International Journal of Computer Applications (0975 - 8887)
Volume 65 - No. 15, March 2013

Existence, Uniqueness and Stability of Neutral Stochastic Functional Integro-differential Evolution Equations with Infinite Delay

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ABSTRACT

This article presents the results on existence, uniqueness and stability of mild solutions to neutral stochastic functional evolution integro-differential equations with non-Lipschitz condition and Lipschitz condition. The existence of mild solutions for the equations are discussed by means of semigroup theory and theory of resolvent operator. Under some sufficient conditions, the results are obtained by using the method of successive approximation and Bihari's inequality. Moreover, an example is given to illustrate our results.

Keywords:

Resolvent operator, Evolution operator, Existence, Uniqueness, Stability, Successive approximation, Bihari's inequality

1. INTRODUCTION

Neutral stochastic differential equation occurs in many areas of science and engineering have attain much attention in the past decades. The partial integro-differential equations has wide applications in the field of mechanical, electrical and so on., and refer [12]. For abstract model of partial integro-differential equations with resolvent operators, see for instance [6, 8, 11]. The deterministic model often fluctuate due to noise. Under this circumstance, we move the deterministic model problems to stochastic model problems, for more details reader may refer [3, 7, 10, 18]. The existence and uniqueness of the neutral stochastic differential equations with infinite delay have been studied by many authors [5, 16]. Recently, the authors have established the problem with Lipschitz and non-Lipschitz condition, we suggest [2, 17, 20, 22] and reference therein.
On other hand, stochastic differential equations are well known problem in many areas of engineering and science. There are only few works on existence, uniqueness and stability of stochastic differential systems have been established [1, 16, 17, 22]. The stochastic systems with resolvent operators has occur in different applications such as heat equation, viscoelasticity and many other physical phenomena, see for instance [15]. The study of existence, uniqueness and stability of stochastic functional differential with resolvent operator is an unprocessed issue and it is also the motivation of this paper.
In [1] Anguraj et al. studied the impulsive stochastic neutral functional differential equations under non-Lipschitz condition and Lipschitz condition, whereas A. Lin et al. [17] have established on neutral impulsive stochastic integro-differential equations with infinite delay via fractional operators and H. Bin Chen [5] have proved the existence and uniqueness for the solution of neutral stochastic functional differential equations with infinite delay, then A. Vinodkumar [22] have examine the existence, uniqueness and stability results of impulsive stochastic semilinear functional differential equations with infinite delay. Recently, Y. Ren [21] have described the existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with poisson jumps and infinite delay. Moreover, the study was conducted on stability through the continuous dependence on the initial values by means of Bihari's inequality. For more details reader may refer [2, 10, 19].
Inspired by the above mentioned works [5, 8, 22], the purpose of this paper is to study the existence, uniqueness and stability for neutral stochastic functional integro-differential equations of the form

$$\begin{aligned} d[x(t) + g(t, x_t)] &= A(t)\big[x(t) + g(t, x_t)\big]dt \\ &+ \left[\int_0^t f(t,s)\big[x(s) + g(s, x_s)\big]ds + h(t, x_t)\right]dt \\ &+ \sigma(t, x_t)dw(t), \quad t \in J := [0, T], \qquad (1) \\ x_0 &= \varphi \in \mathcal{B}. \qquad (2) \end{aligned}$$

Here, the state $x(\cdot)$ takes the values in a real separable Hilbert space H with inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, $A(t)$ is the linear operators generates a linear evolution systems $\{R(t,s),\ t \geq 0\}$ on H, and $f(t,s), t \in J$ is a bounded linear operator. The history $x_t : (-\infty, 0] \to H, x_t(\theta) = x(t+\theta)$, for $t \geq 0$, belongs to the phase space $\mathcal{B}$, which will be described axiomatically in Preliminaries. Suppose $\{w(t); t \geq 0\}$ is a given K-valued Brownian motion with a finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by Wiener process w. we are also employing the same notation $\|\cdot\|$ for the norm $\mathcal{L}(K,H)$, where $\mathcal{L}(K,H)$ denotes the space of all bounded linear operator from K into H. Assume that $h : \mathbb{R}^+ \times \mathcal{B} \to H$ and $\sigma : \mathbb{R}^+ \times \mathcal{B} \to \mathcal{L}_Q(K,H)$, where $\mathbb{R}^+ = [0,\infty)$ are Borel measurable and $g : \mathbb{R}^+ \times \mathcal{B} \to H$ is continuous. Here, $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operator from K into H, which will be defined in Section 2.
The substance of the paper is organized as follows. Section 2, recapitulates some basic definitions, lemmas, notations, and theorems which will be used to develop our results. Section 3 and 4, give several sufficient conditions to prove the existence, uniqueness and stability for the problem (1)-(2) respectively. Section

5 is reserved for an example is to illustrate the efficiency of the obtained results.

2. PRELIMINARIES

Let $(K, \|\cdot\|_K)$ and $(H, \|\cdot\|_H)$ be the two real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_H$, respectively. We denote $\mathcal{L}(K,H)$ be the set of all linear bounded operator from K into H, equipped with the usual operator norm $\|\cdot\|$. In this article, we use the symbol $\|\cdot\|$ to denote norms of operator regardless of the space involved when no confusion possibly arises.
Let $(\Omega, \mathcal{F}, P; H)$ be the complete probability space furnished with a complete family of right continuous increasing σ- algebra $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. An H- valued random variable is an $\mathcal{F}$- measurable function $x(t) : \Omega \to H$ and a collection of random variables $S = \{x(t,\omega) : \Omega \to H \setminus t \in J\}$ is called stochastic process. Usually we write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $\mathrm{Tr}(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $\omega(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\omega_i(t)e_i$, where $\{\omega_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{\omega(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by ω and $\mathcal{F}_t = \mathcal{F}$. Let $\Psi \in \mathcal{L}(K,H)$ and define

$$\|\Psi\|_Q^2 = \mathrm{Tr}(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\Psi e_n\|^2.$$

If $\|\Psi\|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \to H$. The completion $\mathcal{L}_Q(K,H)$ of $\mathcal{L}(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\Psi\|_Q^2 = \langle\Psi,\Psi\rangle$ is a Hilbert space with the above norm topology.
In this work, the axiomatic definition of the phase space $\mathcal{B}$ is introduced by Hale et al. [13]. To establish the axioms of the phase space $\mathcal{B}$, we use the following terminology used in Hinto et al. [14]. The axioms of the space $\mathcal{B}$ are established for $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ into H, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ which satisfies the following axioms:

(A1) If $x : (-\infty, T] \to H$, $T > 0$ is such that $x_0 \in \mathcal{B}$ then for every $t \in [0,T]$, the following conditions hold:
(i) $x_t \in \mathcal{B}$;
(ii) $\|x(t)\| \leq L\|x_t\|_{\mathcal{B}}$;
(iii) $\|x_t\|_{\mathcal{B}} \leq M(t) \sup_{0\leq s\leq t} \|x(s)\| + N(t)\|x_0\|_{\mathcal{B}}$, where $L > 0$ is a constant; $M(\cdot), N(\cdot) : [0,+\infty) \to [1,+\infty)$, is continuous $N(\cdot)$ is locally bounded, and $L, M(\cdot), N(\cdot)$ are independent of $x(\cdot)$.
(A2) For the function $x(\cdot)$ in (A1), x_t is a $\mathcal{B}$-valued continuous function on $[0,b]$.
(A3) The space $\mathcal{B}$ is complete.

The $\mathcal{B}$- valued stochastic process $x_t : \Omega \to \mathcal{B}$, $t \geq 0$, is defined by $x_t(s) = \{x(t+s)(\omega) : s \in (-\infty, 0]\}$ The collection of all strongly measurable, square integrable, H-valued random variables, denoted by $L_2(\Omega, \mathcal{F}, P; H) \equiv L_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2}^2 = E\|x(\cdot,w)\|_H^2$, where E denotes expectation defined by $E(h) = \int_\Omega h(w)dP$. Let $C(J, L_2(\Omega; H))$ be the Banach space of all continuous map from J into $L_2(\Omega; H)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$. An important subspace is given by $L_2^0(\Omega, H) = \{f \in L_2(\Omega, H) : f \text{ is } \mathcal{F}_0 - \text{is measurable}\}$.
Let $\mathcal{Z}$ be the closed subspace of all continuously differentiable process x that belongs to the space $C(J, L_2(\Omega; H))$ consisting of $\mathcal{F}_t$- adapted measurable process such that the $\mathcal{F}_0$-adapted process $\varphi \in L_2^0(\Omega, \mathcal{B})$. Let $\|\cdot\|_{\mathcal{Z}}$ be a seminorm in $\mathcal{Z}$ defined by

$$\|x\|_{\mathcal{Z}} = \left(\sup_{t\in J} \|x_t\|_{\mathcal{B}}^2\right)^{\frac{1}{2}},$$

where

$$\|x_t\|_{\mathcal{B}} \leq N_T E\|\varphi\|_{\mathcal{B}} + M_T \sup\{E\|x(s)\| : 0 \leq s \leq T\},$$

$M_T = \sup_{t\in J}\{M(t)\}$, $N_T = \sup_{t\in J}\{N(t)\}$. It is easy to verify that $\mathcal{Z}$ furnished with the norm topology as defined above, is a Banach space.
The resolvent operator plays an important role in the study of the existence of solutions and to give a variation of constant formula for linear systems. However, need to know when the linear system (3) has a resolvent operator. For more details on resolvent operator, reader may refer [11].
The following assumptions are:

(H1) $A(t)$ generates a strongly continuous semigroup of evolution operators..
(H2) Suppose Y is a Banach space formed from $D(A)$ with the graph norm. $A(t)$ and $f(t,s)$ are closed operators it follows that $A(t)$ and $f(t,s)$ are in the set of bounded linear operators from Y to H, $f(Y,H)$, for $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively. $A(t)$ and $f(t,s)$ are continuous on $0 \leq t \leq T$ and $0 \leq s \leq t \leq T$, respectively, into $B(Y,H)$.

To obtain the results, consider the integro-differential abstract Cauchy problem

$$\begin{aligned} dx(t) &= \left[A(t)x(t) + \int_0^t f(t,s)x(s)ds\right]dt, \\ & 0 \leq s \leq t \leq T, \\ x(0) &= x_0 \in H. \end{aligned} \tag{3}$$

DEFINITION 1. *[11]*A family of bounded linear operator $R(t,s) \in \mathcal{P}(H)$, $0 \leq s \leq t \leq T$ is called a resolvent operator for

$$\frac{dx}{dt} = A(t)\left[x(t) + \int_0^t f(t,s)x(s)ds\right],$$

if

(i) $R(t,s)$ is strongly continuous in s and t. $R(t,t) = I$, the identity operator on H. $\|R(t,s)\| \leq Me^{\beta(t-s)}$ $t, s \in J$ and M, β are constants;
(ii) $R(t,s)Y \subset Y$, $R(t,s)$ is strongly continuous in s and t on Y;
(iii) For $y \in Y$, $R(t,s)y$ is continuously differentiable in s and t, and for $0 \leq s \leq t \leq T$,

$$\frac{\partial}{\partial t}R(t,s)y = A(t)R(t,s)y + \int_s^t f(t,r)R(r,s)ydr,$$
$$\frac{\partial}{\partial s}R(t,s)y = -R(t,s)A(s)y - \int_s^t R(t,r)f(r,s)ydr,$$

with $\frac{\partial}{\partial t}R(t,s)y$ and $\frac{\partial}{\partial s}R(t,s)y$ are strongly continuous on $0 \leq s \leq t \leq T$. Here $R(t,s)$ can be extracted from the evolution operator of the generator A(t).

For the family $\{A(t) : 0 \leq t \leq T\}$ of linear operators, the following restrictions are imposed:

(B1) The domain $D(A)$ of $\{A(t) : 0 \leq t \leq T\}$ is dense in X and independent of t, $A(t)$ is closed linear operator;
(B2) For each $t \in [0,T]$, the resolvent $R(\lambda, A(t))$ exists for all λ with $Re\lambda \leq 0$ and there exists $K > 0$ so that $\|R(\lambda, A(t))\| \leq K/(|\lambda|+1)$;

(B3) There exists $0 < \delta \leq 1$ and $K > 0$ such that $\|(A(t) - A(s))A^{-1}(\tau)\| \leq K|t-s|^{\delta}$ for all $t, s, \tau \in [0,T]$;

(B4) For each $t \in [0,T]$ and some $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, the resolvent $R(\lambda, A(t))$, is a compact operator.

Under these assumptions, the family $\{A(t) : 0 \leq t \leq T\}$ generates a unique linear evolution system, or called linear evolution operator.

DEFINITION 2. *[19]*A two parameter family of bounded linear operators $R(t,s)$, $0 \leq s \leq t \leq T$, on H is called an evolution system if the following two conditions hold

(i) $R(s,s) = I$, $R(t,r)R(r,s) = R(t,s)$, for $0 \leq s \leq r \leq t \leq T$.

(ii) $(t,s) \to R(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

LEMMA 1. *[19]*Assume that $(B1)$-$(B3)$ hold. Then, there exist a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq T$ and a constant $K > 0$ such that

(i) $R(t,s) \leq K$ for $0 \leq s \leq t \leq T$;

(ii) for $0 \leq s \leq t \leq T$, $R(t,s) : H \to Y$ and $t \to R(t,s)$ is strongly differentiable in H.The derivative $\frac{\partial}{\partial t}R(t,s)$ belongs to $L(H)$ and it is strongly continuous on $0 \leq s \leq t \leq T$. Moreover, for all $0 \leq s \leq t \leq T$, it holds

$$\frac{\partial}{\partial t}R(t,s) + A(t)R(t,s) = 0;$$

$$\|\frac{\partial}{\partial t}R(t,s)\| = \|A(t)R(t,s)\| \leq \frac{K}{t-s};$$

$$\|A(t)R(t,s)A(s^{-1})\| \leq K,$$

(iii) for each $y \in Y$ and $t \in J$, $R(t,s)y$ is differentiable with respect to s on $0 \leq s \leq t \leq T$ and $\frac{\partial}{\partial t}R(t,s)y = R(t,s)A(s)y$.

LEMMA 2. *[9]*Let $\{A(t), t \in J\}$ be a family of linear operators satisfying $(B1)-(B4)$. If $\{R(t,s), 0 \leq s \leq t \leq T\}$ is the linear evolution system generated by $\{A(t), t \in J\}$, then $\{R(t,s), 0 \leq s \leq t \leq T\}$ is a compact operator whenever $t - s > 0$.

LEMMA 3. *[4]*Let $T > 0$ and $u_0 \geq 0$, $u(t), v(t)$ be the continuous function on $[0,T]$. Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave continuous and nondecreasing function such that $K(r) > 0$ for $r > 0$. If

$$u(t) \leq u_0 + \int_0^t v(s)K(u(s))ds \quad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq G^{-1}\big(G(u_0) + \int_0^t v(s)ds\big) \text{ for all } t \in [0,T] \text{ such that}$$

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$ for $r \geq 0$ and G^{-1} is the inverse function of G. In particular, moreover if, $u_0 = 0$ and $\int_{0+} \frac{ds}{K(s)} = \infty$, then $u(t) = 0$ for all $t \in [0,T]$.

In order to obtain the stability of the solutions, the following extended Bihari's inequality is used.

LEMMA 4. *[20]*Let the assumption of Lemma 3 holds. If

$$u(t) \leq u_0 + \int_0^t v(s)K(u(s))ds \quad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq G^{-1}\big(G(u_0) + \int_0^t v(s)ds\big) \text{ for all } t \in [0,T] \text{ such that}$$

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$ for $r \geq 0$ and G^{-1} is the inverse function of G.

COROLLARY 1. *[20]*Let the assumption of Lemma 3 hold and $v(t) \geq 0$ for $t \in [0,T]$. If for all $\epsilon > 0$, there exists $t_1 \geq 0$ such that for $0 \leq u_0 \leq \epsilon$, $\int_{t_1}^T v(s)ds \leq \int_{u_0}^{\epsilon} \frac{ds}{K(s)}$ holds, then for every $t \in [t_1, T]$, the estimate $u(t) \leq \epsilon$ holds.

LEMMA 5. *[7]*For any $r \geq 1$ and for arbitrary L_2^0-valued predictable process $\Psi(\cdot)$

$$\sup_{s\in[0,t]} E\|\int_0^s \Psi(u)dw(u)\|_X^{2r} = (r(2r-1))^r\left(\int_0^t (E\|\Psi(s)\|_{L_2^0}^{2r})ds\right)^r.$$

The following is the definition of the mild solution of the system (1)-(2).

DEFINITION 3. A stochastic process $\{x(t) \in C(J, L_2(\Omega;\ H)), t \in (-\infty, T]\}$, $(0 < T < \infty)$, is said to be a mild solution of the equation (1)-(2) if

(i) $x(t) \in H$ is $\mathcal{F}_t$-adapted;

(ii) for each $t \in J$, $x(t)$ satisfies the following integral equation

$$x(t) = \begin{cases} \varphi(t), \quad \text{for} \quad t \in (-\infty, 0], \\ R(t,0)\big[\varphi(0) + g(0,\varphi)\big] - g(t,x_t) \\ + \int_0^t R(t,s)h(s,x_s)ds \\ + \int_0^t R(t,s)\sigma(s,x_s)dw(s) \\ \quad \text{for a.s} \quad t \in [0,T]. \end{cases} \quad (4)$$

3. EXISTENCE AND UNIQUENESS

In this section, the existence and uniqueness of mild solution of the system (1)-(2) are discussed and worked under the following assumptions:

(H3) There exists a resolvent operator $R(t,s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exists a constant $M_1 > 0$ such that $\|R(t,s)\|^2 \leq M_1$, for all $t \in J$.

(H4) For each $x, y \in \mathcal{B}$ and for all $t \in [0,T]$ such that

$$\|h(t,x_t) - h(t,y_t)\|^2 \vee \|\sigma(t,x_t) - \sigma(t,y_t)\|^2 \leq K(\|x_t - y_t\|_{\mathcal{B}}^2),$$

where $K(\cdot)$ is a concave non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, $K(0) = 0$, $K(u) > 0$, for $u > 0$ and $\int_{0+} \frac{du}{K(u)} = \infty$.

(H5) Assuming that there exists a positive number M_g, such that $M_g < \frac{1}{8}$, for any $x, y \in \mathcal{B}$ and for $t \in [0,T]$, we have

$$\|g(t,x_t) - g(t,y_t)\|^2 \leq M_g\|x_t - y_t\|_{\mathcal{B}}^2.$$

(H6) For all $t \in [0,T]$, it follows that $\sigma(t,0), h(t,0), g(t,y_t) \in L^2$ such that

$$\|\sigma(t,0)\|^2 \vee \|h(t,0)\|^2 \vee \|g(t,0)\|^2 \leq \kappa_0,$$

where $\kappa_0 > 0$ is a constant.

Let us now introduce the successive approximation to equation (4) as follows

$$x^0(t) = \begin{cases} \varphi(t) \text{ for } t \in (-\infty, 0], \\ R(t,0)\varphi(0) \text{ for } t \in [0,T]. \end{cases} \tag{5}$$

and, for $n = 1, 2, \dots$,

$$x^n(t) = \begin{cases} \varphi(t), \text{ for } t \in (-\infty, 0], \\ R(t,0)\big[\varphi(0) + g(0,\varphi)\big] - g(t, x_t^n) \\ + \int_0^t R(t,s)h(s, x_s^{n-1})ds \\ + \int_0^t R(t,s)\sigma(s, x_s^{n-1})dw(s), \\ \text{for a.s, } t \in J, \end{cases} \tag{6}$$

with an arbitrary non-negative initial approximation $x^0 \in C(J, L_2(\Omega; H))$.

THEOREM 1. Assume that $(H3)-(H6)$ hold. Then the system (1)-(2) has unique mild solution $x(t)$ in $C(J, L_2(\Omega; H))$ and

$$E\{\sup_{0\le t\le T} \|x^n(t) - x(t)\|^2\} \to 0, \quad n \to \infty,$$

where $\{x^n(t)\}_{n\ge1}$ are successive approximations (6).

PROOF. Let $x^0 \in C(J, L_2(\Omega; H))$ be a fixed initial approximation to (6). To begin with the assumptions $(H3)-(H6)$ and observing that $\|R(t,s)\|^2 \le M_1$ for some $M_1 \ge 1$ and for all $t \in [0,T]$. Then for any $n \ge 1$, we have

$$\begin{aligned}\|x^n(t)\|^2 &\le 4M_1\|\big[\varphi(0) - g(0,\varphi)\big]\|^2 \\ &+ 8\big[\|g(t,x_t^n) - g(t,0)\|^2 + \|g(t,0)\|^2\big] \\ &+ 8M_1T\int_0^t [\|h(s,x_s^{n-1}) - h(s,0)\|^2 + \|h(s,0)\|^2]ds \\ &+ 8M_1\int_0^t [\|\sigma(s,x_s^{n-1}) - \sigma(s,0)\|^2 + \|\sigma(s,0)\|^2]ds.\end{aligned}$$

Thus,

$$E\|x_t^n\|_{\mathcal{B}}^2 \le \frac{N_1}{1-8M_g} + \frac{8M_1(T+1)}{1-8M_g}E\int_0^t K(\|x_s^{n-1}\|_{\mathcal{B}}^2)ds,$$

where $N_1 = 8M_1\big[E\|\varphi(0)\| + M_gE\|\varphi\|_0^2\big] + 8\big[1 + M_1T(T+1)\big]\kappa_0$.
Given that $K(\cdot)$ is concave and $K(0) = 0$, the following pair of positive constants a and b are found such that

$$K(u) \le a + bu, \text{ for all } u \ge 0.$$

Then, we have

$$\begin{aligned}E\|x_t^n\|_{\mathcal{B}}^2 &\le N_2 + \frac{8M_1(T+1)b}{1-8M_g}\int_0^t E\|x_s^{n-1}\|_{\mathcal{B}}^2 ds \\ &\le N_2 + \frac{8M_1(T+1)b}{1-8M_g}\int_0^t \big[N(t)E\|\varphi\|_{\mathcal{B}}^2 + M(t)\sup_{0\le s\le T} \\ &\qquad E\|x^{n-1}(s)\|^2\big]ds \\ &\le N_2 + \frac{8M_1T(T+1)b}{1-8M_g}N_TE\|\varphi\|_{\mathcal{B}}^2 \\ &\quad + \frac{8M_1(T+1)b}{1-8M_g}\int_0^t \sup_{0\le s\le T} E\|x^{n-1}(s)\|^2 ds,\end{aligned}$$

where $N_2 = \frac{N_1}{1-8M_g} + \frac{8M_1T(T+1)a}{1-8M_g}$.
Therefore,

$$E\|x_t^n\|_{\mathcal{B}}^2 \le N_3 + \frac{8M_1(T+1)b}{1-8M_g}\int_0^t \sup_{0\le s\le T} E\|x^{n-1}(s)\|^2 ds, \tag{7}$$

where, $N_3 = N_2 + \frac{8M_1T(T+1)b}{1-8M_g}N_TE\|\varphi\|_{\mathcal{B}}^2$.
Since

$$E\|x_t^0\|_{\mathcal{B}}^2 \le M_1E\|\varphi(0)\|^2 = N_4 < \infty. \tag{8}$$

Thus,

$$E\|x_t^n\|_{\mathcal{B}}^2 \le N_5 < \infty, \tag{9}$$

for all $n = 0, 1, 2, \dots$, and $t \in [0,T]$. This proves the boundedness of $\{x^n(t),\, n \in \mathbb{N}\}$.
Let us next show that $\{x^n(t)\}$ is a Cauchy sequence in $C(J, L_2(\Omega; H))$. For $n, m \ge 1$, we have

$$\begin{aligned}&E\|x^{n+1}(t) - x^{m+1}(t)\|^2 \\ &\le 3M_gE\|x^{n+1}(t) - x^{m+1}(t)\|_{\mathcal{B}}^2 \\ &+ 3M_1(T+1)\int_0^t K(E\|x^n(s) - x^m(s)\|_{\mathcal{B}}^2)ds \\ &\le \frac{3M_1(T+1)}{1-3M_g}\int_0^t K(E\|x^n(s) - x^m(s)\|_{\mathcal{B}}^2)ds.\end{aligned}$$

Thus

$$\begin{aligned}&\sup_{0\le s\le t} E\|x_s^{n+1} - x_s^{m+1}\|_{\mathcal{B}}^2 \\ &\qquad \le N_6\int_0^t K\Big(\sup_{0\le r\le s} E\|x_r^n - x_r^m\|_{\mathcal{B}}^2\Big)ds,\end{aligned} \tag{10}$$

where $N_6 = \frac{3M_1(T+1)}{1-3M_g}$.
Integrating both sides of equation (10) and applying Jensen's inequality gives that

$$\begin{aligned}&\int_0^t \sup_{0\le l\le s} E\|x_l^{n+1} - x_l^{m+1}\|_{\mathcal{B}}^2 ds \\ &\le N_6\int_0^t\int_0^s K\big(\sup_{0\le r\le l} E\|x_r^n - x_r^m\|_{\mathcal{B}}^2\big)dlds \\ &\le N_6\int_0^t s\int_0^s K\big(\sup_{0\le r\le l} E\|x_r^n - x_r^m\|_{\mathcal{B}}^2\big)\frac{1}{s}dlds \\ &\le N_6t\int_0^t K\big(\int_0^s \sup_{0\le r\le l} E\|x_r^n - x_r^m\|_{\mathcal{B}}^2\frac{1}{s}dl\big)ds.\end{aligned}$$

Then

$$\Phi_{n+1,m+1}(t) \le N_6\int_0^t K\big(\Phi_{n,m}(s)\big)ds, \tag{11}$$

where

$$\Phi_{n,m}(t) = \frac{\int_0^t \sup_{0\le r\le l} E\|x_r^n - x_r^m\|_{\mathcal{B}}^2 ds}{t}.$$

From (9), it is easy see that

$$\sup_{n,m} \Phi_{n,m}(t) < \infty.$$

So letting $\Phi(t) = \limsup_{n,m\to\infty} \Phi_{n,m}(t)$ and enchanting the relation Fatou's lemma, it capitulate that

$$\Phi(t) = N_6\int_0^t K\Big(\Phi(s)\Big)ds.$$

Now, applying the Lemma 3, instantaneously expose $\Phi(t) = 0$ for any $t \in [0,T]$. This further means $\{x^n(t),\, n \in \mathbb{N}\}$ is a Cauchy sequence in $C(J, L_2(\Omega; H))$. So there is an $x \in C(J, L_2(\Omega; H))$ such that

$$\lim_{n\to\infty}\int_0^T \sup_{0\le s\le t} E\|x_s^n - x_s\|_{\mathcal{B}}^2 dt = 0.$$

In addition, by (9), it is easy to follow that $E\|x_t\|_{\mathcal{B}}^2 \leq N_5$. Thus, $x(t)$ is a mild solution to $(1)-(2)$. On the other hand, by (H4) then, letting $n \to \infty$, we can also claim that for $t \in [0,T]$

$$E\| \int_0^t R(t,s)\Big[h(t,x_s^{n-1}) - h(t,x_s)\Big]ds\|_{\mathcal{B}}^2 \to 0,$$
$$E\| \int_0^t R(t,s)\Big[\sigma(t,x_s^{n-1}) - \sigma(t,x_s)\Big]dw(s)\|_{\mathcal{B}}^2 \to 0.$$

On further, by applying (H5) then, for $t \in [0,T]$, that

$$E\|g(s,x_s^n) - g(s,x_s)\|^2 \leq M_g E\|x^n(s) - x(s)\|_{\mathcal{B}}^2 \to 0.$$

At this instant, taking limits in both sides of (6) leads, for $t \geq 0$, to

$$x(t) = R(t,0)\big[\varphi(0) - g(0,\varphi)\big] + g(t,x_t) + \int_0^t R(t,s)h(s,x_s)ds + \int_0^t R(t,s)\sigma(s,x_s)dw(s).$$

This certainly exhibit by the Definition 3 that $x(t)$ is a mild solution of (1)-(2) on the interval $[0,T]$.
Now, the uniqueness of the solutions of (4) is to be proved by the following: Let $x,y \in C(J, L_2(\Omega;H))$ be the two solution of (1)-(2) on some interval $(-\infty,T]$. Then, for $t \in (-\infty,0]$, the uniqueness is obvious and for $0 \leq t \leq T$, we have

$$E\|x(t) - y(t)\|^2 \leq 3M_g E\|x_t - y_t\|_{\mathcal{B}}^2 + 3M_1(T+1)\int_0^t K(E\|x_s - y_s\|_{\mathcal{B}}^2)ds.$$

Thus,

$$E\|x_t - y_t\|_{\mathcal{B}}^2 \leq \frac{3M_1(T+1)}{1-3M_g}\int_0^t K(E\|x_s - y_s\|_{\mathcal{B}}^2)ds \leq N_6\int_0^t K(E\|x_s - y_s\|_{\mathcal{B}}^2)ds.$$

Thus, Bihari's inequality yield that

$$\sup_{t\in[0,T]} E\|x_t - y_t\|_{\mathcal{B}}^2 = 0,\ 0 \leq t \leq T.$$

Thus, $x(t) = y(t)$, for all $0 \leq t \leq T$. Therefore, for all $-\infty \leq t \leq T$, $x(t) = y(t)$. This completes the proof. □

4. STABILITY

In this section, the stability through the continuous dependence on initial values is studied by the following definition.

DEFINITION 4. A mild solution $x(t)$ of the system (1)-(2) with initial value φ is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$E\|x_t - \widehat{x}_t\|_{\mathcal{B}}^2 \leq \epsilon, \text{ whenever } E\|\varphi - \widehat{\varphi}\|^2 \leq \delta, \text{ for all } t \in [0,T],$$

where $\widehat{x}(t)$ is another mild solution of the system (1)-(2) with initial data $\widehat{\varphi}$.

THEOREM 2. Let $x(t)$ and $y(t)$ be the mild solution of the system (1)-(2) with initial values φ_1 and φ_2 respectively. If the assumption of Theorem 1 are satisfied, then the mild solution of the system (1)-(2) is stable in the mean square.

PROOF. By the assumption, $x(t)$ and $y(t)$ are two mild solutions of equations (1)-(2) with initial values φ_1 and φ_2 respectively, then for $0 \leq t \leq T$,

$$\begin{aligned} x(t) - y(t) &= R(t,0)\big(\big[\varphi_1(0) - \varphi_2(0)\big] + \big[g(0,\varphi_1) - g(0,\varphi_2)\big]\big) \\ &+ \big[g(t,x_t) - g(t,y_t)\big] + \int_0^t R(t,s)\big[h(s,x_s) - h(s,y_s)\big]ds \\ &+ \int_0^t R(t,s)\big[\sigma(s,x_s) - \sigma(s,y_s)\big]dw(s). \end{aligned}$$

So, estimating as before, we get

$$\begin{aligned} E\|x(t) - y(t)\|^2 &\leq 5M_1\big[1 + M_g\big]E\|\varphi_1 - \varphi_2\|^2 + 5M_g E\|x_t - y_t\|_{\mathcal{B}}^2 \\ &+ 5M_1(T+1)\int_0^t K(E\|x_s - y_s\|_{\mathcal{B}}^2)ds. \end{aligned}$$

Thus,

$$E\|x_t - y_t\|_{\mathcal{B}}^2 \leq \frac{5M_1\big[1+M_g\big]}{1-5M_g}E\|\varphi_1 - \varphi_2\|^2 + \frac{5M_1(T+1)}{1-5M_g}\int_0^t K(E\|x_s - y_s\|_{\mathcal{B}}^2)ds.$$

Let $K_1(u) = \frac{5M_1(T+1)}{1-5M_g}K(u)$, where K is concave increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0+}\frac{du}{K(u)} = +\infty$. So, $K_1(u)$ is obviously, a concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K_1(0) = 0$, $K_1(u) \geq K(u)$, for $0 \leq u \leq 1$ and $\int_{0+}\frac{du}{K_1(u)} = +\infty$. Now for any $\epsilon > 0$, $\epsilon_1 = \frac{1}{2}\epsilon$, then $\lim_{s\to 0}\int_s^{\epsilon_1}\frac{du}{K_1(u)} = \infty$. So, there is a positive constant $\delta < \epsilon_1$, such that $\int_\delta^{\epsilon_1}\frac{du}{K_1(u)} \geq T$.
Let

$$u_0 = \frac{5M_1\big[1+M_g\big]}{1-5M_g}E\|\varphi_1 - \varphi_2\|^2,$$
$$u(t) = E\|x_t - y_t\|_{\mathcal{B}}^2,\ v(t) = 1,$$

when $u_0 \leq \delta \leq \epsilon_1$. From Corollary 1 we have

$$\int_{u_0}^{\epsilon_1}\frac{du}{K_1(u)} \geq \int_\delta^{\epsilon_1}\frac{du}{K_1(u)} \geq T = \int_0^T v(s)ds.$$

So, for any $t \in [0,T]$, we estimate $u(t) \leq \epsilon_1$ holds. This completes the proof. □

REMARK 1. Consider the following stochastic functional integro-differential systems of the form

$$\begin{aligned} d[x(t) + g(t,x_t)] &= A\big[x(t) + g(t,x_t)\big]dt \\ &+ \Big[\int_0^t f(t,s)\big[x(s) + g(s,x_s)\big]ds + h(t,x_t)\Big]dt \\ &+ \sigma(t,x_t)dw(t), \quad t \in J := [0,T], \end{aligned} \tag{12}$$
$$x_0 = \varphi \in \mathcal{B}, \tag{13}$$

where A is the infinitesimal generator of a strongly continuous semigroup $R(t), t \geq 0$ defined on H. Moreover, g, h, σ are same as defined in (1)-(2). Here $R(t,s) = R(t-s), \quad t > s$. The mild solution of (12)-(13) is as follows

$$x(t) = \begin{cases} \varphi(t), \quad \text{for} \quad t \in (-\infty,0], \\ R(t)\big[\varphi(0) + g(0,\varphi)\big] - g(t,x_t) \\ + \int_0^t R(t-s)h(s,x_s)ds \\ + \int_0^t R(t-s)\sigma(s,x_s)dw(s) \\ \text{for a.s} \quad t \in [0,T]. \end{cases} \tag{14}$$

REMARK 2. In [8] the author has studied existence and uniqueness of the stochastic integro-differential systems for (12)-(13) with finite delay. Here, the nature of work is to study the existence, uniqueness and stability results of the equation (12)-(13) with infinite delay. The result of equation (12)-(13) as obtained by the following hypothesis.

(H7) There exists a resolvent operator $R(t-s)$ which is compact and continuous in the uniform operator topology for $t > s$. Further, there exists a constant $M_1 > 0$ such that $\|R(t)\|^2 \leq M_1$, for all $t \in J$.

THEOREM 3. Assume that $(H4) - (H6)$ and $(H7)$ hold. Then the system (12)-(13) has unique mild solution $x(t)$ in $C(J, L_2(\Omega; H))$ and

$$E\{\sup_{0\leq t\leq T} \|x^n(t) - x(t)\|^2\} \to 0, \quad n \to \infty,$$

where $\{x^n(t)\}_{n\geq 1}$ are successive approximations (14).

PROOF. The proof is very similar to Theorem 1. Hence, it is omitted. □

THEOREM 4. Let $x(t)$ and $y(t)$ be the mild solution of the system (12)-(13) with initial values φ_1 and φ_2 respectively. If the assumption of Theorem 3 are satisfied, then the mild solution of the system (12)-(13) is stable in the mean square.

PROOF. The proof of this theorem is similar to Theorem 2. □

5. EXAMPLE

Consider the following stochastic partial integro-differential equation of the form

$$d\Big[u(t,\xi) + \int_0^\pi a(y,\xi)u(tsint,y)dy\Big]$$
$$= \frac{\partial^2}{\partial\xi^2}\Big[u(t,\xi) + \int_0^\pi a(y,\xi)u(tsint,y)dy\Big]dt$$
$$+ \Big[\int_0^t f(t-s)\big[u(s,\xi) + \int_0^\pi a(y,\xi)u(tsint,y)dy\big]ds$$
$$+ H(t,u(tsint,\xi))\Big]dt + G(t,u(tsint,\xi))d\beta(t),$$
$$0 \leq \xi \leq \pi,\ \tau > 0, t \in J = [0,T], \tag{15}$$
$$u(t,0) = u(t,\pi) = 0, \quad t \in J, \tag{16}$$
$$u(\theta,\xi) = \varphi(\theta,\xi), \quad \theta \in (-\infty,0], \quad 0 \leq \xi \leq \pi, \tag{17}$$

where $\beta(t)$ denotes a standard cylindrical Wiener process in H defined on a stochastic process $(\Omega, \mathcal{F}, P)$ and $H = L^2([0,\pi])$. To rewrite (15)-(17) into the form (1)-(2), define $A : H \to H$ by $Az = z''$ with domain

$D(A) = \Big\{z \in H, z, z'$are absolutely continuous$z'' \in H, z(0) = z(\pi) = 0\Big\}$.

Then, A generates a strongly continuous semigroup $R(t)$ on H, thus (H1) is true. Moreover, the operator A can be expressed as

$$Az = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \quad z \in D(A),$$

where $z_n(s) = \sqrt{\frac{2}{\pi}}sin(ns)$, $n = 1, 2, \ldots$, is orthonormal set of eigenvectors of A.

In addition, it follows that $R(t)$ is compact for every $t > 0$ and $\|R(t)\| \leq e^{-t}$ for every $t \geq 0$.

Now, we define an operator $A(t) : D(A) \subset H \to H$ by

$$A(t)x(\xi) = Ax(\xi) + b(t,\xi)x(\xi).$$

Let $b(\cdot)$ be continuous and $b(t,\xi) \leq -\gamma(\gamma > 0)$, for every $t \in \mathbb{R}$. Then, the system

$$\begin{cases} u'(t) = A(t)u(t), & t \geq s, \\ u(s) = x \in H \end{cases}$$

has an associated evolution family, given by

$$R(t,s)x(\xi) = [R(t-s)e^{\int_s^t b(\varsigma,\xi)d\varsigma}x](\xi)$$

From the above expression, it follows that $R(t,s)$ is a compact operator and for every $t, s \in J$ with $t > s$

$$\|R(t,x)\| = e^{-(1+\gamma)(t-s)}.$$

The assumption of the following conditions hold:

(i) The function b is measurable and

$$\int_0^\pi a^2(y,\xi)dyd\xi < \infty.$$

(ii) The function $\frac{\partial}{\partial t}b(y,\xi)$ is measurable $b(y,0) = b(y,\pi) = 0$ and let

$$M_g = \Big[\int_0^\pi \int_0^\pi \big(\frac{\partial}{\partial t}a(y,\xi)\big)^2 dyd\xi\Big]^{\frac{1}{2}} < \infty.$$

Let $\alpha < 0$, define the phase space

$$\mathcal{B} = \Big\{\phi \in C((-\infty,0],H) : \lim_{\theta\to -r} e^{\alpha\theta}\phi(\theta) \quad \text{exists in H}\Big\},$$

and let $\|\phi\|_{\mathcal{B}} = \sup_{\theta\in(-\infty,0]} \{e^{\alpha\theta}\|\phi(\theta)\|_{L_2}\}$. Then, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space and satisfied axioms(1)-(2) with $L = 1$, $N(t) = e^{-\alpha t}$, $M(t) = \max\{1, e^{-\alpha t}\}$. Thus for $(t,\phi) \in J \times \mathcal{B}$, where $\phi(\theta)(\xi) = \varphi(\theta,\xi)$, $(\theta,\xi) \in (-\infty,0] \times [0,\pi]$.

Suppose that conditions (i) and (ii) are verified, then the problem (5.1)-(5.3) can be represent as the abstract neutral stochastic integro-differential equation of the form (1)-(2), as follows

$$g(t,x_t) = \int_0^\pi a(y,\xi)u(tsint,y)dy,\ h(t,x_t) = H(t,u(tsint,\xi)),$$
$$\sigma(t,x_t) = G(t,u(tsint,\xi))$$

The below results are consequence of Theorem 1 and Theorem 2 respectively.

PROPOSITION 1. If the hypothesis (H1)-(H6) hold, then there exists a unique mild solution of u of the system (15)-(17).

PROPOSITION 2. If all the hypothesis of Proposition 1 hold, then the mild solution u of the system (15)-(17) is stable in the mean square.

6. CONCLUSION

In this paper, the existence, uniqueness and stability of neutral stochastic integro-differential evolution equations with infinite delay are discussed by using phase space axioms. The results are obtained by using the method of successive approximation and Bihari's inequality. Finally, an example is illustrated for the effectiveness of the results.

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