac{-\lambda C'(1)B_1^{*'}(0)P_{00}}{1+\lambda C'(1)[V_1^{*'}(0)+B_1^{*'}(0)]+\lambda C'(1)[PV_2^{*'}(0)+\theta B_2^{*'}(0)]} \quad (37)$$

$$P_2(1) = \frac{-\lambda\theta C'(1)B_2^{*'}(0)P_{00}}{1+\lambda C'(1)\left[B_1^{*'}(0)+\theta B_2^{*'}(0)+V_1^{*'}(0)+PV_2^{*'}(0)\right]} \quad (38)$$

$$V_1(1) = \frac{-\lambda C'(1)V_1^{*'}(0)P_{00}}{1+\lambda C'(1)\left[B_1^{*'}(0)+\theta B_2^{*'}(0)+V_1^{*'}(0)+PV_2^{*'}(0)\right]} \quad (39)$$

$$V_2(1) = \frac{-\lambda PC'(1)V_2^{*'}(0)P_{00}}{1+\lambda C'(1)\left[B_1^{*'}(0)+\theta B_2^{*'}(0)+V_1^{*'}(0)+PV_2^{*'}(0)\right]} \quad (40)$$

By substituting (37), (38), (39) and (40) in (36), we get

$$P_{00} = 1+\lambda C'(1)\left[B_1^{*'}(0)+B_2^{*'}(0)+V_1^{*'}(0)+PV_2^{*'}(0)\right] \quad (41)$$

P_{00} is the steady state probability that the server is idle. Hence if $\rho < 1$ be the stability condition under which the steady states solution exits.

$$\rho = \lambda C'(1)[E(B_1)+\theta E(B_2)+E(V_1)+pE(V_2)] \quad (42)$$

where

$E(Bi) = -Bi^{*'}(0)$ $\quad E(Vi) = -V_i^{*'}(0)$ $\quad$ for i = 1, 2 .

Let $P_q(z)$ denote the probability generating function of the queue size irrespective of the server state.

Then adding equations (32), (33), (34) and (35) we obtain

$$P_q(z) = P_1(z) + P_2(z) + V_1(z) + V_2(z) \quad (43)$$

$$P_q(z) = \frac{(z-1)P_{00}}{[Z-V_1^*(R)B_1^*(R)(1-\theta+\theta B_2^*(R))(1-P+PV_2^*(R))]} \quad (44)$$

Substituting for P_{00} from (41) into (44), we have completely and explicitly determined $P_q(z)$ the probability generating function of the queue size.

6. THE MEAN QUEUE SIZE AND THE MEAN WAITING TIME

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz}P_q(z) \quad \text{at } z = 1$$

Since this formula gives 0/0 form, then we write $P_q(z)$ given in (44) as $P_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are given by

$$N(z) = (z-1)$$

$$D(z) = [Z - V_1^*(R)B_1^*(R)\left(1-\theta+\theta B_2^*(R)\right)\left(1-P+PV_2^*(R)\right)$$

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$= \lim_{z\to 1}\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \tag{45}$$

where primes and double primes in (45) denote first and second derivatives at z = 1, respectively. Carrying out the derivatives at z = 1 we have

,

$$N'(1) = 1 \tag{46}$$

$$N''(1) = 0 \tag{47}$$

$$D'(1) = [1+\lambda C'(1)(V_1^{*\prime}(0) + B_1^{*\prime}(0) + \theta B_2^{*\prime}(0) + PV_2^{*\prime}(0))] \tag{48}$$

$$\begin{aligned} D''(1) = & -\lambda^2\left(C'(1)\right)^2\left[V_1^{*\prime\prime}(0) + B_1^{*\prime\prime}(0) + \theta B_2^{*\prime\prime}(0) + PV_2^{*\prime\prime}(0)\right] \\ & -2\lambda^2\left(C'(1)\right)^2\left[V_1^{*\prime}(0)B_1^{*\prime}(0) + P\theta B_2^{*\prime}(0)PV_2^{*\prime}(0)\right] \\ & -2\lambda^2\left(C'(1)\right)^2\left[V_1^{*\prime}(0)B_1^{*\prime}(0)][\theta B_2^{*\prime}(0) + PV_2^{*\prime}(0)\right] \\ & +\lambda C''(1)\left[V_1^{*\prime}(0) + B_1^{*\prime}(0) + \theta B_2^{*\prime}(0) + PV_2^{*\prime}(0)\right] \end{aligned} \tag{49}$$

Then if we substitute the values of $N'(1)$, $N''(1)$, $D'(1)$ and $D''(1)$ from (46), (47), (48) and (49) into (45) we obtain L_q in a closed form.

$$L_q = \frac{\begin{array}{c} \lambda^2\left(C'(1)\right)^2\left[V_1^{*\prime\prime}(0) + B_1^{*\prime\prime}(0) + \theta B_2^{*\prime\prime}(0) + PV_2^{*\prime\prime}(0)\right] \\ +2\lambda^2\left(C'(1)\right)^2\left[V_1^{*\prime}(0)B_1^{*\prime}(0) + P\theta B_2^{*\prime}(0)PV_2^{*\prime}(0)\right] \\ +2\lambda^2\left(C'(1)\right)^2\left[V_1^{*\prime}(0)+B_1^{*\prime}(0)][\theta B_2^{*\prime}(0) + PV_2^{*\prime}(0)\right] \\ -\lambda C''(1)\left[V_1^{*\prime}(0) + B_1^{*\prime}(0) + \theta B_2^{*\prime}(0) + PV_2^{*\prime}(0)\right] \end{array}}{2[1+\lambda C'(1)(V_1^{*\prime}(0) + B_1^{*\prime}(0) + \theta B_2^{*\prime}(0) + PV_2^{*\prime}(0))]} \tag{50}$$

Further, the mean waiting time of a customer could be found using $Wq = \frac{L_q}{\lambda}$

7. PARTICULAR CASES:

Case 1. Assume service and vacation time follows exponential distribution.

The most common distribution for the service time and vacation is the exponential distribution. For this distribution, the rate of first phase of service $\mu_1 > 0$, second phase of service $\mu_2 > 0$ and the rate of first phase of vacation completions is $\beta_1 > 0$, second phase of vacation completions $\beta_2 > 0$ and we have

$$B_1^{*\prime}(0) = -\frac{1}{\mu_1}, B_2^{*\prime}(0) = -\frac{1}{\mu_2}, B_1^{*\prime\prime}(0) = \frac{2}{\mu_1^2}, \ B_2^{*\prime\prime}(0) = \frac{2}{\mu_2^2}$$

$$V_1^{*\prime}(0) = -\frac{1}{\beta_1}, V_2^{*\prime}(0) = -\frac{1}{\beta_2}, V_1^{*\prime\prime}(0) = \frac{2}{\beta_1^2}, \quad V_2^{*\prime\prime}(0) = \frac{2}{\beta_2^2}$$

$$B_1^*(R) = \frac{\mu_1}{R+\mu_2}, B_2^*(R) = \frac{\mu_2}{R+\mu_2}, V_1^*(R) = \frac{\beta_1}{R+\beta_1}, V_2^*(R) = \frac{\beta_2}{R+\beta_2}$$

The idle probability p_{00}, the probability generating function of the number of customers in the queue $P_q(z)$ and the mean queue size L_q can be simplified to the following expressions

$$P_{00} = 1 - \frac{\lambda C'(1)}{\mu_1\mu_2\beta_1\beta_2}[\beta_1\beta_2(\mu_2+\theta\mu_1) + \mu_1\mu_2(\beta_2+P\beta_1)] \tag{51}$$

$$P_q(z) = \frac{(Z-1)(R+\mu_1)(R+\beta_1)(R+\mu_2)(R+\beta_2)P_{00}}{Z(R+\mu_1)(R+\beta_1)(R+\mu_2)(R+\beta_2) - \mu_1\beta_1[R(1-\theta)+\mu_2][R(1-p)+\beta_2]} \quad (52$$

$$L_q =$$

$$\frac{2\lambda^2\left(C'(1)\right)^2\left[\begin{array}{c}\beta_2^2\beta_1^2(\mu_2^2+\theta\mu_1^2) + \mu_1^2\mu_2^2(\beta_2^2+P\beta_1^2) + \beta_1\beta_2\,\mu_1\mu_2[\mu_2\beta_2 + p\theta\beta_1\,\mu_1 \\ +(\beta_1+\mu_1)(p\mu_2+\theta\beta_2)]\end{array}\right] + \lambda C''(1)\beta_1\beta_2\,\mu_1\mu_2[\beta_1\beta_2(\mu_2+\theta\mu_1) + \mu_1\mu_2(\beta_2+P\beta_1)}{2\mu_1\mu_2\beta_1\beta_2[\mu_1\mu_2\beta_1\beta_2 - \lambda C'(1)(\beta_2\beta_1(\mu_2+\theta\mu_1) + \mu_1\mu_2(\beta_2+P\beta_1))]} \tag{53}$$

Case 2. The server has no optional vacation

When the server has no optional vacation, we let p = 0. Using this in the main result of (41), (44) and (50) we have

$$P_{00} = 1 - \frac{\lambda C'(1)}{\mu_1\mu_2\beta_1}[\beta_1(\mu_2+\theta\mu_1) + \mu_1\mu_2] \tag{54}$$

$$P_q(z) = \frac{(Z-1)(R+\mu_1)(R+\mu_2)(R+\beta_1)P_{00}}{Z(R+\mu_1)(R+\mu_2)(R+\beta_1) - \mu_1\beta_1[R(1-\theta)+\mu_2]} \tag{55}$$

$$L_q = \frac{2\lambda^2\left(C'(1)\right)^2[\mu_1^2\mu_2^2 + \left[\beta_1^2(\mu_2^2+\theta\mu_1^2) + \beta_1\mu_1\mu_2(\mu_2+\theta(\mu_1+\beta_1))\right] + \lambda C''(1)\beta_1\mu_1\mu_2[\mu_1\mu_2+\beta_1(\mu_2+\theta\mu_1)]}{2\mu_1\mu_2\beta_1[\mu_1\mu_2\beta_1 - \lambda C'(1)(\beta_1(\mu_2+\theta\mu_1)+\mu_1\mu_2)]} \tag{56}$$

(50)

8. NUMERICAL RESULTS

For the purpose of a numerical result, we choose The following arbitrary values: In table -I $C'(1) = 0.4$, $C''(1) = 0.03$, $\beta_1 = 4$, $\beta_2 = 5$, p = 0.1, $\theta = 0.2$, $\mu_1 = 2$, $\mu_2 = 4$ and in table –II, $\beta_1 = 2$, $\beta_2 = 3$, $\mu_1 = 4$ while λ varies from 0.1 to 1.0 such that the steady state condition is satisfied. The tables I-II give computed values of various states of the server, the idle time, the utilization factor, the mean queue size and customers mean wating time in the queue.

The table I-II clearly shows as long as we increase the arrival rate, the server idle time decreases while the utilization factor, the mean queue size and the mean waiting of the customers all increases.

Table (1): Computed values of various queue characteristics.

λ	P_{00}	ρ	L_q	W_q
0.1	0.967206	0.032800	0.002111	0.021112
0.2	0.934400	0.065600	0.006109	0.030544
0.3	0.901600	0.098400	0.012198	0.040661
0.4	0.868800	0.131200	0.020617	0.051542
0.5	0.836000	0.164000	0.031639	0.063278
0.6	0.803200	0.196800	0.045583	0.075971
0.7	0.770400	0.229600	0.062822	0.089746
0.8	0.737600	0.262400	0.083796	0.104745
0.9	0.704800	0.295200	0.109027	0.121141
1.0	0.672000	0.328000	0.139137	0.139137

Table (II): Computed values of various queue characteristics (p = 0).

λ	P_{00}	ρ	L_q	W_q
0.1	0.967333	0.032667	0.000847	0.008471
0.2	0.934667	0.065333	0.003524	0.017619
0.3	0.902000	0.098000	0.008255	0.027516
0.4	0.869333	0.130667	0.015299	0.038247
0.5	0.836667	0.163333	0.024955	0.049910
0.6	0.804000	0.196000	0.037571	0.062618
0.7	0.771333	0.228667	0.053552	0.076504
0.8	0.738667	0.261333	0.073359	0.091724
0.9	0.706000	0.294000	0.097617	0.108463
1.0	0.673333	0.326667	0.126944	0.126944

9. CONCLUSION

This paper clearly analyses a two phase queueing system with optional service and optional vacation. An application of this model can be found in mobile network where the messages are in batch form, the service may have two phases such that first phase is essential but the second phase may be chosen randomly as optional. The probability generating function of the number of customers in the queue is found using the supplementary variable technique. Also by some numerical approaches the validity of the results are examined.

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